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On the Uniqueness of Kernels

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Let S be a compact orientable surface. For any graph G embedded on S and any closed curve D on S we define $\mu_G(D)$ as the minimum number of intersections of G and D', where D' ranges over all closed curves freely homotopic to D. We call G a *kernel* if $\mu_{G'} \neq \mu_G$ for each proper minor G' of G. We prove that if G and G' are kernels with $\mu_G = \mu_{G'}$ (in such a way that each face of G is an open disk), then G' can be obtained from G by a series of the following operations: (i) homotopic shifts over S; (ii) taking the surface dual graph; (iii) ΔY-exchange (i.e., replacing a vertex v of degree 3 by a triangle connecting the three vertices adjacent to v, or conversely).

1. FORMULATION OF THEOREM 1

Let S be a compact surface, and let G be a graph embedded on S (without crossing edges). For each closed curve D on S we define

$$\mu_G(D) := \min_{D' \sim D} \operatorname{cr}(G, D'). \tag{1}$$

Here $\operatorname{cr}(G, D')$ denotes the number of times D' intersects G (i.e., $\operatorname{cr}(G, D') = |\{z \in S^1 \mid D'(z) \in G\}|$). The minimum ranges over all closed cuvers D' freely homotopic to D. [A closed curve is a continuous function $D: S^1 \to S$ (where S^1 denotes the unit circle in the complex plane). Two closed curves D and D' are *freely homotopic*, in notation: $D \sim D'$, if there exists a continuous function $\Phi: [0, 1] \times S^1 \to S$ such that $\Phi(0, z) = D(z)$ and $\Phi(1, z) = D'(z)$ for all $z \in S^1$.]

Observe that the function μ_G is invariant under the following operations on G:

$$\begin{aligned} &\text{(i) homotopic shifts of } G \text{ over } S; \\ &\text{(ii) replacing } G \text{ by a surface dual } G^* \text{ of } G; \\ &\text{(iii) } \Delta Y\text{-exchanges in } G. \end{aligned} \tag{2}$$

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Here we use the following terminology. Graph G' arises by a *homotopic shift* of G over S (or *is homotopic to* G) if there exists a continuous function Φ: $[0,1]\times G \to S$ so that (i) $\Phi(0, y) = y$ for each $y \in G$; (ii) for each $x \in [0, 1]$, $\Phi(x, \cdot)$ is a one-to-one function on G; (iii) $\Phi(1, \cdot)$ maps G onto G'. [We consider G and G' as subspaces of S.]

We say that graph G^* is a (*surface*) *dual* of G if (i) each face of G is an open disk; (ii) each face of G contains exactly one vertex of G^*, and $V(G^*) \cap G = \emptyset$; (iii) each edge of G^* crosses exactly one edge of G, and each edge of G crosses exactly one edge of G^*, while there are no further intersections of G and G^*. [By $V(\cdot)$ and $E(\cdot)$ we mean the vertex set and edge set of $\cdot$.] So G has a surface dual if and only if each face of G is an open disk. Moreover, G has only one surface dual up to homotopic shifts.

If v is a vertex of G of degree 3, a ΔY-exchange (at v) replaces v and the three edges incident with v by a triange connecting the three vertices adjacent to v (thus forming a triangular face). We also call the converse operation (replacing a triangular face by a "star" with three rays) a ΔY-exchange.

Note moreover that if G' is a minor of G then $\mu_{G'} \leqslant \mu_G$ (i.e., $\mu_{G'}(D) \leqslant \mu_G(D)$ for each closed curve D). Here a *minor* of G arises by a series of deletions of edges, and contradictions of non-loop edges. If we contract an edge, the graph arising is naturally embedded again on S (unique up to homotopic shifts).

Now we call G a *kernel* (on S) if $\mu_{G'} \neq \mu_G$ for each proper minor G' of G. [*Proper* means that we delete or contract at least one edge of G.]

The main result of this paper is that, if S is orientable and each face of G is an open disk, then kernels are uniquely determined by the function μ_G, up to the operations (2):

THEOREM 1. *Let G and G' be kernels on the compact orientable surface S, in such a way that each face of G is an open disk. If $\mu_G = \mu_{G'}$ then G' can be obtained from G by a series of operations* (2).

Note 1. The minimum of $\mu_G(D)$ taken over all homotopic nontrivial closed curves D is called the *representativity* or *face-width* of G. This parameter has been studied recently in relation to minimal genus embeddings of graphs and to graph minors and disjoint paths, by, among others, Archdeacon [2], Fiedler, Huneke, Richter, and Robertson [8], Robertson and Seymour [15], Robertson and Thomas [16], Robertson and Vitray [17], and Thomassen [24]. Note that each graph of given representativity, such that each proper minor has a smaller representativity, is a kernel.

Note 2. The ΔY-operation was studied (for planar graphs) by Steinitz [20] (cf. [21]), who called it the θ-process. More recently, attention has been given by, among others, Akers [1], Epifanov [6], Grünbaum [9], Lehman [11], and Truemper [25].

Note 3. Scott Randby [13] proved the theorem above in the case where S is the projective plane. We do not know if it holds for nonorientable surfaces in general.

Neither do we know if the condition that each face of G is an open disk is necessary. Our proof below shows that we may relax this condition to the weaker condition that no loop of G is (as a closed curve) freely homotopic to a closed curve not intersecting G. In particular, if G has no loops at all, the statement also holds.

2. Tight Graphs

We next formulate an analogous result for so-called tight graphs, a result that actually will be shown to imply Theorem 1. Tight graphs were introduced in [18].

Let H be a graph embedded on the compact surface S. For each closed curve D on S we denote

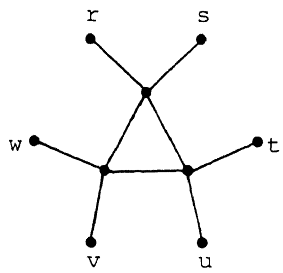
$$\mu'_H(D) := \min_{\substack{D' \sim D \\ D' \subseteq S \setminus V(H)}} \operatorname{cr}(H, D'). \tag{3}$$

Here the minimum ranges over all closed curves D' freely homotopic to D so that D' does not intersect the vertex set $V(H)$ of H.

Let H be 4-regular. The function μ'_H is clearly invariant under the following operations on H:

$$\begin{aligned} &\text{(i) homotopic shifts of } H \text{ over } S;\\ &\text{(ii) } \Delta\nabla\text{-exchanges in } H. \end{aligned} \tag{4}$$

A *$\Delta\nabla$-exchange* replaces a triangular face, adjacent to r, s, t, u, v, w as in



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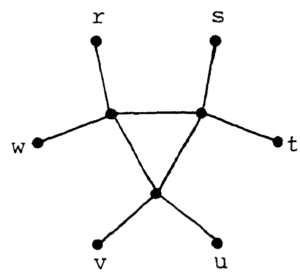
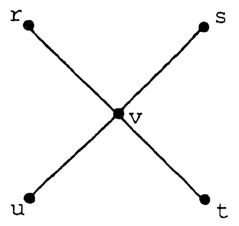


Moreover, we define an *opening* (at v) as replacing a neighbourhood of vertex v of H of degree 4:



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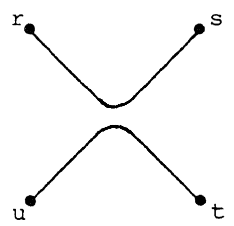


(So at one vertex v there are two possible openings.) If this operation creates a loop without a vertex, we add a new vertex on the loop.

We call a graph H' an *opening* of H if H' arises from H by a series of openings. Note that if H' is an opening of H, then $\mu'_{H'} \leqslant \mu'_H$. We call a 4-regular graph H *tight* (on S) if $\mu'_{H'} \neq \mu'_H$ for each proper opening H' of H. [*Proper* means that we open H at least once.] (In [18] we defined "tight" for each eulerian graph, but in this paper we restrict ourselves to tight 4-regular graphs.)

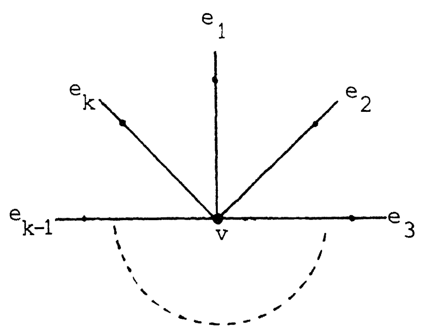
The following theorem says that, if S is orientable, then 4-regular right graphs are uniquely determined by the function μ'_H, up to the operations (4):

THEOREM 2. *Let H and H' be tight 4-regular graphs on the compact orientable surface S. If $\mu'_H = \mu'_{H'}$ then H' can be obtained from H by a series of operations* (4).

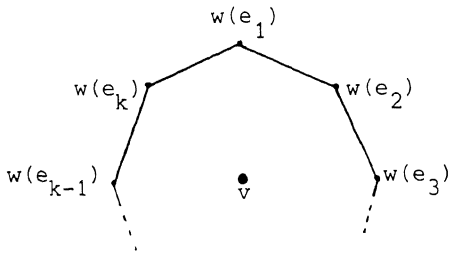
3. Reduction of Theorem 1 to Theorem 2

We show a relation between kernels and tight graphs, which allows us to reduce Theorem 1 to Theorem 2. It is based on constructing the *medial graph* $H(G)$ of G, introduced by Steinitz [20] (cf. [21]), who called it the ω-process, and in reverse form by Tait [22] (cf. [23]).

For any graph G embedded on a surface S, $H(G)$ is constructed as follows. Choose an arbitrary point $w(e)$ "in the middle of" e, for each edge e of G. These points form the vertex set of $H(G)$. For each vertex v of G, there will be edges of $H(G)$ forming a circuit connecting the points $w(e)$ on edges e incident with v. That is, we consider a neighbourhood N (homeomorphic to an open disk) of v:



If $e_1, \ldots, e_k$ denote the edges incident with v in cyclic order, $H(G)$ has edges connecting the pairs $\{w(e_1), w(e_2)\}$, $\{w(e_2), w(e_3)\}, \ldots, \{w(e_{k-1}), w(e_k)\}$, $\{w(e_k), w(e_1)\}$, drawn in N as in



We do this for every vertex v. This makes the 4-regular graph $H(G)$. Note that $\mu'_{H(G)} = 2\mu_G$.

In fact, $H(G)$ determines G up to homotopy and duality:

PROPOSITION 1. *Let G and G' be graphs embedded on the compact surface S so that each face of G is an open disk. Then $H(G)$ and $H(G')$ are homotopic, if and only if G' is homotopic to G or to its dual G^*.*

Proof. This follows directly from the fact that G can be reconstructed from $H(G)$, up to homotopy and duality. ∎

Moreover, we have:

PROPOSITION 2. *Let G be a graph embedded on the compact surface S so that each face of G is an open disk. Then G is a kernel, if and only if $H(G)$ is tight.*

Proof. One easily checks that deletion and contraction of an edge e of G corresponds to the two ways of opening $H(G)$ at vertex $w(e)$. So if G' is a proper minor of G then $H(G')$ is (homotopic to) a proper opening of

$H(G)$. This implies that if $H(G)$ is tight, then for each proper minor G' of G,

$$\mu_{G'} = \tfrac{1}{2}\mu'_{H(G')} \neq \tfrac{1}{2}\mu'_{H(G)} = \mu_G. \tag{5}$$

So G is a kernel.

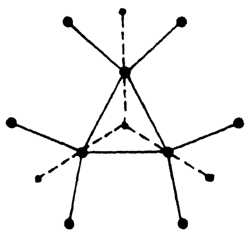
Conversely, if G is a kernel, then $H(G)$ is tight. For suppose to the contrary that we can open $H(G)$ at vertex $w(e)$, say, obtaining a graph H' with $\mu'_{H'} = \mu'_{H(G')}$. This would contradict the fact that G is a kernel, since the opening would correspond to a deletion or contraction of edge e in G, without changing μ_G, unless it corresponds to contracting e while e is a loop. Let D be the closed curve following loop e. Since G is a kernel, D is not nullhomotopic (otherwise we could delete e from G without modifying μ_G). Now $\mathrm{cr}(H', D) = 0$. Hence $\mu_G(D) = \frac{1}{2}\mu'_{H(G)}(D) = \frac{1}{2}\mu'_{H'}(D) = 0$, contradicting the fact that each face of G is an open disk. ▮

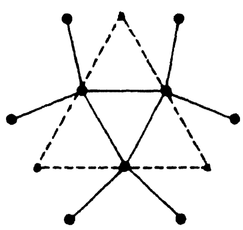
We cannot delete the condition in Proposition 2 that each face of G is an open disk, as on the torus S, the graph G consisting of one vertex with one non-nullhomotopic loop attached, is a kernel, but $H(G)$ is not tight ($H(G)$ consists of one vertex with two non-nullhomotopic loops (of the same homotopy) attached).

Finally we have:

PROPOSITION 3. *Let G and G' be graphs embedded on the compact surface S. If $H(G')$ arises from $H(G)$ by one $\Delta\nabla$-exchange, then G' arises from G by one ΔY-exchange, up to homotopy and duality.*

Proof. This follows from Proposition 1 and by considering the following two figures (where the uninterrupted lines are edges of $H(G)$ or $H(G')$, and the interrupted lines are edges of G or G'):





▮

Proposition 2 and 3 directly yield:

PROPOSITION 4. *Theorem 2 implies Theorem 1.*

Proof. If G and G' are kernels on the compact orientable surface S, so that each face of G and of G' is an open disk, then by Proposition 2, $H(G)$

and $H(G')$ are tight graphs. If $\mu_G = \mu_{G'}$ then $\mu'_{H(G)} = 2\mu_G = 2\mu_{G'} = \mu'_{H(G')}$. So by Theorem 2, $H(G)$ and $H(G')$ arise from each other by homotopic shifts and $\Delta\nabla$-exchanges. So by Proposition 3, G and G' arise from each other by homotopic shifts, duality, and ΔY-exchanges. ▮

4. Reduction of Theorem 2 to a Lemma

We now reduce Theorem 2 to a lemma on closed curves on a compact orientable surface. This lemma will be proved in Section 6 (Section 5 contains some preliminaries on hyperbolic plane geometry).

Let H be a 4-regular graph on a compact surface S. The *straight decomposition* of H is the decomposition of the edges of H into closed curves $C_1, ..., C_k$ in such a way that each edge is traversed exactly once by these curves, and that in each vertex w of H, if e_1, e_2, e_3, e_4 are the edges incident with w in cyclic order, then e_1, w, e_3 are traversed consecutively (in one way or the other), and similarly, e_2, w, e_4 are traversed consecutively (in one way or the other).

The straight decomposition is unique up to the choice of the beginning vertex of the curves, up to reversing the curves, and up to permuting the indices of $C_1, ..., C_k$.

We call a system $C_1, ..., C_k$ of closed curves *minimally crossing* if each C_i has the minimum number of self-intersections (among all closed curves freely homotopic to C_i), and each two C_i and C_j have the minimum number of intersections with each other (among all closed curves freely homotopic to C_i and C_j, respectively; taking $i \neq j$).

To be more precise, define for closed curves $C, D: S^1 \to S$:

$$\begin{aligned} \mathrm{cr}(C) &:= |\{(y, z) \in S^1 \times S^1 \mid C(y) = C(z), y \neq z\}|, \\ \min \mathrm{cr}(C) &:= \min\{\mathrm{cr}(C') \mid C' \sim C\}, \\ \mathrm{cr}(C, D) &:= |\{(y, z) \in S^1 \times S^1 \mid C(y) = D(z)\}|, \\ \min \mathrm{cr}(C, D) &:= \min\{\mathrm{cr}(C', D') \mid C' \sim C, D' \sim D\}. \end{aligned} \tag{6}$$

Then $C_1, ..., C_k$ are minimally crossing if $\mathrm{cr}(C_i) = \min \mathrm{cr}(C_i)$ and $\mathrm{cr}(C_i, C_j) = \min \mathrm{cr}(C_i, C_j)$ for all i, j with $i \neq j$.

A closed curve $C: S^1 \to S$ is *primitive* if C is not freely homotopic to D^n, for some closed curve $D: S^1 \to S$ and some $n \geqslant 2$. (Here D^n is the closed curve defined by $D^n(z) := D(z^n)$ for all $z \in S^1$.)

A key result of [18] is:

Proposition 5. *Let H be a 4-regular graph on the compact orientable* *surface S. Then H is tight if and only if the straight decomposition of H is a minimally crossing collection of primitive closed curves.*

In fact, the assertion holds for any eulerian graph on S.

As is shown in [18], it is not difficult to derive from Proposition 5:

PROPOSITION 6. *Let H be a tight 4-regular graph on the compact orientable surface S, with straight decomposition* $C_1, ..., C_k$. *Then for each closed curve D on S,*

$$\mu'_H(D) = \sum_{i=1}^{k} \min \operatorname{cr}(C_i, D). \tag{7}$$

Moreover, in [19] we derived from the results in [18]:

PROPOSITION 7. *Let* $C_1, ..., C_k$ *and* $C'_1, ..., C'_{k'}$ *be primitive closed curves on the compact orientable surface S. Then the following are equivalent:*

(i) $k = k'$, *and there exists a permutation* π *of* $\{1, ..., k\}$ *so that for each* $i = 1, ..., k$, $C_i \sim C'_{\pi(i)}$ *or* $C_i^{-1} \sim C'_{\pi(i)}$;

(ii) *for each closed curve D on S,*

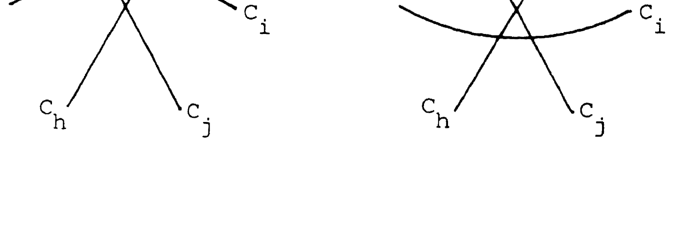
$$\sum_{i=1}^{k} \min \operatorname{cr}(C_i, D) = \sum_{i=1}^{k'} \min \operatorname{cr}(C'_i, D). \tag{8}$$

[The implication (i) $\Rightarrow$ (ii) is trivial.]

In order to prove Theorem 2, let H and H' be tight 4-regular graphs on the compact orientable surface S, with $\mu'_H = \mu'_{H'}$. Let $C_1, ..., C_k$ and $C'_1, ..., C'_{k'}$ be the straight decompositions of H and H', respectively. By (7), we know that (8) holds. Therefore, we may assume that $k = k'$, and that $C'_1 \sim C_1, ..., C'_k \sim C_k$. In fact, $C_1, ..., C_k$ can be moved to $C'_1, ..., C'_k$ in a finite number of steps, each step of which is one of the following:

(i) homotopic shifts of $C_1, ..., C_k$ so that during the shifting, no two crossings coincide and no new crossings are introduced; (9)

(ii) a "jump" of C_i over a crossing of C_h and C_j as in



(h, i, and j need not to be different). We assume that $C_1, ..., C_k$ do not intersect the triangle enclosed by C_h, C_i, and C_j.

If we transform $C_1, ..., C_k$ by applying a series of operations (9), we say that $C_1, ..., C_k$ are *moved by jumps*.

Since each jump corresponds to a $\Delta\nabla$-exchange in the underlying graph, it now suffices to prove:

LEMMA. *Let $C_1, ..., C_k$ and $C'_1, ..., C'_k$ be minimally crossing systems of primitive closed curves on the compact orientable surface S, such that $C_1 \sim C'_1, ..., C_k \sim C'_k$ and such that no point of S is covered more than twice by $C_1, ..., C_k$ or more than twice by $C'_1, ..., C'_k$. Then $C_1, ..., C_k$ can be moved by jumps to $C'_1, ..., C'_k$, possibly after permuting indices.*

Before proving this lemma in Section 6, we first give in Section 5 some prliminaries on hyperbolic plane geometry.

Note 4. The "jump" defined in (9) is similar to the type III move studied by Reidemeister [14] (cf. [10]) for knots and links.

5. THE HYPERBOLIC PLANE

In proving the lemma, we make use of the representation of the universal covering surface of S as the hyperbolic plane (if S is not the 2-sphere and not the torus). This representation was introduced by Poincaré [12] (cf. [5, 7]). Here we review some elements of this representation which we use in our proof.

Let $U = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit open disk in the complex $\mathbb{C}$. A set of points of U is called a *hyperbolic line* if it is the intersection of U with C, where C is a circle or (straight euclidean) line in $\mathbb{C}$ crossing the boundary of U orthogonally. The set U together with the set of hyperbolic lines makes the *hyperbolic plane*. Each two distinct points in U are contained in a unique hyperbolic line.

There exists a metric d on U so that the topology induced by d coincides with the usual topology on U, and so that the hyperbolic lines in U are the geodesics of d. That is, for any three points x, y, z in U one has: x, y, z are, in this order, on a hyperbolic line, if and only if $d(x, y) + d(y, z) = d(x, z)$.

For any $x \in U$ and any hyperbolic line l there is a unique point $y \in l$ so that $d(x, y) = d(x, l)$. If $x \notin l$, then y is also the unique point on l so that the hyperbolic line through x and y is orthogonal to l.

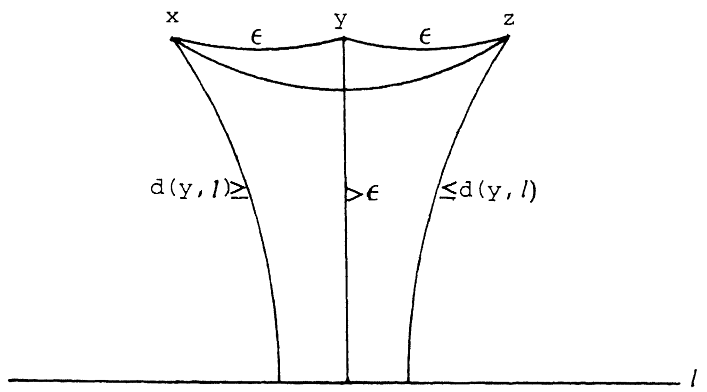
An *isometry* on U is a homeomorphism ϕ: $U \to U$ so that $d(\phi(x), \phi(y)) = d(x, y)$ for all $x, y \in U$. So an isometry brings hyperbolic lines to hyperbolic lines.

We will also use the following elementary fact from hyperbolic plane geometry. For any $\varepsilon > 0$ there exists a $\zeta > 0$ with the following property:

if l is a hyperbolic line and $x, y, z \in U$ so that:

$$\begin{aligned} &\text{(i)} \quad d(y, l) > \varepsilon; \\ &\text{(ii)} \quad d(x, l) \leqslant d(y, l) \text{ and } d(z, l) \leqslant d(y, l); \\ &\text{(iii)} \quad d(x, y) = d(y, z) = \varepsilon, \end{aligned} \tag{10}$$

then $d(x, z) < d(x, y) + d(y, z) - \zeta$.



The hyperbolic plane can be considered as a *universal covering surface* of a compact orientable surface S (of genus $\geqslant 2$). It implies that there exists a "projection" function $\psi: U \rightarrow S$ with the following properties (cf. also Baer [3]):

(i) Each $u \in U$ has a neighbourhood N homeomorphic to an open disk such that $\psi \mid N$ is one-to-one;

(ii) If $u, u' \in U$ and $\psi(u) = \psi(u')$, then there exists an isometry $\phi: U \rightarrow U$ so that $\phi(u) = u'$ and $\psi \circ \phi = \psi$.

(iii) For each closed curve $C: S^1 \rightarrow S$ and each $u \in \psi^{-1}[C(1)]$, there exists a unique continuous function $C': \mathbb{R} \rightarrow U$ such that $C'(0) = u$ and $\psi \circ C'(x) = C(\exp(2\pi i x))$ for each $x \in \mathbb{R}$. [C' is called a *lifting* of C to U.]

(iv) For each closed curve $C: S^1 \rightarrow S$ there exists a closed curve $C': S^1 \rightarrow S$ such that $C' \sim C$ and for each lifting L' of C', the set $L'[\mathbb{R}]$ is hyperbolic line. The closed curve C' is unique (up to orientation-preserving homeomorphism on S^1, i.e., up to replacing C' by $\tau \circ C'$, where $\tau: S^1 \rightarrow S^1$ is an orientation-preserving homeomorphism.) [We call C' a *geodesic curve.*] (11)

(v) If C' is a geodesic curve homotopic to $C: S_1 \to S$, then for each lifting L of C there exists a lifting L' of C' so that $\overline{L[\mathbb{R}]}$ and $\overline{L'[\mathbb{R}]}$ have the same two intersection points with the boundary bd(U) of U. [We say that L' is *parallel* to L.] The functions L and L' are periodic in the following sense: there exists an isometry $\phi: U \to U$ so that $\psi \circ \phi = \psi$ and so that $\phi(L(x)) = L(x+1)$ and $\phi(L'(x)) = L'(x+1)$ for each $x \in \mathbb{R}$.

6. Proof of the Lemma

I. We first prove an auxiliary proposition. Let $\bar{U}$ be the closed unit disk in $\mathbb{C}$, let $L_1, ..., L_t, L'_1, ..., L'_t$ be simple curves on $\bar{U}$ with end points on the boundary *bd*(U) of U, so that L_i and L'_i have the same pair of end points ($i = 1, ..., t$); so that L_i and L_j have at most one intersection, being a crossing in U, and similarly L'_i and L'_j have at most one intersection, being a crossing in U ($i, j = 1, ..., t$; $i \neq j$); and so that no three of the L_i pass one and the same point, and similarly, no three of the L'_i pass one and the same point. Then:

Proposition 8. *$L_1, ..., L_t$ can be moved by jumps in U to $L'_1, ..., L'_t$.*

(*Moving by jumps* is defined in a manner similar to that above.)

Proof. We apply induction on t. We may assume that $L_1, ..., L_t$ are straight euclidean line segments (since if we can move $L'_1, ..., L'_t$ to straight line segments, then by transitivity, any two choices for $L'_1, ..., L'_t$ can be moved to each other). Moreover, we may assume that $L'_1 = L_1$.

Without loss of generality, L_1 crosses $L'_2, ..., L'_n$, in this order ($n \leqslant t$). Let L_1 cross $L_{p(2)}, ..., L_{p(n)}$ in this order, for some permutation p of $\{2, ..., n\}$. We assume we have moved $L'_1, ..., L'_t$ by jumps so that

$$\text{the number of pairs } (i, j) \text{ with } i < j \text{ and } p(j) < p(i) \text{ is as small as possible.} \tag{12}$$

If p is the identity, we may assume that $L_j \cap L_1 = L'_j \cap L_1$ for $j = 2, ..., n$. Then by the induction hypothesis applied to the two parts into which L_1 divides U, we can move $L'_1, ..., L'_t$ by jumps to $L_1, ..., L_t$.

If p is not the identity, there exists an i with $2 \leqslant i < n$ so that $p(i+1) < p(i)$. We may assume that the crossing points x and y of L'_i and L'_{i+1} with L_1 are very close to each other (to be specified). Now by the induction hypotheses applied to the two parts into which L_1 divides U, we can move $L'_2, \ldots, L'_t$ by jumps, without changing the crossing points with L_1, in such a way that finally each of $L'_2, \ldots, L'_t$ is piecewise linear, with only one bend in the crossing point with L_1 (if any). Choosing x and y close enough to each other ensures that L'_1, L'_i, L'_{i+1} form a triangle not intersected by any of the other curves. Now we can do a jump at this triangle (that is, we move the crossing point of L'_i and L'_{i+1} to the other side of L'_1). This however would decrease the number (12), contradicting our assumption.

II. To prove the lemma, we may assume that S is not the 2-sphere. We first consider the case where S is not the torus either. Let $C''_1, \ldots, C''_k$ be geodesic curves on S homotopic to $C_1, \ldots, C_k$, respectively (cf. (11)(iv)). It might be that C''_i and C''_j coincide for $i \neq j$. Let $\mathscr{L}$ denote the collection of liftings of $C''_1, \ldots, C''_k$, considered as hyperbolic lines.

Let X be the set of points of U covered by more than one $l \in \mathscr{L}$. We choose $\rho > 0$ small enough that if $x \in X$ and $l \in \mathscr{L}$ with $x \notin l$, then $d(x, l) > 2\rho$. (The existence of such a $\rho > 0$ follows easily from (11)(i) and (11)(ii), using the fact that $X = \psi^{-1}[Y]$ for the finite set Y of crossing points of $C''_1, \ldots, C''_k$ on S.)

It follows that the closed balls $\overline{B(x, \rho)}$ of radius ρ and center $x \in X$ are pairwise disjoint. Moreover, each such ball intersects $\bigcup \mathscr{L}$ in a "star."

Next choose $\varepsilon > 0$ small enough that $\varepsilon < \rho$ and that each two distinct components of $\bigcup \mathscr{L} \setminus \bigcup_{x \in X} B(x, \rho)$ have distance $> \varepsilon$. (Again, the existence of such an ε follows from the symmetry of U and $\mathscr{L}$.)

We now move $C_1, \ldots, C_k$ by jumps, so as to obtain $\tilde{C}_1 \sim C_1, \ldots, \tilde{C}_1 \sim C_k$, with the property that each lifting $\tilde{L}$ of $\tilde{C}_i$ is at a distance at most ε from the lifting of C''_i that is parallel to $\tilde{L}$ (cf. (11)(v)).

To describe this moving, suppose lifting L of C_i is not contained in the ε-neighborhood of lifting $L' \in \mathscr{L}$ parallel to L. Choose a point $u \in L$ which maximizes $d(u, L')$ (such a u exists, as L and L' are periodic—cf. (11)(v)). So $d(u, L') > \varepsilon$. Consider the closed ball $\overline{B(u, \varepsilon)}$ of radius ε and with center u. By Proposition 8 above, we can move the intersections of the liftings of $C_1, \ldots, C_k$ with $\overline{B(u, \varepsilon)}$ by jumps within $B(u, \varepsilon)$, fixing the points on the boundary of $B(u, \varepsilon)$, in such a way that after moving, these intersections are hyperbolic line segments. (We make "small" deviations to prevent three of these line segments from going through one point—"small" to be specified below.)

Since ψ restricted to $\overline{B(u, \varepsilon)}$ is one-to-one, we can reproduce these moves on S, giving a move of $C_1, \ldots, C_k$ by jumps. Let us call this a *local move*. We show:

PROPOSITION 9. *After a finite number of local moves, each lifting L of each C_i is contained in the ε-neighbourhood of the line in $\mathscr{L}$ parallel to L.*

Proof. Let $\zeta > 0$ satisfy (10). Then at each local move, the sum of the lengths of the C_i is decreased by at least ζ (taking the deviations small enough). Here, without loss of generality, we take the C_i piecewise linear (i.e., the liftings are piecewise linear in hyperbolic geometry). The *length* of C_i is the length of one period of its lifting.

As this sum remains nonnegative, we can apply only a finite number of local moves. ∎

So by a finite number of local moves we can shift $C_1, ..., C_k$ so that each lifting L of each C_i is contained in the ε-neighbourhood of the line in $\mathscr{L}$ parallel to L. We can shift $C'_1, ..., C'_k$ similarly.

Now for each $L' \in \mathscr{L}$ there is a number $k_{L'}$ so that there are $k_{L'}$ curves among $C_1, ..., C_k$ with a lifting in the ε-neighbourhood of L'. These $k_{L'}$ liftings are pairwise disjoint, since $C_1, ..., C_k$ are minimally crossing. Similarly, there are $k_{L'}$ pairwise disjoint liftings of $C'_1, ..., C'_k$ contained in the ε-neighbourhood of L'. By permuting indices we may assume that these $k_{L'}$ curves have indices in the same order in $C_1, ..., C_k$ as in $C'_1, ..., C'_k$.

Consider next a closed ball $\overline{B(x, \rho)}$ with $x \in X$. We may assume that if L' passes through x, then the $k_{L'}$ liftings of $C_1, ..., C_k$ in the ε-neighbourhood of L' intersect the boundary of $\overline{B(x, \rho)}$ exactly twice, and similarly for the $k_{L'}$ liftings of $C'_1, ..., C'_k$. Moreover, we may assume that the points of intersection on the boundary of $\overline{B(x, \rho)}$ are the same for $C_1, ..., C_k$ as for $C'_1, ..., C'_k$. Hence by Proposition 8, we can move the liftings of $C_1, ..., C_k$ by jumps on $B(x, \rho)$ so that they coincide on $B(x, \rho)$ with the liftings of $C'_1, ..., C'_k$. Reproducing these shifts on S, we finally obtain that $C_1, ..., C_k$ are moved by jumps to $C'_1, ..., C'_k$.

III. We finally show that the lemma is also true in the case where S is the torus. In fact, this can be reduced easily to the double torus case (genus 2). To see this, put a "small" handle somewhere on the torus S, where the feet of the handle are close enough to each other that both are contained in the same component of $S \setminus (C_1 \cup \cdots \cup C_k \cup C'_1 \cup \cdots C'_k)$. Then also on the new surface S', C_i and C'_i are freely homotopic ($i = 1, ..., k$). This follows from the fact that on the torus S, there are two distinct ways of shifting C_i to C'_i. Now we saw above that on S' we can move $C_1, ..., C_k$ to $C'_1, ..., C'_k$ by jumps. This implies that the same can be done on S. ∎

Note 5. The lemma also implies the following theorem:

THEOREM 3. *Let $C_1, ..., C_k$ and $C'_1, ..., C'_k$ be minimally crossing collections of primitive closed curves on a compact orientable surface S, such that $C_i \sim C'_i$ for $i = 1, ..., k$. Then we can shift $C_1, ..., C_k$ to $C'_1, ..., C'_k$ over S (possibly after permuting subscripts), keeping them minimally crossing* *throughout the shifting process. That is, there exist a permutation p of* $\{1, ..., k\}$ *and continuous functions* $\Phi_1, ..., \Phi_k: [0, 1] \times S^1 \to S$ *such that*:

(i) $\Phi_i(0, z) = C_i(z)$ *and* $\Phi_i(1, z) = C'_{p(i)}(z)$ *for all* $z \in S^1$ *and all* $i \in \{1, ..., k\}$;

(ii) *for each* $x \in [0, 1]$, *the collection of curves* $\Phi_1(x, \cdot), ..., \Phi_k(x, \cdot)$ *is minimally crossing.*

This generalizes a theorem of Baer [4], where $C_1, ..., C_k$ are simple and pairwise disjoint.

Note 6. The method described in this paper yields a construction of kernels. Let S be a compact orientable surface, and let $C_1, ..., C_k$ be a minimally crossing system of primitive closed curves on S. Assume that each face of the graph $H := C_1 \cup \cdots \cup C_k$ is an open disk, and that the faces of H can be colored black and white so that adjacent faces have different colors. (This last is equivalent to each curve D on S having an even number of crossings with $C_1, ..., C_k$.) Now H is equal to the medial graph $H(G)$ of some graph G on S. Then G is a kernel, and each kernel arises in this way (provided that each face of the kernel is an open disk).

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Note added in proof. Proposition 8 was proved before by G. Ringel, Teilungen der Ebene durch Geraden oder topologische Geraden, *Math. Z.* **64** (1955), 79–102 (Satz 26).

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