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Relaxations of Vertex Packing

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A polynomially computable upper bound for the weighted independence number of a graph is studied. This gives rise to a convex body containing the vertex packing polytope of the graph. This body is a polytope if and only if the graph is perfect. As an application of these notions, we show that the maximum weight independent set of an h-perfect graph can be found in polynomial time. © 1986 Academic Press, Inc.

1. VERTEX PACKING AND ITS RELAXATIONS

Throughout the paper we assume that all graphs we consider have no loops, no multiple edges, and are connected. For our purposes, these assumptions can be made without loss of generality.

Let G be a graph and $w: V(G) \to \mathbb{R}_+$ any weighting of its nodes. Let $\alpha(G; w)$ denote the maximum weight of an independent set. It is well known that to determine $\alpha(G; w)$ even in the special case when $w \equiv 1$ is $\mathcal{NP}$-hard.

A well-known approach to study $\alpha(G; w)$ is to introduce the *vertex packing polytope* VP(G), which is defined as the convex hull of incidence vectors

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of independent sets of nodes. Then $\alpha(G; w)$ can be obtained as the maximum of the linear function $w^T x$ for $x \in \mathrm{VP}(G)$. For this observation to be of any use, however, we need a description of $\mathrm{VP}(G)$ as the solution set of a system of linear inequalities.

The following sets of linear inequalities are obviously all valid for $\mathrm{VP}(G)$:

$$x_i \geqslant 0 \qquad \text{for all} \quad i \in V(G), \tag{1.1}$$

$$x_i + x_j \leqslant 1 \qquad \text{for all} \quad ij \in E(G). \tag{1.2}$$

It is also easy to see that all *integral* solutions of (1.1)–(1.2) are incidence vectors of independent sets of nodes of G.

(1.3) PROPOSITION. *The inequalities* (1.1)–(1.2) *are sufficient to describe* $\mathrm{VP}(G)$ *if and only if* G *is bipartite.*

This assertion follows, e.g., from the results of Egerváry [3]. If G is, say, a triangle, then the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a solution of (1.1)–(1.2) but does not belong to $\mathrm{VP}(G)$. So to describe $\mathrm{VP}(G)$ we need some further inequalities. The example of the triangle suggests the following set of inequalities, which are also obviously valid:

$$\sum_{i \in K} x_i \leqslant 1 \qquad \text{for all complete subgraphs } K \text{ of } G. \tag{1.4}$$

Unfortunately, even (1.4) is not enough to characterize $\mathrm{VP}(G)$. Those graphs for which it is were characterized by Fulkerson [6] and Chvátal [2]. A graph G is called *perfect* if for each induced subgraph G' of G, the chromatic number of G' equals the maximum size of complete subgraphs of G'. For examples and various properties of perfect graphs, see Golumbic [8] and Lóvasz [13].

(1.5) THEOREM. *The inequalities* (1.1) *and* (1.4) *are sufficient to describe* $\mathrm{VP}(G)$ *if and only if* G *is perfect.*

We shall call the solution set of (1.1) + (1.4) the *fractional vertex packing polytope* of the graph G and denote it by $\mathrm{FVP}(G)$.

To get a better description of the relationship between vertex packing and fractional vertex packing polytopes let us introduce the following notion (Fulkerson [5]). Let $P \subseteq \mathbb{R}^n_+$ be any non-empty closed convex set with the following property: if $x \in P$ and $0 \leqslant x' \leqslant x$ then $x' \in P$. The *antiblocker* of P is defined as the set

$$\mathrm{AB}(P) := \{x \in \mathbb{R}^n_+ : y^T x \leqslant 1 \text{ for all } y \in P\}.$$

It is easy to see that the antiblocker is also a non-empty closed convex set with the same property and that its antiblocker is the original set P. It is also easy to see that if P is a polyhedron then so is its antiblocker. It is easy to check that if $\bar{G}$ denotes the complement of the graph G, then $\mathrm{FVP}(\bar{G})$ is the antiblocker of $\mathrm{VP}(G)$ and vice versa. Note that it follows from this observation and Theorem (1.5) that the *complement of a perfect graph is perfect* (Lovász [11]).

It follows from Theorem (1.5) that $\mathrm{VP}(G) \neq \mathrm{FVP}(G)$ if G is an odd circuit. In fact in this case, the point $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is in $\mathrm{FVP}(G)$ but not in $\mathrm{VP}(G)$. This example suggests a new class of inequalities valid for $\mathrm{VP}(G)$:

$$\sum_{i \in V(G)} x_i \leqslant \frac{|V(C)| - 1}{2} \qquad \text{for each odd circuit } C \text{ in } G. \tag{1.6}$$

These inequalities are in general neither stronger nor weaker than the "clique constraints" (1.4). Motivated by Theorem (1.5) Chvátal [2] suggested analogs of the notion of perfectness.

A graph is *t-perfect* if $\mathrm{VP}(G)$ is the solution set of $(1.1) + (1.2) + (1.6)$. A weaker notion is the following: a graph G is called *h-perfect* if $\mathrm{VP}(G)$ is the solution set of $(1.1) + (1.4) + (1.6)$. Various classes of t-perfect and h-perfect graphs are known; e.g., all series-parallel graphs and all graphs arising from a bipartite graph by the contraction of an edge are t-perfect (Fonlupt and Uhry [4]). Recently Gerards and Schrijver [7] generalized both of these results by showing that every graph which does not contain a homeomorph of K_4 such that all 4 cycles corresponding to triangles in the original K_4 are odd is t-perfect.

Let us define the polytope $\mathrm{CVP}(G)$ (for, say, *circuit-constrained vertex packing polytope*) as the solution set of the system of inequalities $(1.1) + (1.2) + (1.6)$. So the graph is t-perfect iff $\mathrm{CVP}(G) = \mathrm{VP}(G)$, and h-perfect iff $\mathrm{CVP}(G) \cap \mathrm{FVP}(G) = \mathrm{VP}(G)$.

A nice property of $\mathrm{CVP}(G)$ is that one can optimize any linear function over it in polynomial time, as we shall show in Section 3. Hence it follows that we can find a maximum weight independent set in every t-perfect graph in polynomial time. In contrast with this, the problem of optimizing a linear objective function over $\mathrm{FVP}(G)$ is $\mathcal{NP}$-hard (Grötschel, Lovász, and Schrijver [9]). But for perfect graphs, i.e., if $\mathrm{VP}(G) = \mathrm{FVP}(G)$, one can find a maximum weight independent set in polynomial time [9]. This algorithm is less immediate than the algorithm for t-perfect graphs, and it involves a weighted version $\vartheta(G, x)$ of an upper bound $\vartheta(G)$ on the independence number of a graph, introduced by Lovász [12].

In this paper we first study the function $\vartheta(G; x)$ in greater detail. This leads us to a convex body $\mathrm{TH}(G)$ such that $\mathrm{VP}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{FVP}(G)$. This set $\mathrm{TH}(G)$ is in general not a polytope; in fact we shall prove that it is a polytope if and only if G is perfect. But in return it has many nice properties: among others, we show that $\mathrm{TH}(\bar{G})$ is the antiblocker of $\mathrm{TH}(G)$ and that any linear objective function can be maximized over $\mathrm{TH}(G)$ in polynomial time. It follows then by the general results of [9] that any linear objective function can be optimized over $\mathrm{TH}(G) \cap \mathrm{CVP}(G)$ in polynomial time. In particular, it follows that a maximum weight independent set can be found in an h-perfect graph in polynomial time.

We end this introduction with a warning that our algorithms involve the Ellipsoid Method and therefore they are not meant to be practical. It is a challenging problem to find combinatorial (and hopefully practical) algorithms for maximum weight independent sets in perfect, t-perfect, and h-perfect graphs.

2. The ϑ Function of a Graph

Let $G=(V,E)$ be a graph. An *orthonormal representation* of G is a sequence $(u_i: i \in V)$ such that $u_i \in \mathbb{R}^N$ for some N, $\|u_i\|=1$ for all $i \in V$, and $u_i^T u_j = 0$ for all pairs (i, j) of non-adjacent nodes. Let $x\colon V \to \mathbb{R}_+$ be any weighting of V. We then define

$$\vartheta(G; x) := \min\left(\max_{i \in V} \frac{x_i}{(c^T u_i)^2}\right),$$

where the minimum is taken over all vectors c with $\|c\|=1$ and all orthonormal representations (u_i) of G. If $x_i=0$ then we take $x_i/(c^T u_i)^2=0$ even if $c^T u_i=0$. If $x_i>0$ but $c^T u_i=0$ then we take $x_i/(c^T u_i)^2=+\infty$. It is easy to see that $\vartheta(G;x)>0$ if $x \neq 0$ and $\vartheta(G;x)<+\infty$. Furthermore, if $x_i=0$ for some i then $\vartheta(G;x)=\vartheta(G-i, x|_{V-i})$.

For the case when each $x_i=1$, this function $\vartheta(G)$ was introduced by Lovász [12] in order to estimate the Shannon capacity of a graph. In [9] the weighted version was also introduced and used to derive a polynomial-time algorithm to find maximum weight independent sets in perfect graphs. Here we shall prove several basic properties of $\vartheta(G;x)$. Many of these are straightforward generalizations of results from Lovász [12] to the weighted case; in these cases, we shall not give the proofs.

Let us introduce some notation. For $x=(x_i: i \in V) \in \mathbb{R}^V$, we set $\bar{x} := (\sqrt{x_i}: i \in V) \in \mathbb{R}^V$ and $X := \mathrm{diag}(\sqrt{x_i}: i \in V) \in \mathbb{R}^{V \times V}$. Let $\mathscr{A}(G)$ denote the set of all symmetric matrices $A \in \mathbb{R}^{V \times V}$ such that $(A)_{ii}=1$ for all i and also $(A)_{ij}=1$ for all non-adjacents pairs (i,j) of nodes. Let $\mathscr{B}(G)$ denote the set of those positive semidefinite symmetric matrices $B \in \mathbb{R}^{V \times V}$ for which $\mathrm{Tr}\, B=1$ and $(B)_{ij}=0$ for all adjacent pairs (i,j) of nodes. We denote by $\Lambda(A)$ the largest eigenvalue of a symmetric matrix A. Then one has the following formulas for $\vartheta(G;x)$:

(2.1) THEOREM. $\vartheta(G; x) = \min\{\Lambda(XAX): A \in \mathscr{A}(G)\}$.

Proof. Straightforward extension from the unweighted case. ∎

(2.2) THEOREM. $\vartheta(G; x) = \max\{\bar{x}^T B \bar{x}: B \in \mathscr{B}(G)\}$.

Proof. Straightforward extension from the unweighted case. ∎

Note that Theorems (2.2) and (2.1) provide a min-max formula and thereby—in some sense—a good characterization of $\vartheta(G; x)$.

(2.3) THEOREM. $\vartheta(G; x) = \max \sum_{i \in V} x_i (d^T v_i)^2$, *where the maximum is taken over all vectors* d *with* $\|d\| = 1$ *and over all orthonormal representations* (v_i) *of the complement of* G.

Proof. Straightforward extension from the unweighted case. ∎

Let us prove some properties of ϑ. These will follow quite easily from the characterizations given by the previous theorems.

(2.4) LEMMA. $\vartheta(G; x) \cdot \vartheta(\bar{G}; y) \geqslant x^T y$ *for all* $x, y \geqslant 0$.

Proof. The lemma is obvious if $x = 0$. Suppose $x \neq 0$, then clearly $\vartheta(G; x) > 0$. Let c be a unit vector and let (u_i) be an orthonormal representation of G such that $\vartheta(G; x) = \max_{i \in V} x_i / (c^T u_i)^2$.

Then by Theorem (2.3),

$$\vartheta(\bar{G}; y) \geqslant \sum_{i \in V} y_i (c^T u_i)^2 \geqslant \sum_{i \in V} y_i \frac{x_i}{\vartheta(G; x)} = x^T y \cdot \frac{1}{\vartheta(G; x)},$$

from which the assertion follows. ∎

(2.5) LEMMA. *For every vector* $x \in \mathbb{R}_+^V$, $x \neq 0$, *there exists a vector* $y \in \mathbb{R}_+^V$, $y \neq 0$, *such that* $\vartheta(G; x) \cdot \vartheta(\bar{G}; y) = x^T y$.

Proof. Let d be a unit vector and (v_i) an orthonormal representation of $\bar{G}$ such that $\vartheta(G; x) = \sum_{i \in V} x_i (d^T v_i)^2$. Choose $y_i = (d^T v_i)^2$. Clearly $y \neq 0$. Then by definition,

$$\vartheta(\bar{G}; y) \leqslant \max_{i \in V} \frac{y_i}{(d^T v_i)^2} = 1,$$

and so

$$x^T y = \sum_{i \in V} x_i y_i = \vartheta(G; x) \geqslant \vartheta(G; x) \cdot \vartheta(\bar{G}; y).$$

By Lemma (2.4), we have equality. ∎

(2.6) LEMMA. *If* $x \geqslant y \geqslant 0$, *then* $\vartheta(G; x) \geqslant \vartheta(G; y)$. *Moreover, for any* $x, y \geqslant 0$, *and any* $\alpha \geqslant 0$, *we have*

$$\vartheta(G; \alpha x) = \alpha \vartheta(G; x)$$

and

$$\vartheta(G; x + y) \leqslant \vartheta(G; x) + \vartheta(G; y).$$

Proof. The first two assertions are obvious. To verify the third, let d be a unit vector and let (v_i) be an orthonormal representation of $\bar{G}$ such that $\vartheta(G; x + y) = \sum_{i \in V} (x_i + y_i)(d^T v_i)^2$. Then

$$\vartheta(G; x + y) = \sum_{i \in V} x_i (d^T v_i)^2 + \sum_{i \in V} y_i (d^T v_i)^2 \leqslant \vartheta(G; x) + \vartheta(G; y). \quad \blacksquare$$

Lemma (2.6) implies that $\vartheta(G; \cdot)$ can be viewed as a "norm" on the non-negative orthant of $\mathbb{R}^V$. Moreover, Lemma (2.4) can be interpreted as a type of mixed Cauchy–Schwarz inequality for the "norms" $\vartheta(G; \cdot)$ and $\vartheta(\bar{G}; \cdot)$ with respect to the Euclidean inner product.

(2.7) LEMMA. *Let d be a unit vector and (v_i) an orthonormal representation of $\bar{G}$ such that $\vartheta(G; x) = \sum_{i \in V} x_i (d^T v_i)^2$. Then*

$$\sum_{i \in V} x_i (d^T v_i)\, v_i = \vartheta(G; x) \cdot d.$$

Proof. By Theorem (2.3) we have

$$\sum_{i \in V} x_i (y^T v_i)^2 \leqslant \vartheta(G; x)$$

for all unit vectors y, and so the left-hand side is maximized by $y = d$. But the left-hand side can be written as a quadratic form in y:

$$\sum_{i \in V} x_i (y^T v_i)^2 = y^T \left(\sum_{i \in V} x_i v_i v_i^T \right) y.$$

As is well known, the maximum of this quadratic form is attained at an eigenvector, say, y, and the maximum value is the corresponding eigenvalue:

$$\left(\sum_{i \in V} x_i v_i v_i^T \right) y = \vartheta(G; x)\, y. \quad \blacksquare$$

Finally, let us recall the following result from [9]; see also (Grötschel, Lovász, and Schrijver [10]).

(2.8) THEOREM. *Given a graph G, a weighting $x: V \to \mathbb{Q}_+$, and rational $\varepsilon > 0$, one can compute in polynomial time a rational matrix $B \in \mathscr{B}(G)$ such that $\bar{x}^T B \bar{x} \geqslant (1-\varepsilon)\,\vartheta(G; x)$.*

Let us remark that the input size of the problem is $|V(G)|^2 +$ the number of digits in the binary expansion of each numerator and denominator in x and ε. It is necessary to allow an error ε since it may happen that all matrices $B \in \mathscr{B}(G)$ maximizing $x^T Bx$ are irrational. This is the case, e.g., when $x \equiv 1$ and G is a pentagon, since then $\vartheta(G; x) = \sqrt{5}$.

Let us also remark that for any vector $x \in \mathbb{R}^V_+$,

$$\max\{x_i : i \in V(G)\} \leqslant \vartheta(G; x) \leqslant \sum_{i \in V(G)} x_i.$$

This in particular implies that in Theorem (2.8) we could as well consider an absolute error instead of a relative error.

We may use this result to derive an algorithmic version of Lemma (2.5), which we will use in the next section.

(2.9) LEMMA. *Given a graph G, a vector $x \in \mathbb{Q}^V_+$, and a rational $\varepsilon > 0$, one can find in polynomial time another vector $y \in \mathbb{Q}^V_+$, $y \neq 0$, such that*

$$\vartheta(G; x)\,\vartheta(\bar{G}; y) \leqslant (1+\varepsilon)\,x^T y.$$

Proof. We may assume $\varepsilon < 1$. By Theorem (2.8), we can find in polynomial time a matrix $B = (b_{ij}) \in \mathscr{B}(G)$ such that $\vartheta(G; x) \leqslant (1+\varepsilon/3)\,\bar{x}^T B \bar{x}$. Without loss of generality we may assume that $b_{ii} > 0$ for all i. Set $t := \bar{x}^T B \bar{x}$, $u := B\bar{x}$, and $z_i := u_i^2 / t^2 b_{ii}$. We claim that $x^T z \geqslant 1$ and $\vartheta(\bar{G}; z) \leqslant 1/t$. To show the first of these inequalities, we use the Cauchy-Schwarz inequality:

$$\begin{aligned} x^T z &= \sum_i x_i \frac{u_i^2}{t^2 b_{ii}} = \Big(\sum_i b_{ii}\Big)\Big(\sum_i \frac{x_i u_i^2}{t^2 b_{ii}}\Big) \\ &\geqslant \Big(\sum_i \sqrt{b_{ii}}\, \frac{\sqrt{x_i}\, u_i}{t\sqrt{b_{ii}}}\Big)^2 = \frac{1}{t^2}\Big(\sum_i \sqrt{x_i}\, u_i\Big)^2 = 1. \end{aligned}$$

The second inequality follows by considering the matrices $Z := \operatorname{diag}(\sqrt{z_i}: i \in V)$ and $U := \operatorname{diag}(1/u_i: i \in V)$. Let J be the all ones matrix and I be the identity matrix then

$$A := J + \frac{1}{t} Z^{-2} - tUBU \in \mathscr{A}(\bar{G})$$

and

$$\frac{1}{t} I - ZAZ = -ZJZ + tZUBUZ = \frac{1}{t} Z(XJ - tU)^T B(XJ - tU)\,Z$$

is positive semidefinite. (The last equation above follows from $JXBXJ = tJ$ and $JXBU = J = UBXJ$.) Hence $\Lambda(ZAZ) \leqslant 1/t$ and so by Theorem (2.1), $\vartheta(\bar{G}; z) \leqslant \Lambda(ZAZ) \leqslant 1/t$.

To finish our argument, let us round each weight z_i to a non-negative rational number $y_i = z_i + h_i$ such that $0 \leqslant h_i < \min\{1, \varepsilon/9t\ \|A\|\ \|Z\|^2\}\ z_i$. Set $Y := \mathrm{diag}(\sqrt{y_i}: i \in V)$, then $\|Y\| \leqslant 2\ \|Z\|$ and

$$\begin{aligned}\Lambda(YAY) &\leqslant \Lambda(ZAZ) + \Lambda(YAY - ZAZ)\\ &\leqslant \frac{1}{t} + \|YAY - ZAZ\|\\ &\leqslant \frac{1}{t} + \|YAY - YAZ\| + \|YAZ - ZAZ\|\\ &\leqslant \frac{1}{t} + \|Y\|\ \|A\|\ \|Y - Z\| + \|Y - Z\|\ \|A\|\ \|Z\|\\ &\leqslant \frac{1}{t} + 3\ \|Z\|\ \|A\|\ \|Y - Z\|\\ &\leqslant \frac{1}{t} + 3\ \|Z\|^2\ \|A\|\ \|YZ^{-1} - I\|\\ &< \frac{1}{t} + \frac{\varepsilon}{3t}\end{aligned}$$

and hence

$$\vartheta(G; x)\ \vartheta(\bar{G}; y) \leqslant t\left(1 + \frac{\varepsilon}{3}\right)\left(\frac{1}{t} + \frac{\varepsilon}{3t}\right) < 1 + \varepsilon \leqslant (1 + \varepsilon)\ x^T y. \quad \blacksquare$$

3. A Non-Polyhedral Relaxation of Vertex Packing

Given a graph $G = (V, E)$, define the following set of vectors:

$$\mathrm{TH}(G) := \{x \in \mathbb{R}_+^V : \vartheta(\bar{G}; x) \leqslant 1\}.$$

From the properties of ϑ, it will be easy to derive the following equivalent definition of this set.

(3.1) Theorem. $\mathrm{TH}(G) = \{x \in \mathbb{R}_+^V : y^T x \leqslant \vartheta(G; y)$ *for all* $y \in \mathbb{R}_+^V\}$.

Proof. (I) Let $x \in \mathrm{TH}(G)$, i.e., let $x \geqslant 0$ and $\vartheta(\bar{G}; x) \leqslant 1$. Then for any $y \geqslant 0$,

$$y^T x \leqslant \vartheta(G; y)\, \vartheta(\bar{G}; x) \leqslant \vartheta(G; y)$$

by Lemma (2.4).

(II) Conversely, assume that $x \geqslant 0$ and $y^T x \leqslant \vartheta(G; y)$ holds of all $y \geqslant 0$. By Lemma (2.5), there exists a vector $y \geqslant 0$, $y \neq 0$ such that

$$\vartheta(\bar{G}; x)\, \vartheta(G; y) = y^T x \leqslant \vartheta(G; y)$$

and hence

$$\vartheta(\bar{G}; x) \leqslant 1,$$

i.e., $x \in \mathrm{TH}(G)$. ∎

The formula in Theorem (3.1) represents $\mathrm{TH}(G)$ as the intersection of (infinitely many) halfspaces. We can obtain another such representation which avoids $\vartheta(G; y)$ if we use that $\vartheta(\bar{G}; x) = \max \sum_{i \in V} x_i (c^T u_i)^2$, where $\|c\| = 1$ and (u_i) is an orthonormal representation of G. Hence we obtain the following description of $\mathrm{TH}(G)$:

(3.2) Theorem. $\mathrm{TH}(G)$ *consists of those vectors* $x \in \mathbb{R}^V$ *which are nonnegative and which satisfy* $\sum_{i \in V} (c^T u_i)^2 x_i \leqslant 1$ *for every* $c \in \mathbb{R}^N$ *with* $\|c\| = 1$ *and every orthonormal representation* (u_i) *of* G *in* $\mathbb{R}^N$.

Note that all the "clique constraints" (1.4) occur here: for, if $K \subseteq V$ spans a complete subgraph of G, then let c and u_i $(i \in V - K)$ be mutually orthogonal unit vectors and $u_i = c$ for $i \in K$. Then (u_i) is an orthonormal representation of G and $\sum_{i \in V} x_i (c^T u_i)^2 = \sum_{i \in K} x_i$, so we get the clique constraint belonging to K. Since 0 and also all the unit vectors trivially satisfy the inequalities in Theorem (3.2), it follows that $\mathrm{TH}(G)$ is full-dimensional.

(3.3) Corollary. $\mathrm{TH}(G)$ *is a convex set.*

Note that by Lemma (2.6), if $0 \leqslant x \leqslant y$ and $y \in \mathrm{TH}(G)$ then $x \in \mathrm{TH}(G)$. The following result gives the antiblocker of $\mathrm{TH}(G)$.

(3.4) Corollary. $\mathrm{AB}(\mathrm{TH}(G)) = \mathrm{TH}(\bar{G})$.

Proof. Let $y \in \mathrm{AB}(\mathrm{TH}(G))$. Then $x^T y \leqslant 1$ for each $x \in \mathrm{TH}(G)$. Let $x \geqslant 0$, $x \neq 0$. Then $x_0 = x / \vartheta(\bar{G}; x) \in \mathrm{TH}(G)$ (and so $x_0^T y \leqslant 1$). But then $x^T y = \vartheta(\bar{G}; x)\, x_0^T y \leqslant \vartheta(\bar{G}; x)$ and hence $y \in \mathrm{TH}(\bar{G})$.

Conversely, let $y \in \mathrm{TH}(\bar{G})$, then for any $x \in \mathrm{TH}(G)$ we have $x^T y \leqslant \vartheta(\bar{G}; x) \leqslant 1$ and so $y \in \mathrm{AB}(\mathrm{TH}(G))$. ∎

From this antiblocking relation, we obtain from Theorem (3.2) a further description of $\mathrm{TH}(G)$, which is perhaps conceptually the simplest, but not so well-suited to work with. (Note that from this description, it is not even straightforward to see that $x \in \mathrm{TH}(G)$ and $0 \leqslant y \leqslant x$ imply $y \in \mathrm{TH}(G)$.)

(3.5) THEOREM. *For any graph G, $\mathrm{TH}(G) = \{((d^T v_i)^2 : i \in V(G)) \in \mathbb{R}_+^{V(G)}: \|d\| = 1,\ (v_i)$ is an orthonormal representation of $\bar{G}\}$.*

As a further application of Theorem (3.1), we show

(3.6) THEOREM. $\mathrm{VP}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{FVP}(G)$.

Proof. Let x be the incidence vector of an independent set A of nodes. Then $\vartheta(\bar{G}; x) = \vartheta(\bar{G}[A]; \mathbb{1}) = 1$ since $\bar{G}[A]$ is a complete graph. So every vertex of $\mathrm{VP}(G)$ is contained in $\mathrm{TH}(G)$. By convexity, $\mathrm{VP}(G) \subseteq \mathrm{TH}(G)$.

Applying this to the complement graph, we obtain that $\mathrm{VP}(\bar{G}) \subseteq \mathrm{TH}(\bar{G})$. Taking antiblockers, we obtain that $\mathrm{FVP}(G) \supseteq \mathrm{TH}(G)$. ∎

The next theorem will imply that $\mathrm{TH}(G)$ is in general not a polytope. Let $P \subseteq \mathbb{R}^n$ be a convex set of dimension n and $a^T x \leqslant \alpha$ an inequality ($a \in \mathbb{R}^n$). We say that $a^T x \leqslant \alpha$ determines a facet of P if it is valid for all $x \in P$ and, moreover, the set $\{x \in P: a^T x = \alpha\}$ has dimension $n-1$.

(3.7) THEOREM. *If an inequality determines a facet of* $\mathrm{TH}(G)$ *then it is a positive multiple of one of the non-negativity constraints* (1.1) *or one of the clique constraints* (1.4).

Proof. Suppose that $a^T x \leqslant \alpha$ determines a facet of $\mathrm{TH}(G)$, and let z be a point in the relative interior of $F := \{x \in \mathrm{TH}(G): a^T x = \alpha\}$. Then either $z_i = 0$ for some i or $\vartheta(\bar{G}; z) = 1$. In the first case $a^T x \leqslant \alpha$ is trivially equivalent to $x_i \geqslant 0$. So suppose that $\vartheta(\bar{G}; z) = 1$. By Theorem (2.3) there exists an orthonormal representation (u_i) of G and a unit vector c such that

$$\sum_{i \in V} z_i (c^T u_i)^2 = 1.$$

Since

$$\sum_{i \in V} x_i (c^T u_i)^2 \leqslant 1 \tag{3.8}$$

is a valid inequality for all $x \in \mathrm{TH}(G)$, it follows that it must be equivalent with $a^T x \leqslant \alpha$, and hence we may assume that $\alpha = 1$ and $a_i = (c^T u_i)^2$. We also see that

$$\sum_{i \in V} x_i (c^T u_i)^2 = 1 \tag{3.9}$$

for all $x \in F$.

By Lemma (2.7)

$$\sum_{i \in V} x_i(c^T u_i)\, u_i = c$$

holds for all $x \in F$, i.e.,

$$\sum_{i \in V} x_i(c^T u_i)(u_i)_j = c_j$$

for all $j \in V$. Since F is $(n-1)$-dimensional, these equations must follow from (3.9), and hence $c_j(c^T u_i)^2 = (c^T u_i)(u_i)_j$. If we consider any $i \in V$ such that $c^T u_i \neq 0$ then this implies that $(c^T u_i)\, c = u_i$. Since $\|u_i\| = \|c\| = 1$, this yields that $u_i = \pm c$. Clearly, we may assume that $u_i = c$.

So we see that for each i, either $c^T u_i = 0$ or $u_i = c$. Let $u_i = c$ for $i \in K$ and $c^T u_i = 0$ for $i \in V - K$. Then K is a complete subgraph, since for any two nodes $i, j \in K$, $u_i^T u_j = c^2 = 1 \neq 0$. So (3.8) is just

$$\sum_{i \in K} x_i \leqslant 1,$$

which proves the theorem. ∎

(3.10) COROLLARY. TH(G) *is a polytope iff G is perfect.*

Proof. If G is perfect then VP(G) = TH(G) = FVP(G) by Theorems (1.5) and (3.6). Conversely, suppose that TH(G) is a polytope. Then TH(G) is the solution set of all inequalities determining a facet of TH(G). By Theorem (3.7), all these inequalities occur in (1.1) and (1.4), and so TH(G) = FVP(G). Since TH$(\bar{G})$ = AB(TH(G)) is also a polytope, it follows that TH$(\bar{G})$ = FVP$(\bar{G})$ and so, taking antiblockers, TH(G) = VP(G). So VP(G) = FVP(G) and by Theorem (1.5), G is perfect. ∎

Two immediate corollaries are the following.

(3.11) COROLLARY. TH(G) = VP(G) *iff G is perfect.*

(3.12) COROLLARY. TH(G) = FVP(G) *iff G is perfect.*

4. OPTIMIZATION OVER TH(G) AND CVP(G)

We are going to show that every linear objective function can be maximized over TH(G) as well as over CVP(G) in polynomial time. Let us formulate this task more precisely, by recalling two definitions from [9].

Let K be a non-empty convex compact set in $\mathbb{R}^n$, and for $z \in \mathbb{R}^n$ let $d(z, K) := \min\{\|z - x\| : x \in K\}$.

(4.1) STRONG OPTIMIZATION PROBLEM. *Given a vector* $c \in \mathbb{Q}^n$, *find a rational vector* $z \in K$ *which maximizes* $c^T x$ *over* K.

(4.2) WEAK OPTIMIZATION PROBLEM. *Given a vector* $c \in \mathbb{Q}^n$ *and a positive rational number* ε, *find a vector* $z \in \mathbb{Q}^n$ *such that* $d(z, K) \leqslant \varepsilon$ *and* $c^T z \geqslant \max\{c^T x \colon x \in K\} - \varepsilon$.

This latter problem is needed because the vector maximizing $c^T x$ over a nonpolyhedral convex body like TH(G) may not be rational.

We start with a lemma.

(4.3) LEMMA. *For any* $c \geqslant 0$,

$$\max\{c^T x \colon x \in \mathrm{TH}(G)\} = \vartheta(G; c).$$

Proof. By Theorem (3.1), $c^T x \leqslant \vartheta(G; c)$ is valid for TH(G), so the maximum above cannot be larger than $\vartheta(G; c)$ On the other hand, if $0 < t < \vartheta(G; c)$ then $(1/t)c \notin \mathrm{TH}(\bar{G})$ and hence by Corollary (3.4) there exists an $x \in \mathrm{TH}(G)$ such that $(1/t)\, c^T x > 1$, i.e., $c^T x > t$. ▮

(4.4) THEOREM. *Let* $G = (V, E)$ *be a graph. Then the weak optimization problem for* TH(G) *can be solved in polynomial time.*

Proof. Let $c \in \mathbb{Q}^V$ and $\varepsilon > 0$. We may assume that $c \geqslant 0$ since if $c_i < 0$ for some $i \in V$ then obviously $x_i = 0$ for every vector x maximizing $c^T x$ and so we can delete i from G.

By Lemma (2.9), we can find in polynomial time a vector $0 \neq d \in \mathbb{Q}_+^V$ such that $\vartheta(\bar{G}; d)\, \vartheta(G; c) \leqslant (1 + \varepsilon)\, c^T d$. Then for $y := (1/\vartheta(\bar{G}, d))\, d$ we have that $y \in \mathrm{TH}(G)$ and $c^T y \geqslant (1/(1+\varepsilon))\, \vartheta(G; c) = (1/(1+\varepsilon)) \max\{c^T x \colon x \in \mathrm{TH}(G)\}$. By Theorem (2.8) we can compute a rational approximation of y with arbitrary precision. ▮

Our next lemma is a preparation for the treatment of CVP(G). This result is well known, but for the sake of completeness we give a short proof.

(4.5) LEMMA. *Let* $G = (V, E)$ *be a graph and let* $z \colon E \to \mathbb{R}_+$ *be "lengths" assigned to its edges. Then a shortest odd circuit in* G *can be found in polynomial time.*

Proof. Replace each $v \in V$ by two points v', v''; for each edge $uv \in E$, connect u' to v'' and u'' to v'. Let G' be the graph obtained this way. Also define a "length" of each edge of G' by $l(u'v'') := l(u''v') := z(uv)$. Find a shortest $v'v''$-path in G' for each $v \in V$. A shortest among all these paths gives a shortest odd circuit in G. ▮

(4.6) THEOREM. *Let $G = (V, E)$ be a graph. Then the strong optimization problem for* CVP(G) *can be solved in polynomial time. Moreover, an optimum vertex solution can be found in polynomial time.*

By Theorems (3.8) and (3.9) of [9], it suffices to prove the following.

(4.7) LEMMA. *Let $G = (V, E)$ be a graph and $y \in \mathbb{Q}^V$ a vector. Then there is an algorithm which concludes in polynomial time with one of the following:*

(a) $y \in \mathrm{CVP}(G)$,

(b) *finding an inequality from* (1.1), (1.2), *or* (1.6) *violated by $x = y$.*

Proof. The inequalities in (1.1) and (1.2) are easily checked by substitution. So we may assume that $y \geqslant 0$, and for each edge $uv \in E$, $y_u + y_v \leqslant 1$.

Define, for each edge $e = uv \in E$, $z_e := 1 - y_u - y_v$. So $z_e \geqslant 0$. Then (1.6) is equivalent to the following set of inequalities:

$$\sum_{e \in E(C)} z_e \geqslant 1 \qquad \text{for each odd circuit } C. \tag{4.8}$$

If we view z_e as the "length" of edge e, then (4.8) says that the length of a shortest odd circuit is at least 1. But a shortest odd circuit can be found in polynomial time by Lemma (4.5). This proves the Lemma and thereby also Theorem (4.6). ∎

(4.9) COROLLARY. *There is a polynomial time algorithm for the maximum weight independent set problem for t-perfect graphs.*

By combining the algorithms in Theorems (4.4) and (4.6), we can prove the following stronger result:

(4.10) THEOREM. *There is a polynomial time algorithm for the maximum weight independent set problem for h-perfect graphs.*

Proof. Let G be an h-perfect graph. Then $\mathrm{VP}(G) = \mathrm{FVP}(G) \cap \mathrm{CVP}(G)$ and hence also $\mathrm{VP}(G) = \mathrm{TH}(G) \cap \mathrm{CVP}(G)$. By Theorems (4.4) and (4.6), we can solve the weak optimization problem for both of TH(G) and CVP(G). Hence the weak optimization problem can also be solved for their intersection by Corollary (3.4) of [9]. But then by Theorem (3.8) of [9], the strong optimization problem can also be solved in polynomial time and by Theorem (3.9) of [9], an optimum vertex can also be found in polynomial time. ∎

References

1. M. Boulala and J. P. Uhry, Polytope des indépendents d'un graphe séries-parallèle, *Discrete Math.* **27** (1979), 225–243.
2. V. Chvátal, On certain polytopes associated with graphs, *J. Combin. Theory Ser. B* **18** (1975), 138–154.
3. E. Egerváry, On combinatorial properties of matrices (Hungarian), *Mat. Lapok* **38** (1931), 16–28.
4. J. Fonlupt and J. P. Uhry, Transformations which preserve perfectness and *h*-perfectness of graphs, *Ann. Discrete Math.* **16** (1982), 83–95.
5. D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Math. Programming* **1** (1971), 168–194.
6. D. R. Fulkerson, On the perfect graph theorem, *in* "Mathematical Programming" (T. C. Hu and S. M. Robinson, Eds.), pp. 69–77, Academic Press, New York, 1973.
7. A. M. H. Gerards and A. Schrijver, Matrices with the Edmonds–Johnson property, preprint, Tilburg University, Holland, June 1984.
8. M. Golumbic, "Algorithmic Graph Theory and Perfect Graphs," Academic Press, New York, 1980.
9. M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1** (1981), 169–197.
10. M. Grötschel, L. Lovász, and A. Schrijver, Polynomial algorithms for perfect graphs, *Ann. Discrete Math.* **21** (1984), 325–356.
11. L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* **2** (1972), 253–267.
12. L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* **25** (1979), 1–7.
13. L. Lovász, Perfect graphs, *in* "More Selected Topics in Graph Theory" (L. W. Beineke and R. J. Wilson, Eds.), pp. 55–87, Academic Press, New York/London, 1983.