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Tangles, tree-decompositions and grids in matroids ☆

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ABSTRACT

A tangle in a matroid is an obstruction to small branch-width. In particular, the maximum order of a tangle is equal to the branch-width. We prove that: (i) there is a tree-decomposition of a matroid that "displays" all of the maximal tangles, and (ii) when M is representable over a finite field, each tangle of sufficiently large order "dominates" a large grid-minor. This extends results of Robertson and Seymour concerning Graph Minors.

1. Introduction

Robertson and Seymour [6] introduced branch-width for graphs and showed that this parameter is characterized by "tangles". Robertson and Seymour also stated that their results extend to matroids [6, p. 190]; the details were later given by Dharmatilake [1] (see, also, [3]). Here we use the definitions given in [3]; we defer these definitions until Section 3. For the purpose of this introduction, a tangle of order θ in M can be thought of as a "θ-connected component" of M. We prove the following two results.

1.1. *Each matroid has a tree-decomposition that "displays" all its maximal tangles.*

This will be made precise in Theorem 9.1, which extends a result in Graph Minors X [6, (10.3)].

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Theorem 1.2. *For each finite field $\mathbb{F}$ and positive integer k there exists an integer θ such that, if M is an $\mathbb{F}$-representable matroid and $\mathcal{T}$ is a tangle in M of order θ, then $\mathcal{T}$ dominates a minor N that is isomorphic to the cycle matroid of a k by k grid.*

The proof is given in Section 7. Theorem 1.2 extends a result of Robertson, Seymour, and Thomas [8, (2.3)]. The term "dominates" is used specifically with respect to grid-minors and is defined in Section 7. To prove Theorem 1.2 we will use the main result of [4] which says that an $\mathbb{F}$-representable matroid with huge branch-width contains a large grid-minor.

These results are technical, but the motivation is to, hopefully, use them in extending the Graph Minors Structure Theorem [7]. For example, for certain fixed binary matroids N, we are interested in the class of binary matroids that do not contain an N-minor. Typically we choose N to be a highly structured matroid, such as: the cycle matroid of a grid, the cycle matroid of a complete graph, or a projective geometry. In such cases N has a unique maximal tangle $\mathcal{T}_N$. Now, if N is a minor of some binary matroid M, then the tangle $\mathcal{T}_N$ "induces" a tangle $\mathcal{T}_M$ in M. Any tangle in M that contains $\mathcal{T}_M$ is said to "dominate" N. Now 1.1 shows that the maximal tangles in M are composed in a tree-like way. This tree structure essentially localizes each maximal tangle in M and shows how M is composed from these local parts. So, to determine the structure of binary matroids with no N-minor, it suffices to determine the local structure of each maximal tangle in M that does not dominate an N-minor. Unfortunately the local structure of tangles that do not dominate N is complicated. This is partly overcome by considering only tangles whose order is much larger than the order of $\mathcal{T}_N$. By Theorem 1.2, each such tangle dominates a huge grid. Supposing that our tangle does not dominate an N-minor, the hope then is that this huge grid-minor will impose local structure on M.

2. Connectivity and branch-width

We assume that the reader is familiar with matroid theory; we use the notation of Oxley [5].

Let λ be a function that assigns an integer value to each subset of a finite set E. We call λ *symmetric* if $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$. We call λ *submodular* if $\lambda(X) + \lambda(Y) \geqslant \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E$. If λ is integer-valued, symmetric, and submodular, then we call λ a *connectivity function on E*. A *connectivity system* is a pair $K = (E, \lambda)$ where λ is a connectivity function on E. A partition (A, B) of $E(K)$ is called a *separation of order $\lambda_K(A)$*.

For a matroid M and $X \subseteq E(M)$, we let $\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M) + 1$. It is straightforward to prove that $K_M = (E(M), \lambda_M)$ is a connectivity system. For a graph G and $X \subseteq E(G)$, we let $\lambda_G(X)$ denote the number of vertices of G that are incident with both an edge of X and an edge of $E(G) - X$. It is also straightforward to prove that $K_G = (E(G), \lambda_G)$ is a connectivity system. Moreover, if G is connected we have for each $X \subseteq E(G)$ that $\lambda_{M(G)}(X) \leqslant \lambda_G(X)$.

Branch-width plays only a minor role in this paper, but we include a definition for completeness. Let K be a connectivity system. A tree is *cubic* if its internal vertices all have degree 3. A *branch-decomposition* of K is a cubic tree T whose leaves are labeled by elements of $E(K)$ such that each element in $E(K)$ labels exactly one leaf of T and each leaf of T receives at most one label from $E(K)$. If T' is a subgraph of T and $X \subseteq E(K)$ is the set of labels of T', then we say that *T' displays X*. The *width* of an edge e of T is defined to be $\lambda_K(X)$ where X is the set displayed by one of the components of $T - \{e\}$. The *width* of T is the maximum among the widths of its edges. The *branch-width* of K is the minimum among the widths of all branch-decompositions of K.

The *branch-width* of a matroid M is the branch-width of its connectivity system $K_M = (E(M), \lambda_M)$.

We remark that there are some trivial graphs G, such as trees, for which K_G and $K_{M(G)}$ have different branch-width. It is, however, conjectured that, if G has a circuit of length at least 2, then K_G and $K_{M(G)}$ have the same branch-width. In Section 6 we prove that this is at least true for n by n grids.

3. Tangles

In this section we review results and definitions from [3].

Let K be a connectivity system. A *tangle* in K of *order* θ is a collection $\mathcal{T}$ of subsets of $E(K)$ such that:

($T1$) For each $B \in \mathcal{T}$, $\lambda_K(B) < \theta$.
($T2$) For each separation (A, B) of order less than θ, $\mathcal{T}$ contains either A or B.
($T3$) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E(K)$.
($T4$) For each $e \in E(K)$, $E(K) - \{e\} \notin \mathcal{T}$.

It is proved in [3, Lemma 3.1] that, to verify that $\mathcal{T}$ is a tangle, we may replace ($T3$) by the following weaker conditions:

($T3a$) If $B \in \mathcal{T}$, $A \subseteq B$, and $\lambda_K(A) < \theta$, then $A \in \mathcal{T}$.
($T3b$) If (A_1, A_2, A_3) is a partition of $E(K)$, then $\mathcal{T}$ does not contain all three of A_1, A_2, and A_3.

Note that throughout this text partitions may have empty members; in particular, ($T3b$) also says that no two members of $\mathcal{T}$ partition $E(K)$.

The following slight variation of [6, (3.5)] was proved in [3, Theorem 3.2].

Theorem 3.1. *Let K be a connectivity system. Then, the maximum order of a tangle in K is equal to the branch-width of K.*

A *tangle* in a matroid M is a tangle in its connectivity system K_M. The following fact is used in the proof of 7.3.1.

Lemma 3.2. *Let $\mathcal{T}$ be a tangle of order θ at least 3 in a matroid M. Then each subset of $E(M)$ with rank less than $\theta - 1$ is in $\mathcal{T}$.*

Proof. Let X be a smallest possible subset in $E(M)$ that is not in $\mathcal{T}$. As $\theta \geqslant 3$ it follows from ($T2$) and ($T4$) that singletons are in $\mathcal{T}$. So X can be partitioned into two smaller sets. By the choice of X these two sets are in $\mathcal{T}$. Hence by ($T3$), $E(M) - X$ is not in $\mathcal{T}$. Thus by ($T2$), $\lambda_M(X) \geqslant \theta$. Note that, for any $Y \subseteq E(M)$, the rank of Y is at least $\lambda_M(Y) - 1$. So X has rank at least $\theta - 1$; as required. □

Let $\mathcal{T}$ be a tangle of order θ in matroid M. For $X \subseteq E(M)$, if X is a subset of a set in $\mathcal{T}$, then we let

$$\phi_{\mathcal{T}}(X) = \min\bigl(\lambda_M(A) - 1:\ X \subseteq A \in \mathcal{T}\bigr),$$

otherwise we let $\phi_{\mathcal{T}}(X) = \theta - 1$. The following result was proved in [3, Lemma 4.3].

Lemma 3.3. *Let M be a matroid and let $\mathcal{T}$ be a tangle in M of order θ. Then $\phi_{\mathcal{T}}$ is the rank-function of a matroid of rank $\theta - 1$.*

This matroid is referred to as the *tangle matroid* of $\mathcal{T}$.

4. New tangles from old

In this section we look at different constructions for tangles. Let $\mathcal{T}$ be a tangle of order θ in a connectivity system K and let $\theta' \leqslant \theta$. Now let $\mathcal{T}'$ be the collection of all sets $A \in \mathcal{T}$ with $\lambda_K(A) < \theta'$. It is straightforward to verify that:

Lemma 4.1. *$\mathcal{T}'$ is a tangle in K of order θ'.*

We say that $\mathcal{T}'$ is the *truncation* of $\mathcal{T}$ to order θ'. Note that if $\mathcal{T}'$ and $\mathcal{T}$ are tangles in K, then $\mathcal{T}'$ is a truncation of $\mathcal{T}$ if and only if $\mathcal{T}' \subseteq \mathcal{T}$.

Let $K = (E, \lambda)$ be a connectivity system and let $X \subseteq E$. We let $K \circ X = ((E - X) \cup \{e_X\}, \lambda')$ where, for each $A \subseteq E - X$, $\lambda'(A) = \lambda(A)$ and $\lambda'(A \cup \{e_X\}) = \lambda(A \cup X)$. It is straightforward to verify that:

Lemma 4.2. *If K is a connectivity system and $X \subseteq E(K)$, then $K \circ X$ is a connectivity system.*

We can also obtain a tangle in $K \circ X$ from a tangle in K.

Lemma 4.3. *Let $\mathcal{T}$ be a tangle of order θ in the connectivity system K and let $X \in \mathcal{T}$. Now let $\mathcal{T}'$ be the collection of subsets of $E(K \circ X)$ such that, for $A \subseteq E(K) - X$, $A \in \mathcal{T}'$ if and only if $A \in \mathcal{T}$; and $A \cup \{e_X\} \in \mathcal{T}'$ if and only if $A \cup X \in \mathcal{T}$. Then $\mathcal{T}'$ is a tangle of order θ in $K \circ X$.*

Proof. Each of the conditions $(T1)$–$(T4)$ for $\mathcal{T}'$ to be a tangle follows directly from the corresponding condition for $\mathcal{T}$. □

A set X of elements in a connectivity system K is called *titanic* if each partition (A_1, A_2, A_3) of X satisfies $\lambda_K(A_i) \geqslant \lambda_K(X)$ for at least one $i = 1, 2, 3$.

The following result is a partial converse of Lemma 4.3; it generalizes a result in Graph Minors X [6, (8.3)].

Lemma 4.4. *Let K be a connectivity system, let $X \subseteq E(K)$ be titanic with $\lambda_K(X) < \theta$, and let $\mathcal{T}'$ be a tangle of order θ in $K \circ X$. Now let $\mathcal{T}$ be the collection of all $A \subseteq E(K)$ such that $\lambda_K(A) < \theta$ and either $A - X \in \mathcal{T}'$ or $(A - X) \cup \{e_X\} \in \mathcal{T}'$. Then $\mathcal{T}$ is a tangle of order θ in K.*

Proof. Let $Y = E(K) - X$ and $L = K \circ X$. Note that $\lambda_L(\{e_X\}) = \lambda_L(Y) = \lambda_K(Y) = \lambda_K(X) < \theta$, so $\{e_X\} \in \mathcal{T}'$. By definition, $\mathcal{T}$ satisfies $(T1)$.

We next prove that $\mathcal{T}$ satisfies $(T2)$. Consider a separation (A, B) of order less than θ in K. Since X is titanic in K, either $\lambda_K(X \cap A) \geqslant \lambda_K(X)$ or $\lambda_K(X \cap B) \geqslant \lambda_K(X)$. By symmetry between A and B, we may assume that $\lambda_K(X \cap A) \geqslant \lambda_K(X)$. Then, by submodularity and symmetry of λ_K, we see that $\lambda_L(Y \cap B) = \lambda_K(Y \cap B) = \lambda_K(A \cup X) \leqslant \lambda_K(A) + \lambda_K(X) - \lambda_K(A \cap X) \leqslant \lambda_K(A) < \theta$. Therefore, as $\mathcal{T}'$ satisfies $(T2)$, one of $Y \cap B = B - X$ or $(Y \cap A) \cup \{e_X\} = (A - X) \cup \{e_X\}$ is in $\mathcal{T}'$. Thus, $\mathcal{T}$ contains B or A, as required. So $\mathcal{T}$ satisfies $(T2)$.

Next consider $(T3a)$. Let $B \in \mathcal{T}$ and $A \subseteq B$ with $\lambda_K(A) < \theta$. Then, by definition, $B - X$ is contained in a set in $\mathcal{T}'$. Since $A \subseteq B$, the union of $(E(K) - A) - X$, $B - X$ and $\{e_X\}$ is $E(L)$. As $\{e_X\}$ in $\mathcal{T}'$ and as $\mathcal{T}'$ satisfies $(T3)$, this implies that $(E(K) - A) - X$ is not contained in a set of $\mathcal{T}'$. So, $E(K) - A \notin \mathcal{T}$. As $\lambda_K(A) < \theta$ and as $\mathcal{T}$ does satisfy $(T2)$ this implies that $A \in \mathcal{T}$, as required. So $\mathcal{T}$ satisfies $(T3a)$.

We next prove by contradiction that $\mathcal{T}$ satisfies $(T3b)$. Let A_1, A_2, and A_3 be members of $\mathcal{T}$ that partition $E(K)$. Then each of $A_1 - X$, $A_2 - X$ and $A_3 - X$ is contained in a set in $\mathcal{T}'$. So, since $E(L)$ cannot be covered by three sets in $\mathcal{T}'$, none of the sets $(A_1 \cap Y) \cup \{e_X\}$, $(A_2 \cap Y) \cup \{e_X\}$, or $(A_3 \cap Y) \cup \{e_X\}$ is in $\mathcal{T}'$. Thus $\mathcal{T}'$ contains each of $A_1 \cap Y$, $A_2 \cap Y$, and $A_3 \cap Y$. Since $A_1 \cap Y$ and $\{e_X\}$ lie in $\mathcal{T}'$, $\mathcal{T}'$ does not contain $Y - A_1$. Now since $\mathcal{T}'$ contains neither $Y - A_1$ nor $(A_1 \cap Y) \cup \{e_X\}$, we have $\lambda_K(Y - A_1) = \lambda_L(Y - A_1) \geqslant \theta > \lambda_K(A_1)$. So, by submodularity and symmetry of λ_K, we get that $\lambda_K(X \cap A_1) \leqslant \lambda_K(X) + \lambda_K(A_1) - \lambda_K(X \cup A_1) = \lambda_K(X) + \lambda_K(A_1) - \lambda_K(Y - A_1) < \lambda_K(X)$. Similarly $\lambda_K(X \cap A_2) < \lambda_K(X)$ and $\lambda_K(X \cap A_2) < \lambda_K(X)$. However this contradicts the fact that X is titanic. Thus $\mathcal{T}$ satisfies $(T3b)$ and, hence, $\mathcal{T}$ is a tangle of order θ in K.

Finally we prove by contradiction that $\mathcal{T}$ satisfies $(T4)$. Suppose $e \in E(K)$ with $E(K) - \{e\} \in \mathcal{T}$. Then at least one of $E(L) - \{e, e_X\} = E(K) - \{e\} - X$ or $E(L) - \{e\} = (E(K) - \{e\} - X) \cup \{e_X\}$ is in $\mathcal{T}'$. As $\mathcal{T}'$ satisfies $(T4)$, this means $E(L) - \{e, e_X\} \in \mathcal{T}'$ and $e \in E(L) - \{e_X\}$. Now we have, as $E(K) - \{e\} \in \mathcal{T}$, that $\lambda_L(\{e\}) = \lambda_K(\{e\}) = \lambda_K(E(K) - \{e\}) < \theta$. So, as $\mathcal{T}'$ satisfies $(T4)$, the singleton $\{e\}$ is in $\mathcal{T}'$. But since also $\{e_X\}$ and $E(L) - \{e, e_X\}$ are in $\mathcal{T}'$, this contradicts that $\mathcal{T}'$ satisfies $(T3)$. So $\mathcal{T}$ does indeed satisfy $(T4)$. □

5. Minors and tangles

Let N be a minor of M and let $\mathcal{T}_N$ be a tangle in N of order θ. Now let $\mathcal{T}_M$ be the collection of all sets $A \subseteq E(M)$ where $\lambda_M(A) < \theta$ and $A \cap E(N) \in \mathcal{T}_N$. The following result is an immediate consequence of definitions.

Lemma 5.1. *$\mathcal{T}_M$ is a tangle in M of order θ.*

We say that $\mathcal{T}_M$ is the *tangle in M induced by $\mathcal{T}_N$*.

Let $f:\mathbb{Z}_+ \to \mathbb{Z}_+$ be a function and $m \in \mathbb{Z}_+$. A matroid M is called (m, f)*-connected* if whenever (A, B) is a separation of order ℓ where $\ell < m$ we have either $|A| \leqslant f(\ell)$ or $|B| \leqslant f(\ell)$.

Let $g(n) = (6^{n-1} - 1)/5$. Note that $g(1) = 0$ and $g(n) = 6g(n-1) + 1$ for all $n > 1$. The main result in this section is the following.

Theorem 5.2. *Let $\mathcal{T}$ be a tangle of order θ in a matroid M. Then there exists a (θ, g)-connected minor N of M and a tangle $\mathcal{T}'$ of order θ in N such that $\mathcal{T}$ is the tangle in M induced by $\mathcal{T}'$.*

We will use the following result from [2, Lemma 3.1].

Lemma 5.3. *Let $f:\mathbb{Z}_+ \to \mathbb{Z}_+$ be a nondecreasing function. If e is an element of an (m, f)-connected matroid M, then $M \setminus e$ or M/e is $(m, 2f)$-connected.*

5.4. Proof of Theorem 5.2. The proof is by induction on $|E(M)|$ with θ fixed; the root of this induction lies in the (θ, g)-connected matroids. Let $\mathcal{T}$ be a tangle of order θ in a matroid M and assume M is not (θ, g)-connected. Choose $m \in \{1, \dots, \theta - 1\}$ as small as possible such that M is not $(m+1, g)$-connected. Then there exists a separation (A, B) of order m with $|A|, |B| > g(m)$. By symmetry we may assume that $A \in \mathcal{T}$. Now let $e \in A$. By Lemma 5.3 and duality, we may assume that M/e is $(m, 2g)$-connected.

5.4.1. *$A - \{e\}$ is titanic in M/e.*

Subproof. When $m = 1$ this is vacuously true. Suppose that $m > 1$ and consider any partition (A_1, A_2, A_3) of $A - \{e\}$. Since $|A| > g(m) = 6g(m-1) + 1$, we have $|A_i| > 2g(m-1)$ for some $i \in \{1, 2, 3\}$. Then, since M/e is $(m, 2g)$-connected, $\lambda_{M/e}(A_i) \geqslant m \geqslant \lambda_{M/e}(A - \{e\})$. Hence $A - \{e\}$ is indeed titanic in M/e. □

5.4.2. *For each $X \subseteq B$, $\lambda_M(X) = \lambda_{M/e}(X)$.*

Subproof. Since M/e is $(m, 2g)$-connected, $\lambda_M(B) = \lambda_{M/e}(B)$. Hence $e \notin \mathrm{cl}_M(B)$. Therefore, for each $X \subseteq B$, $e \notin \mathrm{cl}_M(X)$. So $\lambda_M(X) = \lambda_{M/e}(X)$; as required. □

5.4.3. *For each $X \subseteq E(M)$ with $\lambda_M(X) < \theta$ we have that $X \in \mathcal{T}$ if and only if $X - A \in \mathcal{T}$ or $X \cup A \in \mathcal{T}$.*

Subproof. Let $X \subseteq E(M)$ with $\lambda_M(X) < \theta$. First assume that $X - A \in \mathcal{T}$ or $X \cup A \in \mathcal{T}$. Then, as $A \in \mathcal{T}$, it follows from $(T3)$ that $E(M) - X \notin \mathcal{T}$. Hence $X \in \mathcal{T}$.

For the reverse implication assume now that $X \in \mathcal{T}$. By 5.4.2, $\lambda_M(A) = \lambda_M(B) = \lambda_{M/e}(B - \{e\}) = \lambda_{M/e}(A - \{e\})$. So as A is titanic in M/e either $\lambda_M(A - X) \geqslant \lambda_{M/e}(A - X) \geqslant \lambda_M(A)$ or $\lambda_M(A \cup X) \geqslant \lambda_{M/e}(A \cup X) \geqslant \lambda_M(A)$. If $\lambda_M(A - X) \geqslant \lambda_M(A)$, then by symmetry and submodularity of λ_M we have that $\lambda_M(X - A) = \lambda_M(X \cap B) \leqslant \lambda_M(X) + \lambda_M(B) - \lambda_M(X \cup B) = \lambda_M(X) + \lambda_M(A) - \lambda_M(A - X) \leqslant \lambda_M(X) < \theta$. Hence, if $\lambda_M(A - X) \geqslant \lambda_M(A)$ then it follows from $(T3a)$ that $X - A \in \mathcal{T}$. If $\lambda_M(A \cap X) \geqslant \lambda_M(A)$, then, again by submodularity, $\lambda_M(A \cup X) \leqslant \lambda_M(X) + \lambda_M(A) - \lambda_M(A \cap X) \leqslant \lambda_M(X) < \theta$. So by $(T2)$ either $A \cup X \in \mathcal{T}$ or $B - X \in \mathcal{T}$. However, as $A \in \mathcal{T}$ and $X \in \mathcal{T}$ it follows from $(T3)$ that $B - X \notin \mathcal{T}$. So $A \cup X \in \mathcal{T}$. We conclude that if $X \in \mathcal{T}$ then $X - A \in \mathcal{T}$ or $X \cup A \in \mathcal{T}$. □

Let $\mathcal{T}_1$ be the tangle in $K_M \circ A$ of order θ obtained from $\mathcal{T}$ via Lemma 4.3. By 5.4.2, there is a natural isomorphism between $K_M \circ A$ and $K_{M/e} \circ (A - \{e\})$; let $\mathcal{T}_2$ be the tangle in $K_{M/e} \circ (A - \{e\})$ of order θ that is obtained from $\mathcal{T}_1$ via this isomorphism. In both $K_M \circ A$ and $K_{M/e} \circ (A - \{e\})$ denote the element that is not in B by e'.

Let $\mathcal{T}_3$ be the tangle in M/e of order θ that is obtained from $\mathcal{T}_2$ via Lemma 4.4. Finally let $\mathcal{T}_4$ be the tangle in M that is induced by $\mathcal{T}_3$.

5.4.4. $\mathcal{T} = \mathcal{T}_4$.

Subproof. Let (X, Y) be a separation of M of order less than θ with $e \in Y$. Then each of the following sequence of equivalences follows directly from definitions:

$$\begin{aligned} X \in \mathcal{T}_4 &\iff X \in \mathcal{T}_3 \\ &\iff X - \big(A - \{e\}\big) \in \mathcal{T}_2 \quad \text{or} \quad \big(X - \big(A - \{e\}\big)\big) \cup \{e'\} \in \mathcal{T}_2 \\ &\iff X - A \in \mathcal{T}_1 \quad \text{or} \quad (X - A) \cup \{e'\} \in \mathcal{T}_1 \\ &\iff X - A \in \mathcal{T} \quad \text{or} \quad X \cup A \in \mathcal{T}. \end{aligned}$$

So by 5.4.3, $X \in \mathcal{T}_4$ if and only if $X \in \mathcal{T}$; as required. □

The result now follows easily by applying induction to the tangle $\mathcal{T}_3$ in M/e. □

6. A tangle in a grid

An *n by n grid* is a graph G_n with vertex set $V = \{(i, j)\colon\ i, j \in \{1, \ldots, n\}\}$ where vertices (i, j) and (i', j') are adjacent if and only if either $i = i'$ and $|j - j'| = 1$, or $j = j'$ and $|i - i'| = 1$.

The goal of this section is to prove the existence of a natural tangle of order n in $M(G_n)$. For $i \in \{1, \ldots, n\}$ let P_i denote the path in G_n on vertices $(i, 1), \ldots, (i, n)$ and let Q_i denote the path in G_n on vertices $(1, i), \ldots, (n, i)$. Now we let $\mathcal{T}_n$ denote the collection of all subsets $A \subseteq E(G_n)$ such that $\lambda_{M(G_n)}(A) < n$ and A does not contain any $E(P_i)$ for $i \in \{1, \ldots, n\}$. We will prove, for $n \geqslant 3$:

Lemma 6.1. *$\mathcal{T}_n$ is a tangle in $M(G_n)$ of order n.*

A similar result was proved by Kleitman and Saks; see [6, (7.3)]. They considered tangles in K_{G_n}, whereas we consider tangles in $K_{M(G_n)}$. Our proof follows that of Kleitman and Saks; we need some preliminary results on connectivity.

Let X and Y be disjoint subsets of $E(M)$, we define $\kappa_M(X, Y) = \min(\lambda_M(A)\colon\ X \subseteq A \subseteq E(M) - Y)$. The following result, due to Tutte [9], is an extension of Menger's Theorem.

Theorem 6.2 *(Tutte's Linking Theorem). If S and T are disjoint sets of elements in a matroid M, then there exists a minor N of M such that $E(N) = S \cup T$ and $\lambda_N(S) = \kappa_M(S, T)$.*

The following result was proved in [4].

Lemma 6.3. *Let S and T be disjoint sets of elements of a matroid M. Then there exist sets $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $|S_1| + 1 = |T_1| + 1 = \kappa_M(S_1, T_1) = \kappa_M(S, T)$.*

In order to prove Lemma 6.1, we first need to establish that certain sets of edges in a grid are "highly connected".

Lemma 6.4. *Let $i \in \{1, \ldots, n\}$ and, for each $j \in \{1, \ldots, n\} - \{i\}$, let e_j and f_j be disjoint edges of P_j. Now let $X = \{e_j\colon\ j \in \{1, \ldots, n\} - \{i\}\}$ and let $Y = \{f_j\colon\ j \in \{1, \ldots, n\} - \{i\}\}$. Then $\kappa_{M(G_n)}(X, Y) = n$.*

Proof. Let $D = E(Q_2) \cup \cdots \cup E(Q_{n-1})$ and let $C = E(Q_1) \cup E(Q_n) \cup ((E(P_1) \cup \cdots \cup E(P_n)) - (X \cup Y))$. Now let $H = G_n \setminus D/C$. Note that $H[X]$ and $H[Y]$ are disjoint spanning trees of H. Therefore $n = \lambda_{M(H)}(X) = \kappa_{M(H)}(X, Y) \leqslant \kappa_{M(G_n)}(X, Y) \leqslant |X| + 1 = n$. Thus $\kappa_{M(G_n)}(X, Y) = n$, as required. □

The proofs of the following two results are similar to that of Lemma 6.4; we leave these to the reader.

Lemma 6.5. *Let $i, j \in \{1, \ldots, n\}$. Then $\kappa_{M(G_n)}(P_i, Q_j) = n$. Also, if $i \neq j$, then $\kappa_{M(G_n)}(P_i, P_j) = n$ and $\kappa_{M(G_n)}(Q_i, Q_j) = n$.*

Lemma 6.6. *Let $X \subseteq E(P_1) \cup E(P_n)$ with $|X| \geqslant n-1$ and let $j \in \{1, \ldots, n\}$. Then $\kappa_{M(G_n)}(X, Q_j) = n$.*

We call a set $A \subseteq E(G_n)$ *small* if $\lambda_{M(G_n)}(A) < n$ and A does not contain any of $E(P_1), \ldots, E(P_n)$ or $E(Q_1), \ldots, E(Q_n)$.

Lemma 6.7. *Let (A, B) be a separation of $M(G_n)$ of order less than n. Then one of A and B is small. Moreover, if B is small, then A contains one of $E(P_1), \ldots, E(P_n)$ and one of $E(Q_1), \ldots, E(Q_n)$.*

Proof. By Lemma 6.4, either A or B must contain one of $E(P_1), \ldots, E(P_n)$. Then, by symmetry, either A or B must contain one of $E(Q_1), \ldots, E(Q_n)$. However, by Lemma 6.5, A and B cannot both contain one of $E(P_1), \ldots, E(P_n), E(Q_1), \ldots, E(Q_n)$. □

Note that $\mathcal{T}_n$ trivially satisfies conditions $(T1)$, $(T3a)$, and $(T4)$. By Lemma 6.7, $\mathcal{T}_n$ also satisfies $(T2)$. Thus in order to complete the proof of Lemma 6.1, we need only verify $(T3b)$; this is achieved by the following result.

Lemma 6.8. *For $n \geqslant 3$, $E(G_n)$ cannot be partitioned into three small sets.*

Proof. The proof is by induction on n. The case $n = 3$ is trivial; suppose then that $n \geqslant 4$ and that the result holds for G_{n-1}. Now assume (A_1, A_2, A_3) is a partition of $E(G_n)$ into three small sets.

By symmetry we may assume that Q_n meets A_1 and A_2. (That is, $A_1 \cap E(Q_n)$ and $A_2 \cap E(Q_n)$ are nonempty.) By Lemma 6.7, there is a path Q_j disjoint from A_1. Note that $\kappa_{M(G_n)}(A_1 \cap (E(P_1) \cup E(P_n)), Q_j) \leqslant \lambda_{M(G_n)}(A_1) < n$. Then, by Lemma 6.6, $|A_1 \cap (E(P_1) \cup E(P_n))| < n-1$. Similarly $|A_2 \cap (E(P_1) \cup E(P_n))| < n-1$. Therefore either P_1 or P_n meets A_3; by symmetry, we may assume that P_n meets A_3. Therefore $E(P_n) \cup E(Q_n)$ meets each of A_1, A_2, and A_3.

Note that $G_{n-1} = G_n - (V(P_n) \cup V(Q_n))$. For each $i \in \{1, 2, 3\}$, let $A'_i = E(G_{n-1}) \cap A_i$.

6.8.1. *There exists $k \in \{1, 2, 3\}$ such that $\lambda_{M(G_{n-1})}(A'_k) \geqslant n-1$.*

Subproof. By the induction hypothesis, there exists $k \in \{1, 2, 3\}$ such that A'_k is not small in G_{n-1}. Suppose that $\lambda_{M(G_{n-1})}(A'_k) < n-1$. Then A'_k contains one of $E(P_1) \cap E(G_{n-1}), \ldots, E(P_{n-1}) \cap E(G_{n-1})$ or one of $E(Q_1) \cap E(G_{n-1}), \ldots, E(Q_{n-1}) \cap E(G_{n-1})$. By Lemma 6.7, A_k avoids some path P_i and some path Q_j. Since $E(P_n) \cup E(Q_n)$ meets each of A_1, A_2, and A_3, either $i \neq n$ or $j \neq n$. Thus A'_k avoids one of $E(P_1) \cap E(G_{n-1}), \ldots, E(P_{n-1}) \cap E(G_{n-1})$ or one of $E(Q_1) \cap E(G_{n-1}), \ldots, E(Q_{n-1}) \cap E(G_{n-1})$. So, applying Lemma 6.7 to G_{n-1}, we contradict the assumption that $\lambda_{M(G_{n-1})}(A'_k) < n-1$. □

By Lemma 6.3, there exist $S \subseteq A'_k$ and $T \subseteq E(G_{n-1}) - A'_k$ such that $|S| + 1 = |T| + 1 = \kappa_{M(G_{n-1})}(S, T) \geqslant n-1$. Now, by Tutte's Linking Theorem, there exists a minor H of G_{n-1} such that $E(H) = S \cup T$ and $\lambda_{M(H)}(S) \geqslant n$. Suppose that $H = G_{n-1} \setminus D/C$; we may choose D and C such that D does not contain a cut of G_n. Thus H is connected and S and T are disjoint spanning trees of H; thus $|V(H)| \geqslant n-1$. Now let $H' = G_n \setminus D/H$. Vertices $(1, n)$ and $(n, 1)$ both have a neighbour in $V(H)$ in H'. Note that there exist $e \in (E(P_n) \cup E(Q_n)) \cap A_k$ and $f \in (E(P_n) \cup E(Q_n)) - A_k$. Now there exists a minor H'' of H' such that $S \cup \{e\}$ and $T \cup \{f\}$ are disjoint spanning trees of H''. Thus $\lambda_{M(H'')}(S \cup \{f\}) \geqslant n$. However, this contradicts the fact that $\lambda_M(A_k) < n$. □

7. A grid in a tangle

Let M be a matroid and let N be a minor of M that is isomorphic to the cycle matroid of the n by n grid. Now let $\mathcal{T}_N$ be the tangle in N of order n given by Lemma 6.1 and let $\mathcal{T}_M$ be the tangle in M of order n that is induced by $\mathcal{T}_N$. (We recall that the term "induced" was defined at the start of Section 5 and the term "truncation" was defined at the start of Section 4.) A tangle $\mathcal{T}$ in M is said to *dominate* N if $\mathcal{T}_M$ is a truncation of $\mathcal{T}$. In this section we prove Theorem 1.2. We need the following lemma. (We use the "tangle matroid" which is defined at the end of Section 3.)

Lemma 7.1. *Let $\mathcal{T}$ be a tangle in a matroid M and let $M_{\mathcal{T}}$ be the tangle matroid of $\mathcal{T}$. Now let G_n be the n by n grid and suppose that $N = M(G_n)$ is a minor of M. Then $\mathcal{T}$ dominates N if and only if each of the sets $E(P_1), \dots, E(P_n)$ is independent in $M_{\mathcal{T}}$.*

Proof. Note that, if $\mathcal{T}'$ is the truncation of $\mathcal{T}$ to order n, then $M_{\mathcal{T}'}$ is the truncation of $M_{\mathcal{T}}$ to rank $n-1$. Thus, by possibly truncating, we may assume that $\mathcal{T}$ has order n. Now let $\mathcal{T}_n$ be the tangle in N of order n given by Lemma 6.1 and let $\mathcal{T}_M$ be the tangle in M of order n that is induced by $\mathcal{T}_N$. Thus $\mathcal{T}$ dominates N if and only if $\mathcal{T} = \mathcal{T}_M$. Now $\mathcal{T} \neq \mathcal{T}_M$ if and only if there exists a set $A \in \mathcal{T}$ that contains one of $E(P_1), \dots, E(P_n)$. On the other hand, $E(P_i)$ is independent in $M_{\mathcal{T}}$ if and only if there does not exist $A \in \mathcal{T}$ such that $E(P_i) \subseteq A$. $\square$

We also need the following result from [4].

Theorem 7.2. *There exists an integer-valued function $f(k,q)$ such that for any positive integer k and prime-power q, if M is a $GF(q)$-representable matroid with branch-width at least $f(k,q)$, then M contains a minor isomorphic to $M(G_k)$.*

Note that, if M has a tangle of high order, then M has large branch-width and, hence by Theorem 7.2, M has a big grid as a minor. Unfortunately, this grid-minor need not be dominated by the tangle.

7.3. Proof of Theorem 1.2. Let $g(t) = (6^t - 1)/5$ for any integer $t \geqslant 0$. Let $n = g(k-1) + 2$, let q be the order of $\mathbb{F}$, and let $\theta = f(n,q)$. Now let M be an $\mathbb{F}$-representable matroid and let $\mathcal{T}$ be a tangle in M of order θ. By Theorem 5.2, there exists a (θ, g)-connected minor M_1 of M and a tangle $\mathcal{T}_1$ in M_1 of order θ such that $\mathcal{T}$ is the tangle in M that is induced by $\mathcal{T}_1$. By Theorems 3.1 and 7.2, there exists a minor N of M_1 that is isomorphic to $M(G_n)$. By possibly relabeling, we may assume that $N = M(G_n)$. Now let $P_1, \dots, P_n$ be the vertical paths in G_n, let $M_{\mathcal{T}_1}$ be the tangle matroid of $\mathcal{T}_1$, and let ϕ_1 be the rank-function of $M_{\mathcal{T}_1}$.

7.3.1. $\phi_1(E(P_i)) \geqslant k-1$ *for each* $i \in \{1, \dots, n\}$.

Subproof. Suppose to the contrary that $\phi_1(E(P_i)) < k-1$ for some i. Thus there exists $A \in \mathcal{T}_1$ such that $E(P_i) \subseteq A$ and $\lambda_{M_1}(A) \leqslant k-1$. By definition $|A| \geqslant |E(P_i)| = n-1 > g(k-1)$. Therefore, since M_1 is (θ, g)-connected, $|E(M_1) - A| \leqslant g(k-1) = n-2 \leqslant f(n,q) - 2 < \theta - 1$. Moreover, as $k \geqslant 1$, we have that $\theta \geqslant 3$. Hence by Lemma 3.2, $E(M_1) - A \in \mathcal{T}_1$; contradicting $(T3)$. $\square$

For each $i \in \{1, \dots, k\}$, let A_i be an $M_{\mathcal{T}_1}$-independent subset of $E(P_{1+(i-1)k})$ with $|A_i| = k-1$; as $k^2 - k + 1 \leqslant n$ these sets A_i exist. Now there exists a minor H of G_n such that H is isomorphic to G_k and such that $A_1, \dots, A_k$ are the edge-sets of the vertical paths in H. By Lemma 7.1, $\mathcal{T}_1$ dominates H. Then, since $\mathcal{T}$ is induced by $\mathcal{T}_1$, $\mathcal{T}$ also dominates H. $\square$

8. Tree-decompositions and laminar families

We begin by reviewing some elementary results on laminar families and tree-decompositions. Let E be a set. A partition of E into two sets is called a *separation* of E. Two separations (A_1, A_2) and (B_1, B_2) of a set E are said to *cross* if $A_i \cap B_j \neq \emptyset$ for each i and j in $\{1, 2\}$. A collection $\mathcal{S}$ of separations of E is *laminar* if no two separations in $\mathcal{S}$ cross.

A *tree-decomposition* of E consists of a pair $(T, \mathcal{P})$ where T is a tree and $\mathcal{P} = (P_v\colon\ v \in V(T))$ is a partition of E (where one or more of the P_v may be empty). For any $X \subseteq V(T)$, we let $\mathcal{P}[X]$ denote the set $\bigcup_{v\in X} P_v$. Now, for any $e \in E(T)$, the *separation of E displayed by e* is $(\mathcal{P}[V(T_1)], \mathcal{P}[V(T_2)])$ where T_1 and T_2 are the two components of $T - e$. The following result is both easy and well known.

Lemma 8.1. *If $(T, \mathcal{P})$ is a tree-decomposition of E, then the set of all separations displayed by $(T, \mathcal{P})$ is laminar.*

Let $(T, \mathcal{P})$ be a tree-decomposition of E and let $\mathcal{S}$ be a set of separations of E. We say that $(T, \mathcal{P})$ *represents* $\mathcal{S}$ if $\mathcal{S}$ is the set of separations displayed by $(T, \mathcal{P})$. The following converse to Lemma 8.1 is also well known.

Lemma 8.2. *If $\mathcal{S}$ is a laminar set of separations of E, then there is a tree-decomposition of E that represents $\mathcal{S}$.*

Let K be a connectivity system. A set $X \subseteq E(K)$ is *robust* if for each proper partition (X_1, X_2) of X either $\lambda_K(X_1) > \lambda_K(X)$ or $\lambda_K(X_2) > \lambda_K(X)$. (A partition is *proper* if all its members are nonempty.) A separation (X, Y) of K is *robust* if X and Y are both robust.

Lemma 8.3. *Let K be a connectivity system and let $\mathcal{S}$ be the set of all robust separations of K. Then $\mathcal{S}$ is laminar.*

Proof. Suppose that $(A_1, A_2), (B_1, B_2) \in \mathcal{S}$ cross. By symmetry, we may assume that $\lambda_K(A_1) \leqslant \lambda_K(B_1)$. As λ_K is symmetric, we may assume that $\lambda_K(A_2 \cap B_2) \geqslant \lambda_K(A_1 \cap B_2)$; otherwise swap A_1 and A_2. Then, since B_2 is robust, $\lambda_K(A_2 \cap B_2) > \lambda_K(B_2)$. So symmetry and submodularity of λ_K yield $\lambda_K(A_1 \cap B_1) \leqslant \lambda_K(A_1) + \lambda_K(B_1) - \lambda_K(A_1 \cup B_1) = \lambda_K(A_1) + \lambda_K(B_2) - \lambda_K(A_2 \cap B_2) < \lambda_K(A_1)$. So, since A_1 is robust, $\lambda_K(A_1 \cap B_2) > \lambda_K(A_1)$. Also, as $\lambda_K(B_1) \geqslant \lambda_K(A_1) \geqslant \lambda_K(A_1 \cap B_1)$ and as B_1 is robust, $\lambda_K(A_2 \cap B_1) > \lambda_K(B_1)$. Combining the last two strict inequalities we get $\lambda_K(A_1 \cap B_2) + \lambda_K(A_2 \cap B_1) > \lambda_K(A_1) + \lambda_K(B_1) = \lambda_K(A_1) + \lambda_K(B_2)$. As $\lambda_K(A_2 \cap B_1) = \lambda_K(A_1 \cup B_2)$, this contradicts submodularity. $\square$

9. Tree-representations of maximal tangles

The main result of this section is Theorem 9.1; when applied to the maximal tangles $\mathcal{T}_1, \dots, \mathcal{T}_n$ of the matroid, those that are not truncations of others, it is the result alluded to in the introduction by 1.1.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two tangles in a connectivity system K, neither of which is a truncation of the other, then there exists a *distinguishing separation* (X_1, X_2) with $X_1 \in \mathcal{T}_1$ and $X_2 \in \mathcal{T}_2$.

Theorem 9.1. *Let K be a connectivity system and let $\mathcal{T}_1, \dots, \mathcal{T}_n$ be tangles in K, none of which is a truncation of another. Then there exists a tree-decomposition $(T, \mathcal{P})$ of $E(K)$ such that $V(T) = \{1, \dots, n\}$ and such that the following hold*:

(i) *For each $i \in V(T)$ and $e \in E(T)$ if T' is the component of $T - e$ containing i then $\mathcal{P}[V(T')]$ is not in $\mathcal{T}_i$.*
(ii) *For each pair of distinct vertices i and j of T, there exists a minimum-order distinguishing separation for $\mathcal{T}_i$ and $\mathcal{T}_j$ that is displayed by T.*

Let K and K' be connectivity systems with $E(K) = E(K')$. We call K' a *tie-breaker* for K if for each $X, Y \subseteq E(K)$:

(i) $\lambda_{K'}(X) \neq \lambda_{K'}(Y)$ unless $X = Y$ or $X = E(K) - Y$,
(ii) $\lambda_{K'}(X) < \lambda_{K'}(Y)$ if $\lambda_K(X) < \lambda_K(Y)$.

Lemma 9.2. *Each connectivity system has a tie-breaker.*

Proof. Let K be a connectivity system. We may assume that $E(K) = \{1, \dots, n\}$. Now, for $X \subseteq \{1, \dots, n-1\}$, let $\lambda_L(X) = \sum_{i \in X} 2^i$ and let $\lambda_L(E(K) - X) = \lambda_L(X)$. We leave it to the reader to verify that $L = (E(K), \lambda_L)$ is indeed a connectivity system. Now, for each $X \subseteq E(K)$, we let $\lambda_{K'}(X) = 2^n \lambda_K(X) + \lambda_L(X)$. It is easy to check that $K' = (E(K), \lambda_{K'})$ has the desired properties. □

It is evident that a tangle in a connectivity system K is a tangle in any tie-breaker for K.

Lemma 9.3. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be tangles in a connectivity system K that are incomparable by truncation, let K' be a tie-breaker for K, and let (X_1, X_2) be a distinguishing separation for $\mathcal{T}_1$ and $\mathcal{T}_2$ with minimum order in K'. Then (X_1, X_2) is a robust separation of K'.*

Proof. Suppose otherwise. Then, by symmetry, we may assume that there exists a proper partition (A, B) of X_1 such that $\lambda_{K'}(A) \leqslant \lambda_{K'}(X_1)$ and $\lambda_{K'}(B) \leqslant \lambda_{K'}(X_1)$. Since K' is a tie-breaker, $\lambda_{K'}(A) < \lambda_{K'}(X_1)$ and $\lambda_{K'}(B) < \lambda_{K'}(X_1)$. Condition $(T3a)$ for $\mathcal{T}_1$ implies that $A, B \in \mathcal{T}_1$. Then, by our choice of the distinguishing separation (X_1, X_2), $\mathcal{T}_2$ contains neither $E(K) - A$ nor $E(K) - B$. Then, by $(T2)$, $A, B \in \mathcal{T}_2$. But then $\mathcal{T}_2$ contains each of A, B, and X_2; contrary to $(T3)$. □

Proof of Theorem 9.1. Let K' be a tie-breaker for K. As $\mathcal{T}_1, \dots, \mathcal{T}_n$ are tangles in K', we may assume that $K = K'$. For each $i, j \in \{1, \dots, n\}$ with $i \neq j$ let (X_{ij}, Y_{ij}) be the minimum-order separation of K distinguishing $\mathcal{T}_i$ and $\mathcal{T}_j$ (where we assume that $X_{ij} \in \mathcal{T}_i$). By Lemma 9.3, (X_{ij}, Y_{ij}) is a robust separation of K. Now let $\mathcal{S}$ be the collection of all of these distinguishing separations. By Lemma 8.3, $\mathcal{S}$ is laminar. Then, by Lemma 8.2, there is a tree-decomposition $(T, \mathcal{P})$ of $E(K)$ that represents $\mathcal{S}$. We may assume that if v is a vertex of T with degree 1 or 2, then $P_v \neq \emptyset$ (since, otherwise, we could find a smaller tree-decomposition representing $\mathcal{S}$). This means that the edges of T display proper and distinct separations. It remains to show that there is a bijection between $\mathcal{T}_1, \dots, \mathcal{T}_n$ and $V(T)$ satisfying the conclusion of Theorem 9.1.

For $i = \{1, \dots, n\}$, consider the collection $\mathcal{X}_i$ of nonempty subsets X of $V(T)$ such that $E(K) - \mathcal{P}[X] \in \mathcal{T}_i$ and such that $(\mathcal{P}[X], E(K) - \mathcal{P}[X])$ is displayed by T. Each member of $\mathcal{X}_i$ induces a subtree of T and by $(T3)$ each two members of $\mathcal{X}_i$ intersect. As any collection of pairwise intersecting subtrees of a tree has a common vertex, the members of $\mathcal{X}_i$ have a nonempty intersection. Call that intersection V_i.

Note that by construction of V_i each edge of T that leaves V_i displays a separation (A, B) with $\mathcal{P}[V_i] \subseteq A$ and $B \in \mathcal{T}_i$. From this, $(T2)$, $(T3)$ and the fact that each separation in $\mathcal{S}$ is displayed by T it is straightforward to see that to prove Theorem 9.1 it suffices to show that $(V_1, \dots, V_n)$ is a partition of $V(T)$ into singletons.

The sets $V_1, \dots, V_n$ are pairwise disjoint as for each $i \neq j$ the set $\mathcal{P}[V_i]$ lies in Y_{ij} and the set $\mathcal{P}[V_j]$ lies in $Y_{ji} = X_{ij}$.

It remains to prove that if w in $V(T)$ then $\{w\} = V_i$ for some i. Among the edges incident with w take the one that displays the separation, (X_{ij}, Y_{ij}) say, of largest order. So that order is at most the order of $\mathcal{T}_i$ and of $\mathcal{T}_j$. We may assume that $\mathcal{P}_w \subseteq Y_{ij}$. As no two edges of T display the same separation, all other edges incident with w display a separation of order less than those of $\mathcal{T}_i$ and $\mathcal{T}_j$. By the definition of (X_{ij}, Y_{ij}) these separations do not distinguish $\mathcal{T}_i$ from $\mathcal{T}_j$. Combining that with $(T3)$ for $\mathcal{T}_j$, we see that for each of these separations $\mathcal{P}_w$ is not part of the side that is in $\mathcal{T}_i$. Hence $V_i \subseteq \{w\}$. As V_i is not empty, $\{w\} = V_i$ as claimed. □

We conclude with a simple corollary to Theorem 9.1.

Corollary 9.4. *An m-element connectivity system has at most $\frac{m-2}{2}$ maximal tangles.*

Proof. Let K be an m-element connectivity system and let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be the maximal tangles in K. Now let $(T, \mathcal{P})$ be the tree-decomposition of $E(M)$ given by Theorem 9.1. Let v be a vertex of T of degree d_v. By $(T3)$ and $(T4)$, $d_v + |P_v| \geqslant 4$. Now $4n \leqslant \sum_{i=1}^{n}(d_i + |P_i|) = 2|E(T)| + |E(M)| = 2(n-1) + m$. So $n \leqslant \frac{m-2}{2}$ as claimed. □

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