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Packing and Covering of Crossing Families of Cuts

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Let $\mathscr{C}$ be a *crossing family* of subsets of the finite set V (i.e., if $T, U \in \mathscr{C}$ and $T \cap U \neq \varnothing$, $T \cup U \neq V$, then $T \cap U \in \mathscr{C}$ and $T \cup U \in \mathscr{C}$). If $D = (V, A)$ is a directed graph on V, then a *cut induced by* $\mathscr{C}$ is the set of arcs entering some set in $\mathscr{C}$. A *covering for* $\mathscr{C}$ is a set of arcs entering each set in $\mathscr{C}$, i.e., intersecting all cuts induced by $\mathscr{C}$. It is shown that the following three conditions are equivalent for any given crossing family $\mathscr{C}$:

(P1) For every directed graph $D = (V, A)$, the minimum cardinality of a cut induced by $\mathscr{C}$ is equal to the maximum number of pairwise disjoint coverings for $\mathscr{C}$.

(P2) For every directed graph $D = (V, A)$, and for every length function $l: A \to \mathbb{Z}_+$, the minimum length of a covering for $\mathscr{C}$ is equal to the maximum number t of cuts $C_1, \ldots, C_t$ induced by $\mathscr{C}$ (repetition allowed) such that no arc a is in more than $l(a)$ of these cuts.

(P3) $\varnothing \in \mathscr{C}$, or $V \in \mathscr{C}$, or there are no V_1, V_2, V_3, V_4, V_5 in $\mathscr{C}$ such that $V_1 \subseteq V_2 \cap V_3$, $V_2 \cup V_3 = V$, $V_3 \cup V_4 \subseteq V_5$, $V_3 \cap V_4 = \varnothing$.

Directed graphs are allowed to have parallel arcs, so that (P1) is equivalent to its capacity version. (P1) and (P2) assert that certain hypergraphs, as well as their blockers, have the "$\mathbb{Z}_+$-max-flow min-cut property." The equivalence of (P1), (P2), and (P3) implies Menger's theorem, the König–Egerváry theorem, the König–Gupta edge-colouring theorem for bipartite graphs, Fulkerson's optimum branching theorem, Edmonds' disjoint branching theorem, and theorems of Frank, Feofiloff and Younger, and the present author.

1. INTRODUCTION

Throughout this paper, let V be a finite set. A collection $\mathscr{C}$ of subsets of V is called a *crossing family* if:

$$\text{if } T, U \in \mathscr{C} \text{ and } T \cap U \neq \varnothing \text{ and } T \cup U \neq V, \text{ then } T \cap U \in \mathscr{C} \text{ and } T \cup U \in \mathscr{C}. \tag{1}$$

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For any directed graph $D=(V, A)$, a collection A' of arcs of D is called a *cut induced by* $\mathscr{C}$ if $A'=\delta_A^-(V')$ for some V' in $\mathscr{C}$. ($\delta_A^-(V')$ and $\delta_A^+(V')$ denote the sets of arcs of D entering V' and leaving V', respectively.) A subset A' of A is called a *covering for* $\mathscr{C}$ if each set in $\mathscr{C}$ is entered by at least one arc in A', i.e., if A' intersects each cut induced by $\mathscr{C}$.

Several min–max relations in combinatorics amount to the fact that certain crossing families $\mathscr{C}$ have one of the following properties (P1) and (P2):

(P1) For every directed graph $D=(V, A)$, the minimum cardinality of a cut induced by $\mathscr{C}$ is equal to the maximum number of pairwise disjoint coverings for $\mathscr{C}$.

(P2) For every directed graph $D=(V, A)$, and for every "length" function $l: A \to \mathbb{Z}_+$, the minimum length of a covering for $\mathscr{C}$, is equal to the maximum number t of cuts $C_1, \ldots, C_t$ induced by $\mathscr{C}$ (repetition allowed) such that no arc a of D is in more than $l(a)$ of these cuts.

(Here the *length* of a set of arcs is by definition the sum of the lengths of the arcs in this set. In hypergraph terminology (P2) says: for every directed graph on V, the hypergraph of cuts induced by $\mathscr{C}$ has the $\mathbb{Z}_+$-max-flow min-cut property (cf. Seymour [20, 21]). We allow directed graphs to have parallel arcs, so that (P1) is equivalent to its "capacitated" version: for every directed graph $D=(V, A)$ and for every "capacity" function c: $A \to \mathbb{Z}_+$, the minimum capacity of a cut induced by $\mathscr{C}$ is equal to the maximum number t of coverings $A_1, \ldots, A_t$ for $\mathscr{C}$ (repetition allowed) such that no arc a of D is in more than $c(a)$ of these coverings. In hypergraph language, (P1) is equivalent to: for every directed graph on V, the hypergraph of coverings for $\mathscr{C}$ has the $\mathbb{Z}_+$-max-flow min-cut property. Here and below we assume that min–max relations like (P2) include the case that if the minimum is infeasible (so if in (P2) no coverings exist), then the maximum is unbounded (which means in case of (P2): there exists an empty cut induced by $\mathscr{C}$).)

In this paper we characterize the crossing families which enjoy properties (P1) and (P2). We shall show that, for any crossing family $\mathscr{C}$, (P1) and (P2) are equivalent, and moreover, that $\mathscr{C}$ satisfies (P1) and (P2) if and only if $\phi \in \mathscr{C}$ or $V \in \mathscr{C}$ (two trivial cases), or

(P3) there are no V_1, V_2, V_3, V_4, V_5 $(\neq \varnothing, V)$ in $\mathscr{C}$ such that $V_1 \subseteq V_2 \cap V_3$, $V_2 \cup V_3 = V$, $V_3 \cup V_4 \subseteq V_5$, $V_3 \cap V_4 = \varnothing$.

The forbidden subcollection described in (P3) is illustrated by the Venn diagram of Fig. 1, where a square is meant to represent the complement of its interior.

Applications of the equivalence of (P1), (P2), and (P3) are contained in the following examples.

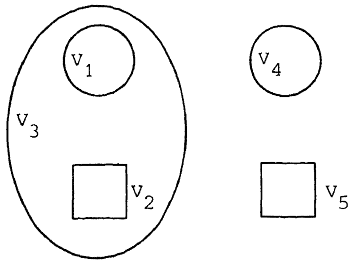

FIGURE 1

EXAMPLE 1 (*$r-s$-paths and $r-s$-cuts*). Let r and s be two fixed elements of V, and let $\mathscr{C}$ be the collection of all subsets of $V\backslash\{r\}$ containing s. Then $\mathscr{C}$ is a crossing family satisfying (P3). For any directed graph $D=(V,A)$, cuts induced by $\mathscr{C}$ are exactly the $r-s$-cuts. Coverings for $\mathscr{C}$ are exactly the sets of arcs containing an $r-s$-path. So (P1) is equivalent to Menger's theorem [16] (and to the max-flow min-cut theorem): the minimum size of an $r-s$-cut is equal to the maximum number of pairwise disjoint $r-s$-paths. (P2) is equivalent to an easy theorem on shortest paths: for any length function $l\colon A \to \mathbb{Z}_+$, the minimum length of an $r-s$-path is equal to the maximum number t of $r-s$-cuts $C_1,\dots,C_t$ such that no arc a is in more than $l(a)$ of these cuts (cf. Fulkerson [7]).

EXAMPLE 2 (*Coverings and edge-colourings in bipartite graphs*). Let $\mathscr{C}$ be a subcollection of $\{\{v\}\mid v\in V\}\cup\{V\backslash\{v\}\mid v\in V\}$. Then $\mathscr{C}$ is a crossing family satisfying (P3). Then property (P1) can easily be seen to be equivalent to a theorem of König [12] and Gupta [10]: let $G=(V,E)$ be a bipartite graph; then the minimum degree of G is equal to the maximum number of colours with which we can colour the edges of G such that in each vertex of G all colours occur at least once. (P2) is equivalent to a theorem of König [13] and Egerváry [4]: let $G=(V,E)$ be a bipartite graph, and let $w\colon E\to\mathbb{Z}_+$ be a weight function; then the minimum weight of an edge set covering all vertices of G is equal to the maximum number t of vertices $v_1,\dots,v_t$ of G (repetition allowed) such that each edge e contains at most $w(e)$ of the v_i. If we take all weights equal to 1, we obtain the well-known König–Egerváry theorem.

EXAMPLE 3 (*Branchings*). Let r be a fixed element of V, and let $\mathscr{C}$ be the collection of all nonempty subsets of $V\backslash\{r\}$. Then $\mathscr{C}$ is a crossing family satisfying (P3). For any directed graph $D=(V,A)$, cuts induced by $\mathscr{C}$ are exactly the *r-cuts* (which are by definition the sets $\delta_A^-(V')$ for $\varnothing\neq V'\subseteq V\backslash\{r\}$). Coverings for $\mathscr{C}$ are exactly the sets of arcs containing an *r-branching* (which are rooted spanning trees with root r). Now (P1) is equivalent to Edmonds' disjoint branching theorem [2]: the minimum size of an r-cut is equal to the maximum number of pairwise disjoint r-branchings.

Property (P2) is equivalent to Fulkerson's optimum branching theorem [8]: given a length function $l: A \to \mathbb{Z}_+$, the minimum length of an r-branching is equal to the maximum number t of r-cuts $C_1,..., C_t$ such that no arc a of D is in more than $l(a)$ of these r-cuts.

EXAMPLE 4 (*Directed cuts and their coverings*). Let $D = (V, A)$ be an acyclic directed graph, in which each pair of source and sink is connected by a directed path. Let $C \subseteq A$ be such that each directed cut of D contains at least k arcs in C. (A *directed cut* is a set $\delta_A^-(V')$ of arcs of D, with $\varnothing \neq V' \neq V$ and $\delta_A^+(V') = \varnothing$.) Then C can be partitioned into sets $C_1,..., C_k$ each intersecting all directed cuts. This result (Feofiloff and Younger [5], Schrijver [18]) follows from the equivalence of (P1) and (P3) by taking $\mathscr{C}$ to be the collection of all nonempty proper subsets V' of V with $\delta_A^+(V') = \varnothing$ (which collection can be easily seen to be a crossing family satisfying (P3)). In this case, (P2) gives a weaker version of the Lucchesi–Younger theorem [15]: the minimum cardinality of a set of arcs intersecting all directed cuts is equal to the maximum number of pairwise disjoint directed cuts (in this case, the "length" version can be easily derived from the cardinality version). This is weaker than the Lucchesi–Younger theorem, as this theorem is not restricted to acyclic directed graphs in which each pair of source and sink is connected by a directed path—see Remark 1.

EXAMPLE 5 (*Strong connectors*). Let $D_0 = (V, A_0)$ be an acyclic directed graph, in which each pair of source and sink is connected by a directed path. Let $D = (V, A)$ be a second directed graph on V. Call a set $A' \subseteq A$ a *strong connector for* D_0 if the directed graph $(V, A_0 \cup A')$ is strongly connected. A set $A' \subseteq A$ is a *cut induced by* D_0 if $A' = \delta_A^-(V')$ for some $V' \subseteq V$ with $\varnothing \neq V' \neq V$ and $\delta_{A_0}^-(V') = \varnothing$. Then the maximum number of pairwise disjoint strong connectors for D_0 is equal to the minimum size of a strong cut induced by D_0. Moreover, for any length function $l: A \to \mathbb{Z}_+$ the minimum length of a strong connector is equal to the maximum number t of cuts $C_1,..., C_t$ induced by D_0 such that no arc a of D is in more than $l(a)$ of these cuts (Schrijver [18]). These results can be seen to contain the min–max relations discussed in Examples 1–4. They follow from the equivalence of (P1), (P2), and (P3) by taking $\mathscr{C}$ to be the collection of all nonempty proper subsets V' of V with $\delta_{A_0}^-(V') = \varnothing$, which is a crossing family satisfying (P3).

EXAMPLE 6 (*Strong connectors again*). The equivalence of (P1), (P2), and (P3) gives the following characterization. Let $D_0 = (V, A_0)$ be an acyclic directed graph. Then the following are equivalent:

(i) for each directed graph $D = (V, A)$, the minimum size of a cut induced by D_0 is equal to the maximum number of pairwise disjoint strong connectors for D_0;

(ii) for each directed graph $D = (V, A)$ and for each length function $l: A \to \mathbb{Z}_+$, the minimum length of a strong connector for D_0 is equal to the maximum number t of cuts $C_1,..., C_t$ induced by D_0 such that no arc a of D is contained in more than $l(a)$ of these cuts;

(iii) each pair of source and sink of D_0 is connected by a directed path in D_0, or $|\{s \in V | s \text{ is a source or a sink of } D_0\}| \leqslant 3$.

This follows by considering $\mathscr{C} := \{V' \subseteq V \mid \varnothing \neq V' \neq V, \delta^-_{A_0}(V') = \varnothing\}$, which is a crossing family. We leave it to the reader to check that (P3) is equivalent to condition (iii). Note that a similar characterization for arbitrary (not necessarily acyclic) directed graphs D_0 follows easily by contracting the strong components of D_0: the properties (i) and (ii) are invariant under such contractions, while (iii) yields a condition on the original D_0.

EXAMPLE 7 (*Intersecting families*). A collection $\mathscr{C}$ of subsets of V is called an *intersecting family* if for all T, U in $\mathscr{C}$ with $T \cap U \neq \varnothing$, also $T \cap U$ and $T \cup U$ belong to $\mathscr{C}$. So each intersecting family is a crossing family. Moreover, it is easy to see that any intersecting family $\mathscr{C}$ either contains $\varnothing$ or V, or satisfies (P3). So each intersecting family satisfies (P1) and (P2), which is the content of a theorem of Frank [6].

In Section 5 we give a proof of the equivalence of (P1), (P2), and (P3). The implications (P1) $\Rightarrow$ (P3) and (P2) $\Rightarrow$ (P3) will follow easily by constructing a counterexample for (P1) and (P2) if the five-set configuration described in (P3) occurs. The implication (P3) $\Rightarrow$ (P2) will be shown with a proof technique set up by Hoffman, Edmonds and Giles, Lovász and Robertson. This consists of showing that there are optimal cut packings which are "cross-free," and that hence by total unimodularity certain LP-problems have integral optimal solutions, which yield the desired result. Here we use an auxiliary theorem characterizing total unimodularity for such cross-free families—see Section 2.

The proof of the implication (P3) $\Rightarrow$ (P1) turns out to be rather complicated. Here we need two other auxiliary theorems, which may be interesting in their own right, and which we give in the Sections 3 and 4.

Remark 1. In (P1) and (P2) we required the min–max relations for cuts and coverings to hold for *all* directed graphs on V. It is a more general problem to characterize pairs $(\mathscr{C}, D)$ of a crossing family $\mathscr{C}$ on V and a directed graph D on V having the properties described in (P1) and (P2), respectively. For example, the Lucchesi–Younger theorem [15], and its extension by Edmonds and Giles [3], assert that if $\mathscr{C}$ is a crossing family on V and no arc of D leaves any set $V' \in \mathscr{C}$, then $(\mathscr{C}, D)$ satisfies the properties described in (P2). However, this class of pairs $(\mathscr{C}, D)$ in general does not have the properties described in (P1) (see Schrijver [17]). So for *fixed* graphs D, (P1) and (P2) are not equivalent.

In Schrijver [19] it is shown that $(\mathscr{C}, D)$ has the property described in (P2), if $\mathscr{C}$ is a crossing family on V, and $D=(V, A)$ is a directed graph such that: if $V_1, V_2, V_3 \in \mathscr{C}$, and $V_1 \subseteq V \backslash V_2 \subseteq V_3$, then no arc of D enters both V_1 and V_3. This generalizes the Lucchesi–Younger theorem.

The more general problem of characterizing pairs $(\mathscr{C}, D)$ of an arbitrary, not necessarily crossing, family $\mathscr{C}$ of subsets of V, together with a directed graph $D=(V, A)$, having the properties described in (P1) or (P2), can easily be seen to be equivalent to the problem of characterizing "hypergraphs with the $\mathbb{Z}_+$-max-flow min-cut property." This is a notoriously difficult problem —see Seymour [20, 21].

Some notation and terminology. Two subsets T and U of V are said to *cross* if $T \cap U \neq \varnothing$, $T \cup U \neq V$, $T \nsubseteq U$, and $U \nsubseteq T$. $\delta_A^-(U)$ and $\delta_A^+(U)$ denote the sets of arcs in A entering U and leaving U, respectively. $d_A^-(U)$ and $d_A^+(U)$ denote the *number* of arcs in A entering U and leaving U, respectively. Other concepts frequently used are *crossing family* (see (1)), *cut*, *covering* (see above), *cross-free family* (see (2)), *intersecting family* (see (7)), and *super-* and *submodular functions* (see (8)).

2. First Auxiliary Theorem

The first auxiliary theorem, necessary for the implication (P3) $\Rightarrow$ (P2), characterizes "cross-free" families which generate totally unimodular matrices in a certain way.

Two subsets T and U of V are said to *cross* if $T \cap U \neq \varnothing$, $T \cup U \neq V$, $T \nsubseteq U$, and $U \nsubseteq T$. A collection $\mathscr{C}$ of subsets of V is called *cross-free* if no two sets in $\mathscr{C}$ cross, i.e., if

$$\text{for all } T, U \text{ in } \mathscr{C}: T \cap U=\varnothing, \quad \text{or} \quad T \cup U=V, \quad \text{or} \quad T \subseteq U, \quad \text{or} \quad U \subseteq T. \tag{2}$$

In particular, each cross-free family is a crossing family.

Let $D=(V, A)$ be the "complete digraph" on V, i.e., A consists of all pairs (u, v) with $u, v \in V$ and $u \neq v$. Let $M_{\mathscr{C}}$ be the matrix with rows and columns indexed by $\mathscr{C}$ and A, respectively, and with

$$\begin{aligned} (M_{\mathscr{C}})_{V', a} &= 1, \qquad \text{if } a \text{ enters } V', \\ &= 0, \qquad \text{otherwise,} \end{aligned} \tag{3}$$

for $V' \in \mathscr{C}$ and $a \in A$.

First Auxiliary Theorem. *For any cross-free family $\mathscr{C}$, the matrix $M_{\mathscr{C}}$ is totally unimodular if and only if $\mathscr{C}$ satisfies* (P3).

Proof. (I) First suppose $M_{\mathscr{C}}$ is totally unimodular, and $\mathscr{C}$ contains a subcollection $\{V_1, V_2, V_3, V_4, V_5\}$ as described in (P3). Choose elements $v_1 \in V_1$, $v_2 \in V \backslash V_2$, $v_4 \in V_4$, $v_5 \in V \backslash V_5$. Consider the submatrix of $M_{\mathscr{C}}$ with rows indexed by V_1, V_2, V_3, V_4, V_5, and with columns indexed by the arcs (v_4, v_1), (v_2, v_1), (v_2, v_4), (v_5, v_4), (v_5, v_2) (cf. Fig. 3 in Section 5). One easily checks that this 5×5 submatrix has determinant ± 2, and hence $M_{\mathscr{C}}$ is not totally unimodular.

(II) Conversely, let the cross-free family $\mathscr{C}$ satisfy (P3). To prove that $M_{\mathscr{C}}$ is totally unimodular, we use the following characterization of Ghouila-Houri [9]: a matrix M is totally unimodular if and only if each collection R of rows of M can be split into classes R_1 and R_2 such that the sum of the rows in R_1, minus the sum of the rows in R_2, is a vector with entries 0, ± 1 only.

To show that $M_{\mathscr{C}}$ fulfills Ghouila-Houri's criterion, choose $\mathscr{C}' \subseteq \mathscr{C}$ (being the index set of the collection R of rows). Without loss of generality we may assume that $\mathscr{C}'$ does not contain $\varnothing$ or V (as they represent all-zero rows).

Now make the directed graph $D' = (\mathscr{C}', A')$, with vertex set $\mathscr{C}'$, where A' consists of all pairs (T, U) such that

$$T, U \in \mathscr{C}', \quad T \subset U, \quad \text{and there is no } W \in \mathscr{C}' \text{ with } \quad T \subset W \subset U. \tag{4}$$

We show that the undirected graph underlying D' is bipartite, which will verify Ghouila-Houri's criterion: if $\mathscr{C}'_1$ and $\mathscr{C}'_2$ are the two colour classes, then any arc $a = (u, v)$ of D will enter a chain of subsets in $\mathscr{C}'$ (as $\mathscr{C}'$ is cross-free), which subsets are alternatingly in $\mathscr{C}'_1$ and $\mathscr{C}'_2$. Hence the sum of the rows of $M_{\mathscr{C}}$ with index in $\mathscr{C}'_1$, minus the sum of the rows with index in $\mathscr{C}'_2$, has an entry 0 or ± 1 in position a.

To show that D' is bipartite, suppose it has an (undirected) circuit of odd length. Since this cycle is odd, and D' has no directed cycles, it follows that there are distinct $U_0, U_1, \ldots, U_k, U_{k+1}$ in $\mathscr{C}'$ with $k \geqslant 3$, such that

$$(U_1, U_0), (U_1, U_2), (U_2, U_3), \ldots, (U_{k-1}, U_k), (U_{k+1}, U_k) \tag{5}$$

are in A'. So U_0 and U_2 are distinct minimal sets in $\mathscr{C}'$ containing U_1 as a subset. As $\mathscr{C}'$ is cross-free, $U_0 \cup U_2 = V$. Similarly, U_{k-1} and U_{k+1} are distinct maximal subsets of U_k, and hence $U_{k-1} \cap U_{k+1} = \varnothing$. As $U_2 \subseteq U_{k-1}$, it follows that $U_1 \subseteq U_0 \cap U_2$, $U_0 \cup U_2 = V$, $U_2 \cup U_{k+1} \subseteq U_k$ and $U_2 \cap U_{k+1} = \varnothing$. However, this configuration is excluded by (P3). ∎

Remark 2. It is not difficult to derive from this theorem the equivalence of (P1), (P2), and (P3) for *cross-free* families, using the results of Hoffman and Kruskal [11] and Berge and Las Vergnas [1] and assuming the implications (P1) $\Rightarrow$ (P3) and (P2) $\Rightarrow$ (P3) (which are not difficult—see Section 5). Indeed, by the theorem, if $\mathscr{C}$ is cross-free and (P3) holds, then $M_{\mathscr{C}}$ is totally unimodular. The matrix $M_{\mathscr{C}}$ has as rows the incidence vectors of the cuts induced by $\mathscr{C}$. By Hoffman and Kruskal's theorem, both sides of the LP-duality equation

$$\begin{array}{llll} \min \sum_{a\in A} l(a)\,x(a) & & = \max \sum_{V'\in\mathscr{C}} y(V') & \\ \text{subject to} & & \text{subject to} & \\ \sum_{a\in\delta^-(V')} x(a) \geqslant 1 & (V'\in\mathscr{C}), & \sum_{V'\in\mathscr{C},\, a\in\delta^-(V')} y(V') \leqslant l(a) & (a\in A), \\ x(a)\geqslant 0 & (a\in A), & y(V')\geqslant 0 & (V'\in\mathscr{C}), \end{array} \tag{6}$$

are attained by integral optimal solutions, for all $l\colon A \to \mathbb{Z}_+$. This follows as the constraint matrix in (6) is the totally unimodular $M_{\mathscr{C}}$. The fact that (6) has integral optimal solutions is equivalent to (P2).

Berge and Las Vergnas showed that if M is a nonnegative totally unimodular matrix (more generally, if M is a "balanced" matrix), and each row of M sums up to at least k, then the columns of M can be split into k classes such that the sum of the columns in any of these classes has all entries at least 1. Since $M_{\mathscr{C}}$ keeps totally unimodular if we repeat or remove columns, (P1) follows from the result of Berge and Las Vergnas.

Remark 3. Using the "tree-representation" of cross-free families, introduced by Edmonds and Giles [3], the first auxiliary theorem is equivalent to the following:

Let $T=(V,A)$ be a directed tree. For each two vertices u and v of T, let $A_{u,v}$ be the set of arcs of T which are directed forwards in the unique path in T from u to v. Let M_T be the $\{0,1\}$-matrix with rows the incidence vectors of the $A_{u,v}$ $(u, v \in V)$. Then M_T is totally unimodular if and only if T is not contractible to the tree given in Fig. 2.

To derive this from the first auxiliary theorem, let $\mathscr{C} = \{V' \subseteq V \mid \text{there is an arc } (u,v) \text{ of } T \text{ such that } V' \text{ is the component of } T\backslash(u,v) \text{ containing } v\}$.

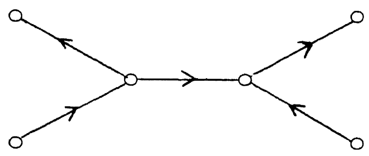

FIGURE 2

Then $\mathscr{C}$ is a cross-free family, and $M_{\mathscr{C}} = M_T$. Moreover, $\mathscr{C}$ contains the five-set configuration described in (P3) if and only if T is contractible to the tree of Fig. 2. This shows that the first auxiliary theorem implies the above characterization. The converse implication is shown similarly.

3. Second Auxiliary Theorem

Our second auxiliary theorem concerns colourings and supermodular functions. A collection $\mathscr{C}$ of subsets of a finite set S is called an *intersecting family* if

$$\text{for all } T, U \text{ in } \mathscr{C} \text{ with } T \cap U \neq \varnothing \text{ we have } T \cap U \in \mathscr{C} \text{ and } T \cup U \in \mathscr{C}. \tag{7}$$

If $\mathscr{C}$ is an intersecting family on S, a function $g\colon \mathscr{C} \to \mathbb{R}$ is called *supermodular* (*on intersecting pairs*) if

$$\text{for all } T, U \text{ in } \mathscr{C} \text{ with } T \cap U \neq \varnothing\colon g(T \cap U) + g(T \cup U) \geqslant g(T) + g(U). \tag{8}$$

Similarly, g is called *submodular* (*on intersecting pairs*) if in (8) the reversed inequality holds.

Clearly, if f and g are supermodular (resp. submodular), then also the function $f + g$ is supermodular (resp. submodular). Moreover, f is supermodular if and only if $-f$ is submodular.

The following observation follows easily with the "sandwich principle":

> if $\mathscr{C}$ is an intersecting family, and $g\colon \mathscr{C} \to \mathbb{R}$ is supermodular and $f\colon \mathscr{C} \to \mathbb{R}$ is submodular, such that $g(T) \leqslant f(T)$ for all T in $\mathscr{C}$, then the collection of sets T in $\mathscr{C}$ with $g(T) = f(T)$ is again an intersecting family. (9)

We shall frequently use the following two submodular functions. If (V, A) is a directed graph, then the set-function $d_A^-(U)$, for $U \subseteq V$, is submodular. If $X_1, \ldots, X_n$ are sets, define for $T \subseteq S$,

$$h_{X_1,\ldots,X_n}(T) := \text{the number of } j = 1, \ldots, n \text{ with } T \cap X_j \neq \varnothing. \tag{10}$$

Then for fixed $X_1, \ldots, X_n$, the function $h_{X_1,\ldots,X_n}$ is a submodular set-function on S.

The following theorem will be applied in proving (P3) $\Rightarrow$ (P1). It contains as direct applications theorems of König, Gupta, and De Werra on edge-colourings of bipartite graphs (see below, after the proof).

SECOND AUXILIARY THEOREM. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be intersecting families on the finite set S, and let $g_1 : \mathscr{C}_1 \to \mathbb{Z}$ and $g_2 : \mathscr{C}_2 \to \mathbb{Z}$ be supermodular on intersecting pairs. Suppose that $g_i(T) \leqslant |T|$ for $i = 1, 2$ and $T \in \mathscr{C}_i$. Then the minimum number of colours needed to colour S such that each set T intersects at least $g_i(T)$ colours (for $i = 1, 2$; $T \in \mathscr{C}_i$), is equal to the maximum of $g_i(T)$ ($i = 1, 2$; $T \in \mathscr{C}_i$) (provided that this maximum is positive).*

(Mathematically, "colouring" is the same as "partitioning," and a "colour" is a class of the partition.)

Proof. Clearly, the maximum does not exceed the minimum. To prove the converse, we use the submodular function defined in (10). Let $k := \max\{ g_i(T) \mid i = 1, 2;\ T \in \mathscr{C}_i\}$. Let $S_1, ..., S_k$ be pairwise disjoint subsets of S such that

$$g_i(T) \leqslant h_{S_1,...,S_k}(T) + |T \backslash (S_1 \cup \cdots \cup S_k)| \tag{11}$$

for $i = 1, 2$, and $T \in \mathscr{C}_i$, and such that

$$|S_1 \cup \cdots \cup S_k| \text{ is as large as possible.} \tag{12}$$

Such $S_1, ..., S_k$ exist, as $S_1 = \cdots = S_k = \varnothing$ satisfies (11). We are finished when we have shown that $S_1 \cup \cdots \cup S_k = S$. Suppose to the contrary there is an s in $S \backslash (S_1 \cup \cdots \cup S_k)$. Then there will exist a j_1 such that if we replace S_{j_1} by $S_{j_1} \cup \{s\}$, then (11) is still satisfied for $i = 1$. Otherwise, for all $j = 1, ..., k$, there would exist a set T_j in $\mathscr{C}_1$ such that

$$g_1(T_j) > h_{S_1,...,S_{j-1},S_j \cup s, S_{j+1},...,S_k}(T_j) + |T_j \backslash (S_1 \cup \cdots \cup S_k \cup s)|. \tag{13}$$

Combined with (11) for the original $S_1, ..., S_k$, this implies that T_j contains s and $T_j \cap S_j \neq \varnothing$, and that (11) holds with equality for $i = 1$ and $T = T_j$. Now the collection of sets T satisfying (11) with equality is an intersecting family (as the left-hand side is supermodular and the right-hand side is submodular; cf. (9)). Hence the union $T_0 := T_1 \cup \cdots \cup T_k$ satisfies (11) with equality. But then

$$g_1(T_0) = h_{S_1,...,S_k}(T_0) + |T_0 \backslash (S_1 \cup \cdots \cup S_k)| \geqslant k + 1 \tag{14}$$

(as T_0 contains s and intersects all S_j). Assertion (14) contradicts the definition of k. Similarly, there exists a j_2 such that if we replace S_{j_2} by $S_{j_2} \cup \{s\}$, then (11) is still satisfied for $i = 2$. Now $j_1 \neq j_2$, since otherwise we could replace S_{j_1} by $S_{j_1} \cup s$, without violating (11) for $i = 1, 2$, contradicting (12). We may assume that $j_1 = 1$ and $j_2 = 2$. Now for $i = 1, 2$ and $T \in \mathscr{C}_i$ one has

$$g_i(T) \leqslant h_{S_3,...,S_k}(T) + |T \backslash (S_1 \cup \cdots \cup S_k \cup s)| + 2. \tag{15}$$

For $i=1$ this follows from the fact that we could augment S_1 with s:

$$\begin{aligned} g_1(T) &\leqslant h_{S_1\cup s, S_2,\ldots,S_k}(T) + |T\backslash(S_1 \cup s \cup S_2 \cup \cdots \cup S_k)| \\ &= h_{S_3,\ldots,S_k}(T) + |T\backslash(S_1 \cup \cdots \cup S_k \cup s)| + h_{S_1\cup s, S_2}(T) \\ &\leqslant h_{S_3,\ldots,S_k}(T) + |T\backslash(S_1 \cup \cdots \cup S_k \cup s)| + 2. \end{aligned} \tag{16}$$

Similarly, (15) is shown for $i=2$. Let $U_1,\ldots, U_m$ be the minimal sets T in $\mathscr{C}_1$ satisfying (15) for $i=1$ with equality (minimal with respect to inclusion). As the collection of sets T in $\mathscr{C}_1$ satisfying (15) with equality (for $i=1$) is an intersecting family (using (9)), the sets $U_1,\ldots, U_m$ are pairwise disjoint. Moreover, as equality in (15) implies equality throughout in (16), we know that $h_{S_1\cup s, S_2}(U_j) = 2$, and hence that $|U_j \cap (S_1 \cup S_2 \cup s)| \geqslant 2$ for $j=1,\ldots, m$.

Similarly, let $W_1,\ldots, W_n$ be the minimal sets in $\mathscr{C}_2$ which satisfy (15) with equality for $i=2$. Again, $W_1,\ldots, W_n$ are pairwise disjoint, and $|W_j \cap (S_1 \cup S_2 \cup s)| \geqslant 2$ for $j=1,\ldots, n$.

Now $S_1 \cup S_2 \cup s$ can be split into classes S_1' and S_2' such that both S_1' and S_2' intersect each of the sets $U_1,\ldots, U_m$, $W_1,\ldots, W_n$. To see this, choose pairs $e_1,\ldots, e_m$, $f_1,\ldots, f_n$ as subsets of $S_1 \cup S_2 \cup s$ such that $e_1 \subseteq U_1,\ldots, e_m \subseteq U_m, f_1 \subseteq W_1,\ldots, f_n \subseteq W_n$. Since $e_1,\ldots, e_m$ are pairwise disjoint, and since $f_1,\ldots, f_n$ are pairwise disjoint, it follows that the "edges" $e_1,\ldots, e_m, f_1,\ldots, f_n$ make up a bipartite graph, with vertex set $S_1 \cup S_2 \cup s$. Then any two-colouring of this bipartite graph gives a splitting into S_1' and S_2' as required.

We finally show that replacing S_1 and S_2 by S_1' and S_2' does not violate (11) for $i=1, 2$, which, however, contradicts the maximality of $|S_1 \cup \cdots \cup S_k|$.

So we have to prove

$$g_i(T) \leqslant h_{S_1', S_2', S_3,\ldots,S_k}(T) + |T\backslash(S_1' \cup S_2' \cup S_3 \cup \cdots \cup S_k)| \tag{17}$$

for $i=1, 2$ and $T \in \mathscr{C}_i$. First let $i=1$, and choose $T \in \mathscr{C}_1$. If T includes one of the U_j as a subset, then T intersects both S_1' and S_2' (as U_j intersects both of these sets). In this case, by (15),

$$\begin{aligned} g_1(T) &\leqslant h_{S_3,\ldots,S_k}(T) + |T\backslash(S_1 \cup \cdots \cup S_k \cup s)| + 2 \\ &= h_{S_1', S_2', S_3,\ldots,S_k}(T) + |T\backslash(S_1' \cup S_2' \cup S_3 \cup \cdots \cup S_k)|. \end{aligned} \tag{18}$$

If T includes none of the U_j, then inequality (15) for $i=1$ is strict (by definition of $U_1,\ldots, U_m$). So if T intersects $S_1' \cup S_2'$, then

$$\begin{aligned} g_1(T) &\leqslant h_{S_3,\ldots,S_k}(T) + |T\backslash(S_1 \cup \cdots \cup S_k \cup s)| + 1 \\ &\leqslant h_{S_1', S_2', S_3,\ldots,S_k}(T) + |T\backslash(S_1' \cup S_2' \cup S_3 \cup \cdots \cup S_k)|. \end{aligned} \tag{19}$$

If T does not intersect $S_1' \cup S_2'$, then

$$\begin{aligned} g_1(T) &\leqslant h_{S_1,\ldots,S_k}(T) + |T\backslash(S_1 \cup \cdots \cup S_k)| \\ &= h_{S_1',S_2',S_3,\ldots,S_k}(T) + |T\backslash(S_1' \cup S_2' \cup S_3 \cup \cdots \cup S_k)|. \end{aligned} \tag{20}$$

The inequality (17) for $i = 2$ is shown similarly. ∎

The theorems of König [12], Gupta [10], and De Werra [22] are essentially equivalent to the case of this Second Auxiliary Theorem where both $\mathscr{C}_1$ and $\mathscr{C}_2$ are partitions of S. Then the theorem is equivalent to: let $G = (V, E)$ be a bipartite graph, and let for each v in V, a_v be an integer with $a_v \leqslant \deg(v)$ (:= the degree of v). Then the edges of G can be coloured with $\max_{v \in V} a_v$ colours in such a way that each vertex of G is touched by at least a_v colours. By taking $a_v = \deg(v)$, König's edge-colouring theorem follows. By taking all a_v equal to the minimum degree of G, Gupta's theorem follows. By taking $a_v = \min\{k, \deg(v)\}$ (where k is a fixed integer), De Werra's result follows.

4. Third Auxiliary Theorem

Also the Third Auxiliary Theorem will be used in proving the implication (P3) ⇒ (P1). Again, a collection $\mathscr{C}$ of subsets of the finite set V is called an *intersecting family* if $T \cap U \in \mathscr{C}$ and $T \cup U \in \mathscr{C}$ whenever $T, U \in \mathscr{C}$ and $T \cap U \neq \varnothing$. Moreover, we use again the notation

$$h_{X_1,\ldots,X_n}(X) = \text{the number of } j = 1,\ldots, n \text{ with } X \cap X_j \neq \varnothing. \tag{21}$$

Recall that for fixed $X_1,\ldots, X_n$, this gives a submodular function, and that also the set-function $d_A^-(U)$ is submodular, for any arc set A (cf. Section 3).

The theorem below extends a result of Edmonds [2] (Edmonds' disjoint branching theorem). It is proved by generalizing a method of Lovász [14] and Frank [6]. Frank showed that each intersecting family has property (P1), which is equivalent to the case $R_1 = \cdots = R_k = \varnothing$ in the following theorem.

Third Auxiliary Theorem. *Let $\mathscr{C}$ be an intersecting family of subsets of V, let $D = (V, A)$ be a directed graph, and let $R_1,\ldots, R_k$ be subsets of V. Suppose that*

$$d_A^-(T) + h_{R_1,\ldots,R_k}(T) \geqslant k \tag{22}$$

for each T in $\mathscr{C}$. Then A can be split into classes $A_1,\ldots, A_k$ such that

$$d_{A_j}^-(T) \geqslant 1 \qquad \text{or} \qquad T \cap R_j \neq \varnothing \tag{23}$$

for each $j = 1,\ldots, k$ and each T in $\mathscr{C}$.

(Here $d_A^-(T)$ denotes the number of arcs in A entering T.)

Proof. The theorem is proved by induction on $\sum_{j=1}^{k} |V\backslash R_j|$. If each R_j intersects each $T \in \mathscr{C}$, the theorem is trivial. So we may assume that R_1 does not intersect some set in $\mathscr{C}$. Let W be a maximal set in $\mathscr{C}$ not intersecting R_1 (maximal with respect to inclusion). Note that (22) implies that for each T in $\mathscr{C}$,

$$d_A^-(T) + h_{R_2,\ldots,R_k}(T) \geqslant k-1. \tag{24}$$

Consider the collection $\mathscr{F}$ of all sets T in $\mathscr{C}$ which have equality in (24). As the left-hand side of (24) is submodular, the collection $\mathscr{F}$ is an intersecting family. Moreover, as (22) holds, each T in $\mathscr{F}$ intersects R_1.

Now select an arc $a = (u, v)$ as follows. If each set in $\mathscr{F}$ is disjoint from W, let a be an arbitrary arc entering W (which exists by (22), as $W \cap R_1 = \varnothing$). If there are sets in $\mathscr{F}$ which intersect W, let U be a minimal set in $\mathscr{F}$ intersecting W (minimal with respect to inclusion). Since

$$\begin{aligned} &d_A^-(U \cap W) + h_{R_1,\ldots,R_k}(U \cap W) \\ &\geqslant k > d_A^-(U) + h_{R_2,\ldots,R_k}(U) \geqslant d_A^-(U) + h_{R_1,\ldots,R_k}(U \cap W), \end{aligned} \tag{25}$$

we know that $d_A^-(U \cap W) > d_A^-(U)$, and so we can choose an arc $a = (u, v)$ entering $U \cap W$ but not entering U, i.e., $u \in U\backslash W$, $v \in U \cap W$.

Now replace R_1 by $R_1 \cup v$, and A by $A\backslash a$. Then

$$d_{A\backslash a}^-(T) + h_{R_1\cup v, R_2,\ldots,R_k}(T) \geqslant k \tag{26}$$

for all T in $\mathscr{C}$. For suppose (26) does not hold. Then

$$d_A^-(T) + h_{R_1,\ldots,R_k}(T) \geqslant k > d_{A\backslash a}^-(T) + h_{R_1\cup v, R_2,\ldots,R_k}(T). \tag{27}$$

Hence a enters T, $v \in T$, and $d_A^-(T) + h_{R_2,\ldots,R_k}(T) = k-1$. So T is in $\mathscr{F}$. In particular, there is a set in $\mathscr{F}$ intersecting W. As $v \in T \cap U$, and T and U are in $\mathscr{F}$, also $T \cap U$ is in $\mathscr{F}$. However, $u \in U\backslash T$, contradicting the minimality of U.

So (26) holds for all T in $\mathscr{C}$. By induction we know that $A\backslash a$ can be split into classes $A_1,\ldots, A_k$ such that

$$d_{A_1}^-(T) + h_{R_1\cup v}(T) \geqslant 1, \qquad d_{A_j}^-(T) + h_{R_j}(T) \geqslant 1 \qquad \text{for} \quad j = 2,\ldots, k, \tag{28}$$

for all T in $\mathscr{C}$. Hence

$$d_{A_1\cup a}^-(T) + h_{R_1}(T) \geqslant 1 \tag{29}$$

for all T in $\mathscr{C}$. Indeed, suppose (29) does not hold. Then $h_{R_1}(T) = d_{A_1\cup a}^-(T) = 0$. This implies $d_{A_1}^-(T) = 0$, and hence by (28) $h_{R_1\cup v}(T) = 1$, i.e., $v \in T$ and $T \cap R_1 = \varnothing$. But then $T \subseteq W$ (otherwise $T \cup W$ would be a larger set in $\mathscr{C}$ disjoint from R_1, contradicting our choice of W). However, this implies that a enters T, and (29) follows.

Combining (28) and (29) gives that $A_1 \cup a, A_2, ..., A_k$ is a splitting of A as required. ∎

5. Main Theorem

Main Theorem. *Let $\mathscr{C}$ be a crossing family of subsets of the finite set V, not containing $\varnothing$ or V. Then conditions* (P1), (P2), *and* (P3) *are equivalent.*

Proof. (I) (P1) $\Rightarrow$ (P3) and (P2) $\Rightarrow$ (P3). Suppose $\mathscr{C}$ contains five sets V_1, V_2, V_3, V_4, V_5 as described in (P3). Let $\mathscr{C}_0 := \{V_1, V_2, V_3, V_4, V_5\}$, and let $\mathscr{C}_1 := \mathscr{C} \backslash \mathscr{C}_0$. Choose elements $v_1 \in V_1$, $v_2 \in V \backslash V_2$, $v_4 \in V_4$, $v_5 \in V \backslash V_5$. Let $D = (V, A)$ be a directed graph, with $A = A_0 \cup A_1$, where

$$A_0 := \{(v_2, v_1), (v_4, v_1), (v_2, v_4), (v_5, v_4), (v_5, v_2)\}, \tag{30}$$

$$A_1 := \{(u, v) \mid u, v \in V \text{ such that } (u, v) \text{ does not enter any } V_i \ (i = 1, ..., 5)\}.$$

The sets $V_1, ..., V_5$ and the arcs in A_0 are given in Fig. 3. Now observe the following:

every set in $\mathscr{C}_0$ is entered by exactly two arcs from A_0, and every arc from A_0 enters exactly two sets in $\mathscr{C}_0$. (31)

This is an easy checking. Moreover:

every set in $\mathscr{C}_1$ is either entered by at least one arc in A_1, or by at least two arcs in A_0. (32)

This can be seen as follows. It follows from the definition (30) of A_1 that a subset U of V is not entered by any arc in A_1 if and only if U belongs to the

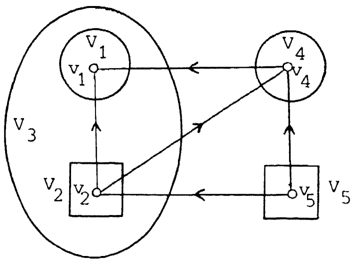

Figure 3

lattice generated by $\mathscr{C}_0$ (with respect to $\subseteq$, $\cap$, and $\cup$). This lattice consists of the sets

$$\begin{gathered}\varnothing, V, V_1, V_2, V_3, V_4, V_5, V_1\cup V_4, V_2\cap V_3,\\ (V_2\cap V_3)\cup V_4, V_3\cup V_4, V_2\cap V_5,\end{gathered} \tag{33}$$

as (33) is closed under taking intersections and unions, and as each set in (33) is generated by $\mathscr{C}_0$. Since each of the sets in (33), except for $\varnothing$ and V, is entered by at least two arcs from A_0, (32) follows.

From (31) and (32) it directly follows that:

> any covering for $\mathscr{C}$ contains at least three arcs from A_0, and any cut induced by $\mathscr{C}$ contains at least one arc from A_1 or at least two arcs from A_0. (34)

Now define a length function l on A by

$$l(a)=1 \quad \text{if } a\in A_0, \qquad l(a)=0 \quad \text{if } a\in A_1. \tag{35}$$

Then by (34) the minimum length of a covering for $\mathscr{C}$ is at least three. However, if U_1, U_2, U_3 are sets in $\mathscr{C}$ such that any arc a enters at most $l(a)$ of U_1, U_2, U_3, then by (35) each U_i is entered by no arc from A_1, and hence, by (32), by at least two arcs from A_0. This is not possible, as $|A_0|=5$.

So negating (P3) implies negating (P2). Similarly, negating (P3) implies negating (P1). To see this, define a capacity function c on A by

$$c(a)=1 \quad \text{if } a\in A_0, \qquad c(a)=2 \quad \text{if } a\in A_1 \tag{36}$$

(or replace any arc in A_1 by two parallel arcs, if one wishes to stick to the cardinality formulation). Then by (34) the minimum capacity of a cut induced by $\mathscr{C}$ is at least 2. However, if C_1 and C_2 are coverings for $\mathscr{C}$ such that no arc a is in more than $c(a)$ of C_1 and C_2, then by (34) both C_1 and C_2 intersect A_0 in at least 3 arcs, contradicting (36).

(II) (P3)$\Rightarrow$(P2). To show the implication (P3)$\Rightarrow$(P2) we use the theory of total dual integrality. Let $\mathscr{C}$ be a crossing family on V satisfying (P3), not containing $\varnothing$ or V. Let $D=(V,A)$ be a directed graph, and let $l\colon A\to\mathbb{Z}_+$. Then condition (P2) can be equivalently formulated as: both optima in the linear programming duality equation

$$\begin{array}{llll}
\min \sum_{a\in A} l(a)\,x(a) & & = \max \sum_{T\in\mathscr{C}} y(T) & \\
\text{subject to} & & \text{subject to} & \\
\sum_{a\in\delta_A^-(T)} x(a)\geqslant 1 & (T\in\mathscr{C}), & \sum_{T\in\mathscr{C},\, a\in\delta_A^-(T)} y(T)\leqslant l(a) & (a\in A),\\
x(a)\geqslant 0 & (a\in A), & y(T)\geqslant 0 & (T\in\mathscr{C}),
\end{array} \tag{37}$$

have integer optimal solutions. By a theorem of Edmonds and Giles [3], to show that both optima in (37) have integer optimal solutions, it suffices to show that the maximization problem in (37) has an integer optimal solution y, for each length function $l\colon A \to \mathbb{Z}_+$.

To show that the maximum in (37) has an integer optimal solution, let y be a, not necessarily integer, optimal solution for the maximum in (37), such that

$$\sum_{T\in\mathscr{C}} y(T)\cdot|T|\cdot|V\backslash T| \tag{38}$$

is as small as possible (such a y exists by simple compactness and continuity arguments). Now consider the collection

$$\mathscr{F} := \{T\in\mathscr{C} \mid y(T)>0\}. \tag{39}$$

We show that $\mathscr{F}$ is cross-free (cf. Section 2).

Indeed, suppose to the contrary there are T, U in $\mathscr{C}$ with $y(T)>0$, $y(U)>0$, and $T\cap U\neq\varnothing$, $T\cup U\neq V$, $T\not\subseteq U$, $U\not\subseteq T$. Let $\varepsilon=\min\{y(T), y(U)\}>0$. Reset y as follows:

$$\begin{aligned} y(T) &:= y(T)-\varepsilon, \qquad & y(T\cap U) &:= y(T\cap U)+\varepsilon,\\ y(U) &:= y(U)-\varepsilon, \qquad & y(T\cup U) &:= y(T\cup U)+\varepsilon, \end{aligned} \tag{40}$$

letting y unchanged in the remaining components. One easily checks that the new y again is a feasible solution for the maximum in (37), with the same objective value as the old y. So y is again an optimal solution. However, the sum (38) is decreased, contradicting our choice of y.

So $\mathscr{F}$ is cross-free. Therefore by the First Auxiliary Theorem (Section 2), the (primal) constraints in (37) with positive dual variable $y(T)$ form a totally unimodular matrix (as it comes from the matrix $M_{\mathscr{F}}$ by duplicating and deleting columns). Hence, by Hoffman and Kruskal's theorem [11], the maximum in (37) has an integer optimal solution.

(III) (P3) $\Rightarrow$ (P1). The remainder of this paper is devoted to proving the implication (P3) $\Rightarrow$ (P1). Let $\mathscr{C}$ be a crossing family of subsets of V satisfying (P3), not containing $\varnothing$ or V. Let $D=(V,A)$ be a directed graph. Suppose the minimum size of a cut induced by $\mathscr{C}$ is more than the maximum number of pairwise disjoint coverings for $\mathscr{C}$, and suppose furthermore that we have chosen this counterexample so that $|\mathscr{C}|+|A|$ is as small as possible.

Let k be the minimum size of a cut induced by $\mathscr{C}$, i.e.,

$$k := \min\{d_A^-(U) \mid U\in\mathscr{C}\}. \tag{41}$$

Moreover, define:

$$\mathscr{C}^{\min} := \text{the collection of minimal sets in } \mathscr{C},$$
$$\mathscr{C}^{\max} := \text{the collection of maximal sets in } \mathscr{C} \tag{42}$$

(minimal and maximal with respect to inclusion), and

$$\mathscr{S} := \{U \in \mathscr{C} \mid \text{there are no } S, T \in \mathscr{C} \text{ such that } S \subseteq U,\ U \cup T = V,\ S \subseteq T\},$$
$$\mathscr{L} := \{U \in \mathscr{C} \mid \text{there are no } S, T \in \mathscr{C} \text{ such that } U \subseteq T,\ U \cap S = \varnothing,\ S \subseteq T\}. \tag{43}$$

Since (P3) holds, we know that $\mathscr{C} = \mathscr{S} \cup \mathscr{L}$.

CLAIM 1. *$\mathscr{S}$ and $\mathscr{L}$ are crossing families.*

Proof of Claim 1. To show that $\mathscr{S}$ is a crossing family, let $U, W \in \mathscr{S}$ cross (i.e., $U \cap W \neq \varnothing$, $U \cup W \neq V$, $U \nsubseteq W$, $W \nsubseteq U$). Then $U \cap W$ and $U \cup W$ belong to $\mathscr{C}$. It is immediate from the definition (43) of $\mathscr{S}$ that $U \cap W$ belongs to $\mathscr{S}$. To show that $U \cup W$ belongs to $\mathscr{S}$, suppose to the contrary that there exist S, T in $\mathscr{C}$ such that $S \subseteq U \cup W$, $T \cup U \cup W = V$ and $S \subseteq T$. Without loss of generality, $S \in \mathscr{C}^{\min}$ and $T \in \mathscr{C}^{\max}$. As T is a maximal set, T crosses neither U nor W (as otherwise $T \cup U$ or $T \cup W$ would be a larger set), which implies:

$$\begin{array}{lll} \text{either} & \text{(i)} & T \cup (U \backslash W) = V, \\ \text{or} & \text{(ii)} & T \cup (U \cap W) = V, \\ \text{or} & \text{(iii)} & T \cup (W \backslash U) = V. \end{array} \tag{44}$$

If (i) holds, then $U \cap W$ and T contradict the fact that U belongs to $\mathscr{S}$. If (iii) holds, then $U \cap W$ and T contradict the fact that W belongs to $\mathscr{S}$. So (ii) holds, i.e., $T \cup (W \cap U) = V$. Since S is a minimal set, it crosses neither U nor W (as otherwise $S \cap U$ or $S \cap W$ would be a smaller set). Hence $S \subseteq U$ or $S \subseteq W$. However, $T \cup U = V$ and $T \cup W = V$, contradicting the fact that both U and W are in $\mathscr{S}$.

Similarly, $\mathscr{L}$ is a crossing family. ∎

CLAIM 2. *If $W \in \mathscr{C}$ and $W \notin \mathscr{C}^{\min} \cup \mathscr{C}^{\max}$, then $d_A^-(W) \geqslant k + 1$.*

Proof of Claim 2.

Case 1. $W \in \mathscr{S}$. Suppose $d_A^-(W) = k$. Let:

$$\begin{aligned} \mathscr{C}' &:= \{U \in \mathscr{C} \mid U \subseteq W\}, \\ \mathscr{C}'' &:= \{U \in \mathscr{C} \mid U \cap W = \varnothing, \text{ or } W \subseteq U, \text{ or } U \cup W = V\}, \\ A' &:= \{a \in A \mid a \text{ enters a set } U \text{ in } \mathscr{C}'\}, \\ A'' &:= \{a \in A \mid a \text{ enters a set } U \text{ in } \mathscr{C}''\}. \end{aligned} \tag{45}$$

So $\mathscr{C}' \cup \mathscr{C}''$ consists of all sets U in $\mathscr{C}$ which do not cross W.

We first show that $A' \cap A'' = \delta_A^-(W)$. As $W \in \mathscr{C}' \cap \mathscr{C}''$, we know that $\delta_A^-(W) \subseteq A' \cap A''$. To see the opposite inclusion, suppose arc a enters both U' in $\mathscr{C}'$ and U'' in $\mathscr{C}''$. Then $U' \cap U'' \neq \varnothing$, and hence $U'' \cap W \neq \varnothing$. So $W \subseteq U''$ or $U'' \cup W = V$. If $U'' \cup W = V$, then $U' \cap U''$ and $U' \cup U''$ contradict the fact that W belongs to $\mathscr{S}$. So $W \subseteq U''$, and hence $U' \subseteq W \subseteq U''$, which implies that a is in $\delta_A^-(W)$.

By definition of k and A' and A'', we know

$$d_{A'}^-(U) \geqslant k \text{ for all } U \text{ in } \mathscr{C}', \text{ and } d_{A''}^-(U) \geqslant k \text{ for all } U \text{ in } \mathscr{C}''. \tag{46}$$

As W is not a minimal or a maximal set in $\mathscr{C}$ we also know

$$|\mathscr{C}'| + |A'| < |\mathscr{C}| + |A| \qquad \text{and} \qquad |\mathscr{C}''| + |A''| < |\mathscr{C}| + |A|. \tag{47}$$

Moreover, $\mathscr{C}'$ and $\mathscr{C}''$ are again crossing families satisfying (P3). As $\mathscr{C}, A$ form a smallest counterexample, we know from (46) and (47) that A' can be split into classes $A_1', \ldots, A_k'$, and A'' can be split into classes $A_1'', \ldots, A_k''$, such that

$$\begin{aligned} d_{A_j'}^-(U) &\geqslant 1 \qquad \text{for } j = 1, \ldots, k \text{ and } U \in \mathscr{C}', \\ d_{A_j''}^-(U) &\geqslant 1 \qquad \text{for } j = 1, \ldots, k \text{ and } U \in \mathscr{C}''. \end{aligned} \tag{48}$$

As the arcs in $\delta_A^-(W) = A' \cap A''$ will be, in each of these partitions, in different classes (as $d_A^-(W) = k$ and $W \in \mathscr{C}' \cap \mathscr{C}''$), we may assume that, for $j = 1, \ldots, k$, A_j' and A_j'' intersect in an arc of $\delta_A^-(W)$. Hence $A_1' \cup A_1'', \ldots, A_k' \cup A_k''$ partition $A' \cup A''$.

Moreover,

$$d_{A_j' \cup A_j''}^-(U) \geqslant 1 \qquad \text{for } j = 1, \ldots, k \text{ and } U \in \mathscr{C}. \tag{49}$$

Indeed, if U does not cross W, then $U \in \mathscr{C}' \cup \mathscr{C}''$, and (49) follows from

(48). If U crosses W, then $U \cap W$ belongs to $\mathscr{C}'$ and $U \cup W$ belongs to $\mathscr{C}''$. Then we have

$$\begin{aligned} d^-_{A'_j \cup A''_j}(U) &\geqslant d^-_{A'_j \cup A''_j}(U \cap W) + d^-_{A'_j \cup A''_j}(U \cup W) - d^-_{A'_j \cup A''_j}(W) \\ &\geqslant d^-_{A'_j}(U \cap W) + d^-_{A''_j}(U \cup W) - 1 \geqslant 1 + 1 - 1 = 1 \end{aligned} \tag{50}$$

(using (48), the submodularity of $d^-_{A'_j \cup A''_j}$ and the fact that $d^-_{A'_j \cup A''_j}(W) = 1$).

So A contains pairwise disjoint sets $A_1, ..., A_k$ such that $d^-_{A_j}(U) \geqslant 1$ for $j = 1, ..., k$ and $U \in \mathscr{C}$. This contradicts the fact that $\mathscr{C}, A$ form a counterexample to (P1).

Case 2. $W \in \mathscr{L}$. This case follows similarly to Case 1, now using

$$\begin{aligned} \mathscr{C}' &:= \{U \in \mathscr{C} \mid U \subseteq W, \text{ or } U \cap W = \varnothing, \text{ or } U \cup W = V\}, \\ \mathscr{C}'' &:= \{U \in \mathscr{C} \mid W \subseteq U\}, \\ A' &:= \{a \in A \mid a \text{ enters a set } U \text{ in } \mathscr{C}'\}, \\ A'' &:= \{a \in A \mid a \text{ enters a set } U \text{ in } \mathscr{C}''\}. \quad \blacksquare \end{aligned} \tag{51}$$

Claim 2 immediately implies

CLAIM 3. *Each arc of D enters a minimal or a maximal set of C.*

Proof of Claim 3. Otherwise we could delete this arc, without violating the condition $d^-_A(U) \geqslant k$ for all U in $\mathscr{C}$ (by Claim 2), thus obtaining a smaller counterexample. $\blacksquare$

Define

$$\begin{aligned} A^\circ &:= \{a \in A \mid a \text{ enters both a set in } \mathscr{C}^{\min} \text{ and a set in } \mathscr{C}^{\max}\}, \\ A' &:= \{a \in A \mid a \text{ enters no set in } \mathscr{C}^{\max}\}, \\ A'' &:= \{a \in A \mid a \text{ enters no set in } \mathscr{C}^{\min}\}. \end{aligned} \tag{52}$$

Claim 3 is equivalent to

$$A^\circ, A', A'' \text{ partition } A. \tag{53}$$

Note that any arc of D enters at most one set in $\mathscr{C}^{\min}$, and at most one set in $\mathscr{C}^{\max}$ (since if it enters both T and U, it also enters $T \cap U \in \mathscr{C}$ and $T \cup U \in \mathscr{C}$).

Our next claim is

CLAIM 4. *Let $a' = (u', v') \in A'$ and $a'' = (u'', v'') \in A''$, such that a' enters a set $S \in \mathscr{C}$ and a'' enters a set $T \in \mathscr{C}$, with $S \subseteq T$. Then there exists a set $U \in \mathscr{C}$ such that $u', v' \in U$ and $u'', v'' \notin U$.*

Proof of Claim 4. Let a', a'', S, T be as above, and suppose such a set U does not exist. Let $a := (u'', v')$ be a new arc. Then

$$d^-_{(A\setminus\{a',a''\})\cup\{a\}}(U) = d^-_A(U) - d^-_{a',a''}(U) + d^-_a(U) \geqslant k \tag{54}$$

for all U in $\mathscr{C}$. Indeed, if $U \notin \mathscr{C}^{\min} \cup \mathscr{C}^{\max}$, then (54) follows from the facts that $d^-_A(U) \geqslant k+1$ (by Claim 2) and $d^-_a(U) + 1 \geqslant d^-_{a',a''}(U)$ (trivially). If $U \in \mathscr{C}^{\min} \cup \mathscr{C}^{\max}$, then $d^-_{a',a''}(U) \leqslant 1$, as a' is in A' and a'' is in A''. Negating (54) will give that $d^-_{a',a''}(U) = 1$ and $d^-_a(U) = 0$. However, if U is a minimal set, then, as $a'' \in A''$, a' enters U. Hence $U \subseteq S$ (as otherwise $U \supset S \cap U \in \mathscr{C}$, contradicting the minimality of U). Therefore, $U \subseteq T$, and hence $u'' \notin U$ and $v' \in U$, contradicting that $d^-_a(U) = 0$. Similarly, if U is a maximal set, then, as $a' \in A'$, a'' enters U. Hence $T \subseteq U$. Therefore $S \subseteq U$, and again $u'' \notin U$ and $v' \in U$, contradicting that $d^-_a(U) = 0$.

Therefore, as $|\mathscr{C}| + |(A\setminus\{a', a''\}) \cup \{a\}| < |\mathscr{C}| + |A|$, we know that $(A\setminus\{a', a''\}) \cup \{a\}$ can be split into classes $A_1, \ldots, A_k$ such that

$$d^-_{A_j}(U) \geqslant 1 \qquad \text{for } j = 1, \ldots, k \text{ and } U \in \mathscr{C}. \tag{55}$$

Without loss of generality, a belongs to A_1. Then $(A_1\setminus\{a\}) \cup \{a', a''\}$, $A_2, \ldots, A_k$ is a splitting of A as required, which would contradict our assumption that $\mathscr{C}, A$ form a counterexample. To see that this is indeed a splitting of A as required, it suffices to show that

$$d^-_{(A_1\setminus\{a\})\cup\{a',a''\}}(U) \geqslant 1 \qquad \text{for} \quad U \in \mathscr{C}. \tag{56}$$

Suppose the left-hand side here is 0. As $d^-_{A_1}(U) \geqslant 1$, it follows that the arc a enters U, but neither a' nor a'' enters U. This however contradicts our assumption that there is no U in $\mathscr{C}$ with $u', v' \in U$ and $u'', v'' \notin U$. ∎

Define

$$\begin{aligned} \mathscr{S}' &:= \{U \in \mathscr{S} \mid \text{no arc in } A'' \text{ enters } U\}, \\ \mathscr{L}' &:= \{U \in \mathscr{L} \mid \text{no arc in } A' \text{ enters } U\}. \end{aligned} \tag{57}$$

It follows directly from this definition and from Claim 1 that

$$\mathscr{S}' \text{ and } \mathscr{L}' \text{ are crossing families.} \tag{58}$$

Next we make

Claim 5. (i) *For all T in $\mathscr{S}$ there exists a T' in $\mathscr{S}'$ such that $T' \subseteq T$ and $\delta^-_A(T') \subseteq \delta^-_A(T)$.*

(ii) *For all T in $\mathscr{L}$ there exists a T' in $\mathscr{L}'$ such that $T \subseteq T'$ and $\delta^-_A(T') \subseteq \delta^-_A(T)$.*

Proof of Claim 5. We first prove (i), by induction on $|T|$. If $d^-_{A''}(T)=0$, we know that $T\in\mathcal{S}'$, and (i) is trivial. Suppose $d^-_{A''}(T)\neq\varnothing$, and take $a''\in A''$ entering T. Let S be a set with

$S\in\mathscr{C}$, $S\subseteq T$, a'' does not enter S, and $|S|$ is as large as possible. (59)

Such a set S exists, as T includes a minimal set of $\mathscr{C}$, which is not entered by a'' (as $a''\in A''$). As $S\subseteq T$ and $T\in\mathcal{S}$, we know $S\in\mathcal{S}$.

Now $\delta^-_A(S)\subseteq\delta^-_A(T)$. For suppose to the contrary that arc a' enters S but not T. Then $a'\in A'$. [Otherwise there is a maximal set W of $\mathscr{C}$ entered by a'. Then $S\subseteq W$ (as otherwise $W\subset S\cup W\in\mathscr{C}$, contradicting the maximality of W). As T belongs to $\mathcal{S}$, we know that $T\cup W\neq V$. Moreover, $T\cap W\neq\varnothing$ (as the head of arc a' belongs to T and W). Hence $T\cap W$ and $T\cup W$ belong to $\mathscr{C}$. As W is maximal, it follows that $T\subseteq W$. So $S\subseteq T\subseteq W$, which contradicts the fact that a' enters both S and W but not T.] Now the premise of Claim 4 is satisfied. Therefore, there exists a set U in $\mathscr{C}$ such that a' is contained in U and a'' is disjoint from U. But now $(S\cup U)\cap T$ belongs to $\mathscr{C}$ (as $S\cap U\neq\varnothing$, $S\cup U\neq V$, and $(S\cap U)\cap T\neq\varnothing$, $(S\cup U)\cup T\neq V$), and is not entered by a''. However, $|(S\cup U)\cap T|>|S|$ (as the tail of a' belongs to $(S\cup U)\cap T$ but not to S), contradicting that we have chosen $|S|$ as large as possible.

Since $|S|<|T|$ (as a'' does not enter S), we know by induction that there exists a set T' in $\mathcal{S}'$ with $T'\subseteq S\subseteq T$ and $\delta^-_A(T')\subseteq\delta^-_A(S)\subseteq\delta^-_A(T)$. This proves (i).

Assertion (ii) of Claim 5 is shown similarly. ∎

In order to apply the Second Auxiliary Theorem, we next introduce two intersecting families and two supermodular functions:

$$\begin{aligned}
\mathscr{A}_1 &:= \{\delta^-_{A^\circ}(U)\mid U\in\mathcal{S}'\},\\
\mathscr{A}_2 &:= \{\delta^-_{A^\circ}(U)\mid U\in\mathscr{L}'\},\\
g_1(B) &:= \max\{k-d^-_{A'}(U)\mid U\in\mathcal{S}',\ \delta^-_{A^\circ}(U)=B\} \qquad \text{for } B\in\mathscr{A}_1,\\
g_2(B) &:= \max\{k-d^-_{A''}(U)\mid U\in\mathscr{L}',\ \delta^-_{A^\circ}(U)=B\} \qquad \text{for } B\in\mathscr{A}_2.
\end{aligned} \tag{60}$$

CLAIM 6. *$\mathscr{A}_1$ and $\mathscr{A}_2$ are intersecting families on A°, and g_1 and g_2 are supermodular on intersecting pairs.*

Proof of Claim 6. To show that $\mathscr{A}_1$ is intersecting, suppose $\delta^-_{A^\circ}(U)\cap\delta^-_{A^\circ}(W)\neq\varnothing$ for $U,W\in\mathcal{S}'$. This means that there is an arc in A° entering both U and W. Hence $U\cap W\neq\varnothing$ and $U\cup W\neq V$, and therefore $U\cap W$ and $U\cup W$ belong to $\mathcal{S}'$ (cf. (58)). We next show

$$\delta^-_{A^\circ}(U)\cap\delta^-_{A^\circ}(W)=\delta^-_{A^\circ}(U\cap W),\qquad \delta^-_{A^\circ}(U)\cup\delta^-_{A^\circ}(W)=\delta^-_{A^\circ}(U\cup W). \tag{61}$$

Showing $\subseteq$ in the first equality is easy: if an arc enters both U and W it enters $U \cap W$. Similarly, showing $\supseteq$ in the second equality is easy: if an arc enter $U \cup W$ it enters at least one of U and W. The reverse inclusions directly follow from the fact that no arc in A° has its tail in $U \backslash W$ and its head in W, or its tail in $W \backslash U$ and its head in U. For suppose that $a \in A^\circ$ has its tail in, say, $U \backslash W$ and its head in W. Let T be the set in $\mathscr{C}^{\max}$ entered by a. Then T does not cross U nor W (as otherwise $T \cup U$ or $T \cup W$ would be larger than T). Hence $T \cup (U \backslash W) = V$. But then $U \cap W \subseteq T$, which contradicts the fact that U belongs to $\mathscr{S}$. This shows (61). The supermodularity of g_1 follows from the submodularity of the function $d^-_{A'}$. It similarly follows that $\mathscr{A}_2$ is an intersecting family, and that g_2 is supermodular on intersecting pairs. ∎

Now we use the Second Auxiliary Theorem. Note that, if $B \in \mathscr{A}_1$, and $U \in \mathscr{S}'$ attains the maximum for $g_1(B)$ in (60), then

$$g_1(B) = k - d^-_{A'}(U) \leqslant d^-_{A^\circ}(U) = |B|, \tag{62}$$

using the fact that no arc in A'' enters U, and that hence $d^-_{A^\circ}(U) + d^-_{A'}(U) = d^-_A(U) \geqslant k$. Similarly, for $B \in \mathscr{A}_2$,

$$g_2(B) \leqslant |B|. \tag{63}$$

Since moreover the values of g_1 and g_2 do not exceed k, by the Second Auxiliary Theorem (Section 3), there exists a partition of A° into classes $A^\circ_1, ..., A^\circ_k$ such that

for each $B \in \mathscr{A}_1$, the number of $j = 1, ..., k$ with $B \cap A^\circ_j \neq \emptyset$ is at least $g_1(B)$;

for each $B \in \mathscr{A}_2$, the number of $j = 1, ..., k$ with $B \cap A^\circ_j \neq \emptyset$ is at least $g_2(B)$. (64)

By definition (60) of $\mathscr{A}_1$, $\mathscr{A}_2$, g_1, g_2 this is equivalent to

for each $U \in \mathscr{S}'$: the number of $j = 1, ..., k$ with $d^-_{A^\circ_j}(U) \geqslant 1$ is at least $k - d^-_{A'}(U)$;

for each $U \in \mathscr{L}'$: the number of $j = 1, ..., k$ with $d^-_{A^\circ_j}(U) \geqslant 1$ is at least $k - d^-_{A''}(U)$. (65)

CLAIM 7. (i) *The arc set A' can be split into classes $A'_1, ..., A'_k$ such that $d_{A^\circ_j \cup A'_j}(U) \geqslant 1$, for $j = 1, ..., k$ and $U \in \mathscr{S}'$.*

(ii) *The arc set A'' can be split into classes $A''_1, ..., A''_k$ such that $d_{A^\circ_j \cup A''_j}(U) \geqslant 1$, for $j = 1, ..., k$ and $U \in \mathscr{L}'$.*

Proof of Claim 7. To prove (i), let $U_1, \ldots, U_t$ be the maximal sets of $\mathscr{S}'$ (maximal with respect to inclusion—they are not necessarily in $\mathscr{C}^{\max}$). For $i = 1, \ldots, t$, let

$$\mathscr{S}_i := \{U \in \mathscr{S}' \mid U \subseteq U_i\}, \quad B_i := \{a \in A' \mid a \text{ enters some set in } \mathscr{S}_i\}. \tag{66}$$

Then the collections $\mathscr{S}_i$ are pairwise disjoint, and the sets B_i are pairwise disjoint. For suppose, say, $U \in \mathscr{S}_1 \cap \mathscr{S}_2$. Then $U \subseteq U_1 \cap U_2$, and hence $U_1 \cap U_2 \neq \varnothing$. Therefore $U_1 \cup U_2 = V$ (as otherwise U_1 and U_2 would cross, which implies that $U_1 \cup U_2$ is in $\mathscr{S}'$, contradicting the maximality of U_1 and U_2). But now $U \subseteq U_1$, $U_1 \cup U_2 = V$, $U \subseteq U_2$, contradicting the fact that U_1 belongs to $\mathscr{S}$.

Suppose, say, $a \in B_1 \cap B_2$. Then there are sets $T_1 \in \mathscr{S}_1$ and $T_2 \in \mathscr{S}_2$ entered by a. Hence T_1 and T_2 cross, and hence $T_1 \cap T_2 \in \mathscr{S}_1 \cap \mathscr{S}_2$, contradicting that $\mathscr{S}_1$ and $\mathscr{S}_2$ are disjoint.

Moreover, each collection $\mathscr{S}_i$ is an intersecting family. Indeed, if two members of $\mathscr{S}_i$ intersect, then they cross, as both are contained in $U_i \neq V$.

Let, for $j = 1, \ldots, k$,

$$R_{ij} := \{v \in V \mid v \text{ is the head of some arc in } A_j^\circ \text{ entering some set } U \text{ in } \mathscr{S}_i\}. \tag{67}$$

Now for fixed $i = 1, \ldots, t$, (65) implies

$$d_{B_i}^-(U) + h_{R_{i1}, \ldots, R_{ik}}(U) \geqslant k, \tag{68}$$

for each $U \in \mathscr{S}_i$. Hence, by the Third Auxiliary Theorem (Section 4), B_i can be split into classes $B_{i1}, \ldots, B_{ik}$ so that

$$d_{B_{ij}}(U) + h_{R_{ij}}(U) \geqslant 1 \qquad \text{for} \quad j = 1, \ldots, k \quad \text{and} \quad U \in \mathscr{S}_i. \tag{69}$$

Now

$$d_{A_j^\circ}(U) \geqslant h_{R_{ij}}(U), \qquad \text{for} \quad j = 1, \ldots, k \text{ and } U \in \mathscr{S}_i. \tag{70}$$

Indeed, if the right-hand side in (70) is 0 it is trivial. If the right-hand side is 1, then $U \cap R_{ij} \neq \varnothing$. Hence there is an arc a in A_j° entering some set S in $\mathscr{S}_i$ and whose head is in U (by Definition (67) of R_{ij}). We may assume $S \subseteq U$ (since otherwise we could replace S by $S \cap U$, as $\mathscr{S}_i$ is an intersecting family), and even that S is in $\mathscr{C}^{\min}$ (as a is in A°). We prove that a enters U (implying that the left-hand side in (70) is at least 1). For suppose to the contrary that the tail of a also belongs to U. As a belongs to A°, there exists $T \in \mathscr{C}^{\max}$ such that a enters T. Then $S \subseteq U$, $T \cup U = V$, $S \subseteq T$ (as follows directly from the minimality of S and the maximality of T). But this contradicts the fact that U belongs to $\mathscr{S}$.

Combining (69) and (70) gives

$$d_{B_{1j}\cup\cdots\cup B_{tj}\cup A_j^\circ}(U)\geqslant 1 \qquad \text{for} \quad j=1,\ldots,k \quad \text{and} \quad U\in\mathscr{S}'. \tag{71}$$

This shows (i). One similarly shows (ii). ∎

Using the splittings of Claim 7, we show

$$d_{A_j^\circ\cup A_j'\cup A_j''}(U)\geqslant 1 \qquad \text{for} \quad j=1,\ldots,k \quad \text{and} \quad U\in\mathscr{C}. \tag{72}$$

Since A°, A', A'' partition A, since $A_1^\circ,\ldots,A_k^\circ$ partition A°, since $A_1',\ldots,A_k'$ partition A', and since $A_1'',\ldots,A_k''$ partition A'', this will show that A can be split as required by (P1), contradicting our assumption that $\mathscr{C}, A$ form a counterexample, and thus proving the implication (P3) $\Rightarrow$ (P1).

To show (72), let $U\in\mathscr{C}$. If $U\in\mathscr{S}'\cup\mathscr{L}'$, then (72) follows from Claim 7. If $U\in\mathscr{S}\backslash\mathscr{S}'$, then $\delta_A^-(U)\supseteq\delta_A^-(U')$ for some $U'\in\mathscr{S}'$ (by Claim 5), and hence (72) follows from the fact that (72) holds for U'. Similarly, the case that $U\in\mathscr{L}\backslash\mathscr{L}'$ follows from Claim 5. ∎

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