

FINDING k PARTIALLY DISJOINT PATHS IN A DIRECTED PLANAR GRAPH

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Abstract. The *partially disjoint paths problem* is: *given:* a directed graph, vertices $r_1, s_1, \dots, r_k, s_k$, and a set F of pairs $\{i, j\}$ from $\{1, \dots, k\}$, *find:* for each $i = 1, \dots, k$ a directed $r_i - s_i$ path P_i such that if $\{i, j\} \in F$ then P_i and P_j are disjoint.

We show that for fixed k , this problem is solvable in polynomial time if the directed graph is planar. More generally, the problem is solvable in polynomial time for directed graphs embedded on a fixed compact surface. Moreover, one may specify for each edge a subset of $\{1, \dots, k\}$ prescribing which of the $r_i - s_i$ paths are allowed to traverse this edge.

1. Introduction

In this paper we show that the following problem, the k *partially disjoint paths problem*, is solvable in polynomial time for directed planar graphs, for each fixed k :

- (1) *given:* a directed graph $D = (V, E)$, vertices $r_1, s_1, \dots, r_k, s_k$ of D , and a set F of pairs $\{i, j\}$ from $\{1, \dots, k\}$,
find: for each $i = 1, \dots, k$, a directed $r_i - s_i$ path P_i in D such that if $\{i, j\} \in F$ then P_i and P_j are disjoint.

Here ‘disjoint’ means vertex-disjoint. So F prescribes the set of pairs of paths that are forbidden to intersect.

This paper extends [15], where *all* pairs of paths are prescribed to be disjoint (so F is the set of all pairs from $\{1, \dots, k\}$). Also the method of [15] based on free groups and cohomology is extended to free partially commutative groups (but also some simplifications of the method in [15] have been included in the present paper).

The partially disjoint paths problem comes up in multi-commodity routing where certain commodities are forbidden to use the same facility, to avoid clashes of conflicting commodities (radio frequencies, soccer fan gangs, chemicals (including gases) through a pipeline network, or time slots in routing on a VLSI-chip).

The disjoint paths problem is well-studied, and generally NP-complete, implying *a fortiori* that the partially disjoint paths problem is generally NP-complete. The problem is NP-complete if we do not fix k , even in the undirected case (Lynch [10]). Moreover, it is NP-complete for $k = 2$ for directed graphs (Fortune, Hopcroft, and Wyllie [8]). This is in contrast to the undirected case (if $\text{NP} \neq \text{P}$), where Robertson and Seymour [14] showed that, for any fixed k , the k disjoint paths problem is polynomial-time solvable for any graph (not necessarily planar). For a survey of results till 2003 we refer to Chapter 70 of [16].

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Our method for the partially disjoint paths problem (1) for directed planar graphs consists of a number of layers and reductions:

- (2) (i) The top layer is to select a homology type for the solution. The number of potentially feasible homology types can be bounded by $(2|E(G)| + 1)^{4k^2}$. This is the only level where the ‘fixed k ’ comes in.
- (ii) For each homology type, one can find in polynomial time a solution of that type, if it exists. The formalism to keep track of homology is that of flows over a ‘graph group’: the group given by generators g_1, \dots, g_k and relations $g_i g_j = g_j g_i$ for all i, j with $\{i, j\} \notin F$.
- (iii) Finding such a solution of the prescribed homology type is done by reduction to a ‘cohomology feasibility problem’ in a (generally nonplanar) *extension* of the planar dual of the input graph. (This is why we need cohomology — homology in the original, planar graph seems not enough, mainly because disjoint paths should not only be edge-disjoint, but also vertex-disjoint.)
- (iv) This cohomology feasibility problem is reduced to a 2-satisfiability problem, whose input is based on a (polynomial) number of ‘pre-feasible’ solutions for the cohomology feasibility problem.
- (v) Finding these pre-feasible solutions forms the bottom layer of the algorithm. It consists of a rather brute-force, but yet polynomial-time, constraint satisfaction method (adapting an instance as long as it is not pre-feasible).

In our description, we start at the bottom and work our way up to the top layer.

The method rests on quite basic combinatorial group theory. The approach allows application of the algorithm where the embedding of the graph in the plane is given in an implicit way, viz. by a list of the cycles that bound the faces of the graph, or alternatively by the clockwise order of edges incident with v , for each vertex v .

Our method directly extends to directed graphs on any fixed compact surface and to inputs where for each edge e a subset K_e of $\{1, \dots, k\}$ is given that prescribes which of the $r_i - s_i$ paths may traverse e . We did not see if the method would extend to a polynomial-time algorithm if, instead of fixing k , we fix the number of faces by which $r_1, s_1, \dots, r_k, s_k$ can be covered.

Our algorithm is a ‘brute force’ polynomial-time algorithm. We did not aim at obtaining the best possible running time bound, as we presume that there are much faster (but possibly more complicated) methods for problem (1) for directed planar graphs than the one we describe in this paper.

We could not avoid that k pops up in the degree of the polynomial. In fact, Cygan, Marx, Pilipczuk, and Pilipczuk [3] recently showed that there exists a constant t , independent of k , such that the k (fully) disjoint paths problem for directed planar graphs is solvable in $O(n^t)$ time, for any fixed k . So k only shows up in the coefficient of the polynomial. In other words, the problem is ‘fixed parameter tractable’. This raises the question if also the partially disjoint paths problem is fixed parameter tractable for directed planar graphs.

In the case of undirected planar graphs, it was shown by Reed, Robertson, Schrijver, and Seymour [13] that the k disjoint paths problem can be solved in *linear* time, for any fixed k . This algorithm utilizes methods from Robertson and Seymour’s graph minors theory. For

general undirected graphs, the k disjoint paths problem is solvable in time $O(n^2)$ for any fixed k ([14], [9]).

2. Graph groups

Our method uses the framework of combinatorial group theory, viz. groups defined by generators and relations. For background literature on combinatorial group theory we refer to Magnus, Karrass, and Solitar [12] and Lyndon and Schupp [11].

In particular we utilize ‘graph groups’. These groups are studied *inter alia* by Baudisch [2], Droms [6], Servatius [17], Wrathall [18], and Esyp, Kazachkov, and Remeslennikov [7]. Specific properties of graph groups that we will use are given in [2] and [7], but we will also need several other properties that seem not to have been considered before, in particular concerning phenomena like convexity and periodicity emanating in graph groups.

We first give some standard terminology. Let g_1, \dots, g_k form an abstract set of *generators*. Call $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$ *symbols*. A *word* (of size t) is a sequence $\alpha_1 \cdots \alpha_t$ where each α_j is a symbol. The empty word (of size 0) is denoted by \emptyset . Define $(g_i^{-1})^{-1} := g_i$, and $(\alpha_1 \cdots \alpha_t)^{-1} := \alpha_t^{-1} \cdots \alpha_1^{-1}$.

Let g_1, \dots, g_k be generators, and let F be a set of unordered pairs $\{i, j\}$ from $[k]$ with $i \neq j$. So $([k], F)$ is an undirected graph. (Throughout this paper: $[k] := \{1, \dots, k\}$.)

Then the group $G = G_F$ is generated by the generators g_1, \dots, g_k , with relations

$$(3) \quad g_i g_j = g_j g_i \text{ for each pair } \{i, j\} \notin F.$$

If $F = \emptyset$ G_F is the *free group* generated by g_1, \dots, g_k . If F consists of *all* pairs, the group G_F is the *free group* generated by g_1, \dots, g_k . If $F = \emptyset$ then G_F is isomorphic to \mathbb{Z}^k . Let 1 denote the unit element of G_F . So $1 = \emptyset$.

The group G_F is called a *graph group*, or a *free partially commutative group*, or a *right-angled Artin group*, or a *semifree group*. (Our definition (3) of graph group differs in a nonessential way from that generally used, where the graph describes the pairs of commuting generators, rather than the pairs of noncommuting generators. Definition (3) is more convenient for our purposes. For instance, it implies that the group G_F is equal to the product of the groups obtained from each component of the graph $([k], F)$.)

2.1. Independent symbols, commuting, reduced words

We review the basics of graph groups, referring to Baudisch [2] and Esyp, Kazachkov, and Remeslennikov [7] for the elaboration of some details.

To describe G_F , call symbols α and β *independent* if $\alpha \in \{g_i, g_i^{-1}\}$ and $\beta \in \{g_j, g_j^{-1}\}$ for some $\{i, j\} \notin F$ with $i \neq j$. So if α and β are independent then $\alpha\beta = \beta\alpha$ and $\beta \neq \alpha^{\pm 1}$. (It follows from (5) below that also the converse implication holds.)

Call words w and v *equivalent* if v arises from w by iteratively:

- (4) (i) replacing $x\alpha\alpha^{-1}y$ by xy or vice versa, where α is a symbol,
- (ii) replacing $x\alpha\beta y$ by $x\beta\alpha y$ where α and β are independent symbols.

By *commuting* we will mean applying (ii) iteratively.

Then the elements of G_F are equivalence classes of words, which we can indicate by words, although one should obviously keep in mind that different words will indicate one group element. We will write $w \equiv v$ if we want to stress that w and v are equal as words. We denote G_F by G if F is clear from the context.

A word w is called *reduced* if it is not equal (as a word) to $x\alpha y\alpha^{-1}z$ for some symbol α independent of y . Note that reducedness is a property of words, and that it is invariant under commuting. We say that a symbol α *occurs in* an element x of G , or that x *contains* α , if α occurs in any reduced word representing x . Two elements x and y of G are called *independent* if any symbol in x and any symbol in y are independent. (In particular, $\beta \neq \alpha^{\pm 1}$ for any symbols α in x and β in y .)

The following is basic — see Lemma 2.3 in [7]:

- (5) *Let w and x be reduced words with $w = x$ as group elements. Then word x can be obtained from w by a series of commutings.*

Define, for $x \in G$, $|x|$ as the size of any reduced word representing x . So $|xy| \leq |x| + |y|$ for all $x, y \in G$.

(5) implies that testing if $w = 1$ is easy: just replace (iteratively) any contiguous subword $\alpha y \alpha^{-1}$ by y where α is a symbol and y is a word independent of α . The final word is empty if and only if $w = 1$. This gives a test for equivalence of words w and x : just test if $w x^{-1} = 1$. So the ‘word problem’ for graph groups is solvable in polynomial time. (In fact it can be solved in linear time — see Wrathall [18].)

It will be convenient to have a mean to emphasize when the concatenation of two reduced words x and y gives again a reduced word (without cancellation as in (4)(i)). In other words, when $|xy| = |x| + |y|$.

To this end, we add an abstract new element $*$ to G and define a multiplication \cdot on $G \cup \{*\}$ as follows. Let $x, y \in G$. Then $x \cdot y := xy$ if $|xy| = |x| + |y|$, and $x \cdot y := *$ if $|xy| < |x| + |y|$. So $x \cdot y$ belongs to G if for any reduced words x' and y' representing x and y one has that the concatenation of x' and y' is reduced. So no symbol in x' cancels out any symbol in y' . If we moreover set $* \cdot x := *$ and $x \cdot * := *$ for all $x \in G \cup \{*\}$, we obtain an associative multiplication \cdot on $G \cup \{*\}$.

The only purpose of introducing $*$ is to have a convenient and formally correct tool to write, for $x, y, z \in G$, $x = y \cdot z$, meaning $x = yz$ and $|x| = |y| + |z|$. By extension, for $x, y_1, \dots, y_n \in G$, $x = y_1 \cdot \dots \cdot y_n$ means $x = y_1 y_2 \dots y_n$ and $|x| = |y_1| + |y_2| + \dots + |y_n|$. That is, in the concatenation of reduced words y_1, \dots, y_n there is no cancellation. The element $*$ will not occur anymore below.

While the multiplication \cdot is associative, it is generally not the case that if $xy = x \cdot y$ and $yz = y \cdot z$ then $xyz = x \cdot y \cdot z$, because in xyz , symbols in x might cancel out symbols in z . Nevertheless, the following holds. Call $y \in G$ a *segment* of $a \in G$ if there exist $x, z \in G$ with $a = x \cdot y \cdot z$. Then:

- (6) *If $xy = x \cdot y$ and $yz = y \cdot z$, then y is a segment of xyz .*

To see this, let x, y , and z be reduced words, and consider the concatenation of x, y, z . In the cancellation to obtain a reduced word, only symbols in x and symbols in z can cancel each other out (since the concatenations x, y and y, z are reduced). So y survives as segment

of xyz .

2.2. The partial order \leq

Most of what follows in this section is, explicitly or implicitly, in [7]. Let $x, y \in G$. We write $x \leq y$ if $x = x \cdot a$ for some a (namely, $a = y^{-1}x$). (If $x \leq y$, x is called in [7] a *left divisor* of y .) So if y is given as reduced word, it means that y can be commuted so that the first $|x|$ symbols in y form x (by (5)). Also, $x \leq y$ if and only if $|y| = |x| + |x^{-1}y|$.

It is easy to derive from the norm properties of $|\cdot|$ that \leq is a partial order. In fact, the partial order \leq is a lattice if we add to G an element ∞ at infinity (Propositions 3.10 and 3.12 in [7]). This follows from the existence of the meet $x \wedge y$ for all $x, y \in G$. Then the join \vee exists for all x, y for which there exists $z \in G$ with $x, y \leq z$ (then $x \vee y$ is the meet of all such z). So adding an element ∞ with $\infty \geq x$ for all x , makes $(G \cup \{\infty\}, \leq)$ to a lattice. Then $x \vee y = \infty$ if there is no $z \in G$ with $z \geq x, y$.

If finite, the join $x \vee y$ can be described as follows (Proposition 3.18 in [7]):

$$(7) \quad \text{Let } x, y \in G \text{ and define } x' := (x \wedge y)^{-1}x \text{ and } y' := (x \wedge y)^{-1}y. \text{ Then } x \vee y < \infty \text{ if and only if } x' \text{ and } y' \text{ are independent.}$$

Moreover:

$$(8) \quad \text{If } x \vee y < \infty \text{ then } x \vee y = (x \wedge y) \cdot x' \cdot y' = x(x \wedge y)^{-1}y.$$

For any $a \in G$ define:

$$(9) \quad a^\downarrow := \{x \in G \mid x \leq a\} \text{ and } a^\uparrow := \{x \in G \mid x \geq a\}.$$

The norm characterization of \leq implies for all x, y, z with $x \leq y, z$:

$$(10) \quad y \leq z \text{ if and only if } x^{-1}y \leq x^{-1}z.$$

Hence for any $a \in G$, the function $a^\uparrow \rightarrow a^{-1}a^\uparrow$ with $b \mapsto a^{-1}b$ for $b \in a^\uparrow$ is an order isomorphism, and therefore:

$$(11) \quad \text{If } b, c \geq a \text{ then } a^{-1}(b \wedge c) = a^{-1}b \wedge a^{-1}c, \text{ and if moreover } b \vee c < \infty, \text{ then } a^{-1}(b \vee c) = a^{-1}b \vee a^{-1}c.$$

Proposition 1. *Let x_1, \dots, x_t be such that $x_i \vee x_j < \infty$ for all i, j . Then $x_1 \vee \dots \vee x_t < \infty$.*

Proof. It suffices to show this for $t = 3$. Suppose $x \vee y \vee z = \infty$ while $y \vee z < \infty$. Let $a := x \wedge (y \vee z)$. As $x \vee y \vee z = \infty$, $a^{-1}x$ and $a^{-1}(y \vee z)$ are dependent. Now, using (11) (with $b := a \vee y$ and $c := a \vee z$) and (8):

$$(12) \quad a^{-1}(y \vee z) = a^{-1}(a \vee y \vee z) = a^{-1}(a \vee y) \vee a^{-1}(a \vee z) = (a \wedge y)^{-1}y \vee (a \wedge z)^{-1}z = (x \wedge y)^{-1}y \vee (x \wedge z)^{-1}z.$$

So $a^{-1}x$ is dependent on at least one of $(x \wedge y)^{-1}y$ and $(x \wedge z)^{-1}z$, say $a^{-1}x$ is dependent on $(x \wedge y)^{-1}y$. Since $x \wedge y \leq a$ this implies that $(x \wedge y)^{-1}x$ is dependent $(x \wedge y)^{-1}y$. So $x \vee y$ is infinite. ■

We say that a symbol α is a *minimal symbol* of $x \in G$ if $\alpha \leq x$ (so α can be commuted so as to become the first symbol). Similarly, α is a *maximal symbol* if $\alpha^{-1} \leq x^{-1}$, i.e., if $x\alpha^{-1} \leq x$ (so α can be commuted so as to become the last symbol).

We will need that for all $x, y, z \in G$:

$$(13) \quad \text{If } y \leq xyz = x \cdot y \cdot z, \text{ then } y \leq xy.$$

This can be seen by induction on $|z|$. Let α be a maximal element of z , let $z' := z\alpha^{-1}$, and suppose that $y \not\leq xyz'$. Let α occur m times in y . As $y \leq xyz$ and $y \not\leq xyz'$, α occurs $m-1$ times in xyz' . This contradicts the fact that α occurs m times in y , hence in $x \cdot y \cdot z'$. This proves (13).

We also will need that for all $x, y, z \in G$:

$$(14) \quad \text{If } x, y \leq z \text{ then } (x \wedge y)^{-1}x \leq y^{-1}z.$$

To see this, let $b := x \vee y = y(x \wedge y)^{-1}x$. As $y \leq b \leq z$, we have $(x \wedge y)^{-1}x = y^{-1}b \leq y^{-1}z$ by (10). proving (14).

Moreover, for all $x, y, z \in G$:

$$(15) \quad \text{If } y^{-1}x \wedge y^{-1}z = 1 \text{ then } (x \wedge z) \vee (x \wedge y) \vee (y \wedge z) = y.$$

To prove this, we can assume (by (11)) that $x \wedge y \wedge z = 1$. Let $a := y \wedge z$, $b := x \wedge z$, and $c := x \wedge y$. Then $a \wedge b = a \wedge c = b \wedge c = 1$. Hence, by (7), a , b , and c are pairwise independent, and $bc \leq x$, $ac \leq y$, $ab \leq z$. Let x' , y' , and z' satisfy $x = bcx'$, $y = acy'$, and $z = abz'$. Since $x \wedge y = c$, we know $y^{-1}x = y'^{-1} \cdot a^{-1} \cdot b \cdot x'$. Hence, as a and b are independent, $y'^{-1} \cdot b \leq y^{-1}x$. Similarly, $y'^{-1} \cdot b \leq y^{-1}z$. As $y^{-1}x \wedge y^{-1}z = 1$, we have $y' = 1$ and $b = 1$. Hence $(x \wedge z) \vee (x \wedge y) \vee (y \wedge z) = a \vee b \vee c = ac = y$.

(We finally remark, but will not need, that the lattice $G \cup \{\infty\}$ is not distributive (if $F \neq \emptyset$), while for each $a \in G$, the sublattice a^\downarrow is distributive.)

2.3. Convex sets

The function $\text{dist}(x, y) := |x^{-1}y|$ is a metric, since, for all $x, y \in G$, (i) $|x| = 0 \iff x = 1$, (ii) $|x^{-1}| = |x|$, and (iii) $|xy| \leq |x| + |y|$. Note that this distance is left-invariant: $\text{dist}(zx, zy) = \text{dist}(x, y)$ for all x, y, z .

We call a subset L of G *convex* if L is nonempty and if $x, z \in L$ and $\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)$ then $y \in L$. Since the distance function is left-invariant, if L is convex also yL is convex, for each $y \in G$.

Proposition 2. *A nonempty subset L of G is convex if and only if*

- $$(16) \quad \begin{aligned} & \text{(i) if } x \leq y \leq z \text{ and } x, z \in L \text{ then } y \in L, \\ & \text{(ii) if } x, y \in L \text{ then } x \wedge y \in L \text{ and, if } x \vee y \text{ is finite, } x \vee y \in L. \end{aligned}$$

Proof. Necessity follows from the facts that if $x \leq y \leq z$ then $\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)$, that $\text{dist}(x, y) = \text{dist}(x, x \wedge y) + \text{dist}(x \wedge y, y)$ and that, if $x \vee y$ is finite then $\text{dist}(x, y) = \text{dist}(x, x \vee y) + \text{dist}(x \vee y, y)$.

To see sufficiency, let $\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)$ with $x, z \in L$. We must show $y \in L$. So $|x^{-1}y| + |y^{-1}z| = |x^{-1}z|$, hence $y^{-1}x \wedge y^{-1}z = 1$. Hence by (15), $(x \wedge z) \vee (x \wedge y) \vee (y \wedge z) = y$. So $x \wedge z \leq x \wedge y \leq x$ and $x \wedge z \leq y \wedge z \leq z$. Therefore, $x \wedge y$ and $y \wedge z$ belong to L and hence y belongs to L . \blacksquare

This implies that a^\uparrow and a^\downarrow are convex. Moreover, each convex set L has a unique minimal element $\min L$. This in fact characterizes convex sets:

(17) *A nonempty subset L of G is convex if and only if for each $a \in G$, aL has a unique minimal element.*

Here necessity follows from (16)(ii). To see sufficiency, let $x, z \in L$ and $y \in G$ such that $\text{dist}(x, y) + \text{dist}(y, z) = \text{dist}(x, z)$. We prove $y \in L$. We may assume $y = 1$ (as the condition is invariant under resetting $L \rightarrow y^{-1}L$). Let a be the unique minimal element in L . So $x \geq a$ and $z \geq a$. On the other hand, $|x| + |z| = |x^{-1}z|$, and hence $x \wedge z = 1$. So $a = 1$, proving (17).

Clearly, the intersection of any number of convex sets is convex again. Moreover, convex sets satisfy the following ‘Helly-property’:

(18) *Let L_1, L_2, L_3 be convex sets with $L_i \cap L_j \neq \emptyset$ for all $i, j = 1, 2, 3$. Then $L_1 \cap L_2 \cap L_3 \neq \emptyset$.*

For choose $x \in L_1 \cap L_2, y \in L_1 \cap L_3, z \in L_2 \cap L_3$. Without loss of generality, $z = 1$ (as we can replace L_1, L_2, L_3 by $z^{-1}L_1, z^{-1}L_2, z^{-1}L_3$). Now $x \wedge y \in L_1 \cap L_2 \cap L_3$.

This proves (18), which implies the following. As usual, define $XY := \{xy \mid x \in X, y \in Y\}$ and $X^{-1} := \{x^{-1} \mid x \in X\}$, for $X, Y \subseteq G$. Then:

(19) *Let L_1, L_2 , and L_3 be convex, with $L_1 \cap L_2 \neq \emptyset$. Then $L_1L_3^{-1} \cap L_2L_3^{-1} = (L_1 \cap L_2)L_3^{-1}$.*

Indeed, trivially, $L_1L_3^{-1} \cap L_2L_3^{-1} \supseteq (L_1 \cap L_2)L_3^{-1}$. To see the reverse inclusion, let $x \in L_1L_3^{-1} \cap L_2L_3^{-1}$. Since $x \in L_1L_3^{-1}$, we know $x^{-1}L_1 \cap L_3 \neq \emptyset$. Similarly, $x^{-1}L_2 \cap L_3 \neq \emptyset$. Since also $L_1 \cap L_2 \neq \emptyset$, (18) gives $x^{-1}L_1 \cap x^{-1}L_2 \cap L_3 \neq \emptyset$. Hence $x \in (L_1 \cap L_2)L_3^{-1}$.

2.4. Ideals and closed sets

A subset I of G an *ideal* if I is nonempty and

- (20) (i) if $y \leq x$ and $x \in I$ then $y \in I$,
(ii) if $x, y \in I$ and $x \vee y$ is finite then $x \vee y \in I$.

Since 1 belongs to any ideal, by comparing (16) and (20) one sees that each ideal is convex. Moreover, for any $L \subseteq G$ and $x \in L$:

(21) *L is convex if and only if $x^{-1}L$ is an ideal.*

Call H *closed* if both H and H^{-1} are ideals. In particular, if H is closed and $x \in H$, then any segment of x belongs to H .

Proposition 3. *If H is closed and $x, y \in G$, then $x \leq y$ implies $\min(xH) \leq \min(yH)$.*

Proof. We can assume that $y = x\alpha$ for some symbol α . Let $c \in H$ with $yc = \min(yH)$. It suffices to show that there exists $d \in H$ with $xd \leq yc$.

If $\alpha^{-1} \leq c$, let $d := \alpha c$, in which case $d \in H$ (as H^{-1} is an ideal) and $xd = yc$. If $\alpha^{-1} \not\leq c$, then let $d := c$, in which case $xd \leq yc$. Indeed, as $yc = \min(yH)$ and H is an ideal, we know $c \leq y^{-1}$. Since $c \leq y^{-1}$ and $\alpha^{-1} \leq y^{-1}$, α^{-1} and c are independent. As moreover $\alpha^{-1} \leq y^{-1}$, we know $\alpha^{-1} \leq c^{-1}y^{-1}$. Therefore, $yc\alpha^{-1} \leq yc$, and hence $xc = y\alpha^{-1}c = yc\alpha^{-1} \leq yc$. \blacksquare

This is used in showing:

(22) *If L is convex and H is closed, then LH is convex. Moreover, $\min(LH) = \min(\min(L)H)$.*

To show that LH is convex, by (17) it suffices to show that LH has a unique minimal element (as each xL is again convex). Let $a = \min(L)$ and choose $c \in H$ with $ac := \min(aH)$. Then for each $x \in L$ and $y \in H$, we have by Proposition 3, as $a \leq x$, $ac = \min(aH) \leq \min(xH) \leq xy$. So ac is the unique minimal element in LH . Hence $\min(LH) = \min(aH) = \min(\min(L)H)$, and we have (22).

This implies:

(23) *If H and H' are closed, then HH' is closed.*

Indeed, H is an ideal, hence convex, hence by (22), HH' is convex. As $1 \in HH'$, it follows that HH' is an ideal. Similarly, $(HH')^{-1}$ is an ideal. So HH' is closed.

This gives for any closed H and $x, x \in G$:

(24)
$$xHz^{-1} = x^\downarrow H(z^\downarrow)^{-1} \cap x^\uparrow H(z^\downarrow)^{-1} \cap x^\downarrow H(z^\uparrow)^{-1} \cap x^\uparrow H(z^\uparrow)^{-1}.$$

This follows from (19) and (22), as x^\uparrow , x^\downarrow , z^\uparrow , and z^\downarrow all are convex, hence $z^\downarrow H^{-1}$ and $z^\uparrow H^{-1}$ are convex. Then (19) gives that the right-hand side in (24) is equal to $xH(z^\downarrow)^{-1} \cap xH(z^\uparrow)^{-1}$. Applying (19) to the inverse of this set, we obtain (24).

2.5. Peaks

An element of G is called a *peak* if it has precisely one maximal symbol. The peaks are precisely the join-irreducible elements of G with respect to \vee . If the maximal symbol equals α , then p is called an α -*peak*. For each $x \in G$ and symbol α , all α -peaks $p \leq x$ are totally ordered by \leq . Moreover:

(25) *Each $x \in G$ is the join of all peaks $p \leq x$.*

To see this, let y be the join of all peaks $p \leq x$. If $y \neq x$, choose a maximal symbol α of $y^{-1}x$. Then α is also a maximal symbol of x . Write $x = \xi_1 \cdot \dots \cdot \xi_n$ with symbols ξ_1, \dots, ξ_n , in such a way that the maximum j for which $\xi_j = \alpha$ is minimized. Then $q := \xi_1 \cdot \dots \cdot \xi_j$

is an α -peak with $q \leq x$. So $q \leq y$. This however contradicts the fact $y^{-1}x$ has maximal symbol α , thus showing (25).

For each $x \in G$ and symbol α , let $|x|_\alpha$ be the number of occurrences of symbol α in x (not considering α^{-1}).

Proposition 4. *Let $x, y \in G$ and let $p \leq x$ and $q \leq y$ be α -peaks satisfying $|p^{-1}x|_\alpha = |q^{-1}y|_\alpha$. Then $|pq^{-1}| \leq |xy^{-1}|$.*

Proof. By induction on $|x| + |y|$. If x is not an α -peak, let β be a maximal symbol of x with $\beta \neq \alpha$. Let $x' := x\beta^{-1}$. If β is also a maximal symbol of y , then we can apply induction to x' and $y' := y\beta^{-1}$, since $x'(y')^{-1} = xy^{-1}$. If β is not a maximal symbol of y then $|x'y^{-1}| \leq |xy^{-1}|$ (as β is not canceled in the concatenation of x and y^{-1}), and hence we can apply induction to x' and y .

So we can assume that x is an α -peak, and similarly that y is an α -peak. If $|p^{-1}x|_\alpha = 0$, then $x = p$ and $y = q$, and we are done. If $|p^{-1}x|_\alpha > 0$, then $x > p$ and $y > q$ and we can apply induction to $x' := xp^{-1}$ and $y' := yq^{-1}$. Note that $|p^{-1}x'|_\alpha = |p^{-1}x|_\alpha - 1 = |q^{-1}y|_\alpha - 1 = |q^{-1}y'|_\alpha$. ■

We use this proposition only in obtaining the following:

$$(26) \quad \text{Let } p \leq r \leq ar \text{ with } p \text{ an } \alpha\text{-peak. Then there exists } a' \text{ with } p \leq a'p, |p^{-1}a'p|_\alpha = |r^{-1}ar|_\alpha, \text{ and } |a'| \leq |a|.$$

This follows by applying Proposition 4 to $x := r$ and $y := ar$, taking for q the (unique) α -peak satisfying $p \leq q \leq ar$ with $|q^{-1}ar|_\alpha = |p^{-1}r|_\alpha$, which shows that we can take $a' := qp^{-1}$. (Note that $|q^{-1}ar|_\alpha = |p^{-1}r|_\alpha$ is equivalent to $|p^{-1}q|_\alpha = |r^{-1}ar|_\alpha$, since $|p^{-1}q|_\alpha + |q^{-1}ar|_\alpha = |p^{-1}ar|_\alpha = |p^{-1}r|_\alpha + |r^{-1}ar|_\alpha$.)

We also need:

$$(27) \quad \text{If } x \leq y \text{ and } \alpha \text{ is a symbol not occurring in } x^{-1}y, \text{ then } |xp^{-1}| \leq |yp^{-1}| \text{ for each } \alpha\text{-peak } p.$$

Indeed, by induction we can assume that $x^{-1}y = \beta$ for some symbol $\beta \neq \alpha$. Then in yp^{-1} , the maximal symbol β of y is not cancelled, since otherwise β would be maximal symbol also of p , hence $\beta = \alpha$, contradicting our assumption. Hence $|xp^{-1}| \leq |x\beta p^{-1}| = |yp^{-1}|$.

2.6. Connectedness and cyclic reducedness

We study periodicity of symbols in elements of G in order to obtain control on ‘stalling’ in the algorithm. For this we need Proposition 8 below — the other results in Sections 2.6–2.9 are only needed to prove Proposition 8.

Call an element b of G *connected* if the generators occurring in b induce a connected subgraph of $([k], F)$. So b is connected if and only if there are no $a, c \in G$ with $b = ac$, $a \neq 1 \neq c$, and a and c independent. Each peak is connected.

Call an element b of G *cyclically reduced* if $b \wedge b^{-1} = 1$. So b is cyclically reduced if and only if $b^2 = b \cdot b$. Also, if b is cyclically reduced, then for each $s \geq 0$: $b^s = b \cdot b \cdot \dots \cdot b$ (cf. [2]).

The following proposition will be used in proving Propositions 6 and 7.

Proposition 5. *Let $c, d \in G$ satisfy $d \leq dc$, $c \leq dc$, and $d \not\leq c$. Suppose that d is connected and that all minimal symbols of c occur in d . Then $|c| \leq |c \wedge d|^2$. If d is moreover cyclically reduced, then $c \leq d^{|c \wedge d|}$.*

Proof. The proof is by induction on $|c|$. Let $c' := (c \wedge d)^{-1}c$. Then $c' \leq c$, by (14), since $c, d \leq dc$. Also, $d \leq dc'$ (as $d \leq dc$ and $c' \leq c$) and $c' \leq dc'$ (by (13), as $c' \leq c \leq dc$ and $dc' \leq dc$). As $d \not\leq c$ and $c' \leq c$, we know $d \not\leq c'$. Moreover, as $c' \leq c$, all minimal symbols of c' occur in c , hence in d .

If $c' \wedge d < c \wedge d$, then by induction $|c'| \leq |c' \wedge d|^2$ and, if d is cyclically reduced, $c' \leq d^{|c' \wedge d|}$. Hence $|c| = |c \wedge d| + |c'| \leq |c \wedge d| + |c' \wedge d|^2 \leq |c \wedge d|^2$ and $c \leq c \vee d = d(c \wedge d)^{-1}c = dc' \leq d^{|c' \wedge d|+1} \leq d^{|c \wedge d|}$, as required.

So we can assume $c' \wedge d = c \wedge d$. As $c, d \leq dc$, $c \vee d < \infty$. So $(c \wedge d)^{-1}c = c'$ and $(c \wedge d)^{-1}d$ are independent. Hence $c' \wedge d = c \wedge d$ and $(c \wedge d)^{-1}d$ are independent. As $c \wedge d \neq d$ and as d is connected, we know $c \wedge d = 1$. So c and d are independent. Since all minimal symbols of c occur in d , this implies $c = 1$ and the bounds are trivial. \blacksquare

2.7. Conjugates

An element c of G is called a *conjugate* of $a \in G$ if $c = x^{-1}ax$ for some $x \in G$. Then:

(28) *For each $a \in G$, each conjugate c of a contains a segment $x^{-1}ax$ with x using only generators occurring in a .*

Indeed, let $c = y^{-1}ay$. Then (28) can be proved by induction on $|y|$. If $z := y \wedge a \neq 1$, replace y by $z^{-1}y$ and a by $z^{-1}az$, and apply induction (this resetting does not change $y^{-1}ay$). So we can assume that $y^{-1}a = y^{-1} \cdot a$ and similarly that $ay = a \cdot y$. Hence a is a segment of $y^{-1}ay$ by (6), proving (28).

We use this in proving:

(29) *If a and b are independent, then each conjugate c of ab contains a segment which is a conjugate of a .*

Indeed, by (28) c has a segment $x^{-1}abx$ with x only using generators occurring in ab . As a and b are independent we can write $x = yz$ with y only using generators occurring in a and z only using generators occurring in b . Hence $y^{-1}ay$ and $z^{-1}bz$ are independent, and so $y^{-1}ay$ is a segment of $x^{-1}abx$.

Proposition 5 implies:

Proposition 6. *Let d be connected and cyclically reduced. Then for each $n \geq 0$, each conjugate c of $d^{n+2|d|}$ contains d^n as segment.*

Proof. Choose $x \in G$ with $c = x^{-1}d^{n+2|d|x}$ and $|x|$ as small as possible. Then $d \not\leq x$, otherwise replacing x by $d^{-1}x$ contradicts the minimality of x . Let $y := x \wedge d^{n+|d|}$. Then $y \leq d^{n+|d|+1}$ and $d \leq dy \leq d^{n+|d|+1}$, hence by (13), $y \leq dy$. Since $y \leq d^{n+|d|}$, all minimal symbols of y occur in d . Hence by Proposition 5, $y \leq d^{|d|}$. So $d^{|d|} = y \cdot a$ for some a . Hence for $z := y^{-1}x$ one has $x^{-1}d^{n+|d|} = z^{-1} \cdot a \cdot d^n$, implying $x^{-1}d^{n+|d|} = x^{-1}d^{|d|} \cdot d^n$.

By symmetry, $d^{n+|d|}x = d^n \cdot d^{|d|}x$. So by (6), d^n is a segment of $x^{-1}d^{|d|}d^n d^{|d|}x = c$. \blacksquare

2.8. Periodicity

We give conditions for the eventual periodicity of a peak:

Proposition 7. *Let q be connected and contain symbol α , and let p be an α -peak with $p \leq pq$. Then there exists an α -peak r and $t \geq 0$ with $p = r \cdot q^t$ and $|r| \leq 2|pqp^{-1}|^2$.*

Proof. Let $a := pqp^{-1}$. Then $|q| \leq |a|$, as $|p| + |q| = |pq| = |ap| \leq |p| + |a|$. If $|p| \leq 2|a|^2$, we can take $r := p$ and $t := 0$. So we can assume that $|p| > 2|a|^2$.

Let $m := |q|_\alpha$, and let p' be minimal with the properties that $p' \leq p$ and $|p'|_\alpha \geq |p|_\alpha - m$. Then $p' = 1$ or p' is an α -peak. By showing that $(p')^{-1}p = q$ we are done, since then we can apply induction, as $p'q(p')^{-1} = pqp^{-1}$.

Define $c := p^{-1}$, $d := q^{-1}$, and $u := c \wedge dc$. As $u \leq c$, $u \leq dc$, and $d \leq dc$, we know $u \leq du = d \cdot u$ (by (13)). Since $a = (dc)^{-1}c$, we have $|a| = |c| + |dc| - 2|u|$, and so, as $|c| = |p| > 2|a|^2$ and $|a| \geq |d|$:

$$(30) \quad 2|u| = |c| + |dc| - |a| = 2|c| + |d| - |a| > 4|a|^2 + |d| - |a| \geq 2|d|^2 + 2|d|.$$

Hence $|u| > |d|^2 + |d| \geq |u \wedge d|^2$. Therefore, $d \leq u$ by Proposition 5.

Let $b := p^{-1}p'$. So we must show $b = d$. Since $q^{-1} = d \leq u \leq c = p^{-1}$ and $m = |q|_\alpha$, we know $p' \leq pq^{-1}$, that is, $d \leq b$. On the other hand, $b \wedge u \leq d$, since $b \wedge u \leq u \leq dc = q^{-1}p^{-1}$, $|b \wedge u|_\alpha \leq m = |q|_\alpha$, and p is a peak. So $b \wedge u = d$, and hence we must show $b \leq u$.

Let $u' := d^{-1}u$. Since $d \leq u \leq du$, we have $u' \leq u$ (by (10)). Moreover, as $b, u \leq c$, $b \vee u < \infty$. Hence $u' \leq u \leq b \vee u = bd^{-1}u = b \cdot u'$. Since $|u'| = |u| - |d| > |d|^2 = |u \wedge b|^2 \geq |u' \wedge b|^2$, Proposition 5 gives $b \leq u'$, implying $b \leq u$. \blacksquare

A g.c.d argument shows:

$$(31) \quad \text{Let } p, r, r', q, q' \text{ be } \alpha\text{-peaks, and let } t, t' \geq 0 \text{ with } p = r \cdot q^t = r' \cdot (q')^{t'} \text{ and } \frac{1}{3}|p|_\alpha \geq |r|, |r'|, |q|, |q'|. \text{ Then there exists } d \text{ such that } q \text{ and } q' \text{ are powers of } d.$$

Indeed, define $g := \max\{|r|, |r'|, |q|, |q'|\}$, $m := |p|_\alpha$, $u := |q|_\alpha$, and $u' := |q'|_\alpha$. So $m \geq 3g$. As p is an α -peak, we can uniquely write $p = p_1 \cdot p_2 \cdot \dots \cdot p_m$, with each p_i being an α -peak. Since r is an α -peak with $|r|_\alpha \leq g$, and since $p = r \cdot q^t$, the sequence $z := (p_{g+1}, \dots, p_m)$ is periodic with period u . Similarly, z is periodic with period u' . Moreover, z has at least $u + u'$ terms, since $m - g \geq 2g \geq u + u'$. This implies² that z is periodic with period $v := \gcd\{u, u'\}$. Let $d := z_{m-v+1} \cdot \dots \cdot z_m$. Then $q = z_{m-u+1} \cdot \dots \cdot z_m = d^{u/v}$ and similarly $q' = d^{u'/v}$.

² If a and b are periods of $x = (x_1, \dots, x_n)$ and $n \geq a + b$, then $a - b$ is a period of x : if $i \leq n - a$ then $x_{i+(a-b)} = x_{(i+a)-b} = x_{i+a} = x_i$; if $n - a < i \leq n - (a - b)$ then $i > b$, hence $x_{i+(a-b)} = x_{(i-b)+a} = x_{i-b} = x_i$.

2.9. A main tool

We now come to a main tool for bounding the complexity of our algorithm (which we will use in Section 3.3).

Proposition 8. *Let p be an α -peak, and let $a, a' \in G$ be such that $p \leq ap$ and $p \leq a'p$ and such that α occurs in $p^{-1}a'p$. If $|p| \geq 8|a|^3$, then each conjugate of a has a segment s satisfying $|p^{-1}aps^{-1}|_\alpha \leq 2|a'|^2$.*

Proof. We can assume $|p^{-1}ap|_\alpha > 2|a'|^2$, as otherwise we can take $s := 1$. This implies $|a| \geq |ap| - |p| = |p^{-1}ap| \geq |a'|$.

Let q be the component of $p^{-1}ap$ that contains α ; that is, q is the element such that $p^{-1}ap = qu$ for some u independent of q , with q connected and containing α . Similarly, let q' be the component of $p^{-1}a'p$ that contains α . Note that $|q| \leq |a|$ and $|q'| \leq |a'|$.

By (27) applied to $x := pq$ and $y := ap$ we have $|pqp^{-1}| \leq |a|$. Hence, by Proposition 7, $p = r \cdot q^t$ for some α -peak r with $|r| \leq 2|pqp^{-1}|^2 \leq 2|a|^2$. Similarly, $p = r' \cdot (q')^{t'}$ for some α -peak r' with $|r'| \leq 2|a'|^2 \leq 2|a|^2$. Now

$$(32) \quad |p|_\alpha \geq t = (|p| - |r|)/|q| \geq (8|a|^3 - 2|a|^2)/|a| \geq 6|a|^2 \geq 3 \max\{|r|, |r'|, |q|, |q'|\},$$

Hence by (31), $q = d^n$ for some d and $n \geq 0$, with $|d| \leq |q'| \leq |a'|$. As $t \geq 1$ by (32), q is cyclically reduced, hence also d is cyclically reduced. As q is connected, also d is connected.

Let c be a conjugate of a . Then c is a conjugate of $p^{-1}ap = qu$, with u independent of q . Hence by (29), c contains a segment c' which is a conjugate of $q = d^n$. Now $n = |q|/|d| \geq |p^{-1}ap|_\alpha/|a'| > 2|a'| \geq 2|d|$. Hence, by Proposition 6, c' contains $s := d^{n-2|d|}$ as segment. Then $qs^{-1} = d^{2|d|}$, hence $|p^{-1}aps^{-1}|_\alpha = |qs^{-1}|_\alpha \leq |qs^{-1}| = 2|d|^2 \leq 2|a'|^2$. \blacksquare

2.10. The function $\mu_{a,H} : x \mapsto \min(a^{-1}x^\uparrow H)$

The following function $\mu_{a,H} : G \rightarrow G$ forms an important ingredient in our algorithm. Fixing $a \in G$ and a closed set $H \subseteq G$, it is defined by

$$(33) \quad \mu_{a,H}(x) := \min(a^{-1}x^\uparrow H)$$

for $x \in G$, which is well-defined as $a^{-1}xH$ is convex by (22). So for each $x \in G$:

$$(34) \quad \mu_{a,H}(x) \leq a^{-1}x.$$

Moreover, for all $x, y \in G$:

$$(35) \quad \mu_{a,H}(x) \leq y \text{ if and only if } a \in x^\uparrow H(y^\downarrow)^{-1},$$

since $\min(a^{-1}x^\uparrow H) \leq y$ if and only if $a^{-1}bh = c$ for some $b \in x^\uparrow$, $h \in H$ and $c \in y^\downarrow$.

Proposition 9. *Let $a \in G$ and $H \subseteq G$ be closed, and set $\mu := \mu_{a,H}$. Then for all $x, y \in G$:*

- $$(36) \quad \begin{aligned} & \text{(i) if } x \leq y \text{ then } \mu(x) \leq \mu(y), \\ & \text{(ii) } \mu(x \wedge y) \leq \mu(x) \wedge \mu(y), \\ & \text{(iii) if } x \vee y \text{ is finite, then } \mu(x \vee y) = \mu(x) \vee \mu(y). \end{aligned}$$

Proof. Since $x \leq y$ implies $x^\uparrow \supseteq y^\uparrow$, we have (i). Then (ii) follows from (i). To see (iii), we have $\mu(x) \vee \mu(y) \leq \mu(x \vee y)$ by (i). In particular, $\mu(x) \vee \mu(y)$ is finite. To see the reverse inequality, set $d := \mu(x) = \min(a^{-1}x^\uparrow H)$ and $e := \mu(y) = \min(a^{-1}y^\uparrow H)$. Then, by (35) and as both d^{-1} and e^{-1} belong to $((d \vee e)^\downarrow)^{-1}$,

$$(37) \quad a \in x^\uparrow H d^{-1} \cap y^\uparrow H e^{-1} \subseteq x^\uparrow H ((d \vee e)^\downarrow)^{-1} \cap y^\uparrow H ((d \vee e)^\downarrow)^{-1} = (x \vee y)^\uparrow H ((d \vee e)^\downarrow)^{-1},$$

where the equality follows from (19) (as $x^\uparrow \cap y^\uparrow = (x \vee y)^\uparrow$). So by (35), $\mu(x \vee y) = \min(a^{-1}(x \vee y)^\uparrow H) \leq d \vee e$. \blacksquare

The composition of functions $\mu_{a,H}$ have the following property. Let $x, a, a' \in G$ and let H and H' be closed. Then

$$(38) \quad \mu_{a',H'}(\mu_{a,H}(x)) \leq \mu_{aa',HH'}(x).$$

Indeed, by the definitions of HH' and $\mu_{aa',HH'}(x)$, there exist $x' \geq x$ and $c \in H, c' \in H'$ with $\mu_{aa',HH'}(x) = (aa')^{-1}x'cc'$. Now $\mu_{a,H}(x) \leq a^{-1}x'c$. Hence, using (36)(i),

$$(39) \quad \mu_{a',H'} \circ \mu_{a,H}(x) \leq \mu_{a',H'}(a^{-1}x'c) \leq (a')^{-1}(a^{-1}x'c)c' = \mu_{aa',HH'}(x).$$

2.11. Polynomial-time algorithms

Let I be an ideal and $x \in G$. Then there is a unique largest element $y \leq x$ with $y \in I$. We can find it in polynomial time if membership of I can be tested in polynomial time:

$$(40) \quad \text{Let } I \text{ be an ideal of which we can test membership in polynomial time. Then for any } x \in G, \text{ we can find the maximal element } y \leq x \text{ with } y \in I \text{ in polynomial time.}$$

To see this, grow a word $y \leq x$ with $y \in I$, starting with $y = 1$. If there is a minimal symbol in $y^{-1}x$ with $y\alpha \in I$, replace y by $y\alpha$. If no such α exists, y is as required.

Note that y is the closest (with respect to dist) element in I to x . Hence, by the left-invariance of the distance function, $x^{-1}y$ is the closest element in $x^{-1}I$ to 1. That is: $x^{-1}y = \min(x^{-1}I)$. Therefore:

$$(41) \quad \text{Let } I \text{ be an ideal of which we can test membership in polynomial time. Then for any } z \in G, \text{ we can find } \min(zI) \text{ in polynomial time.}$$

Note that we can test membership of x^\uparrow and of x^\downarrow in polynomial time. Hence:

$$(42) \quad \text{For any } y, x \in G, \text{ we can find } \min(y^{-1}x^\uparrow) \text{ in polynomial time.}$$

This follows from (41) setting $I := x^{-1}x^\uparrow$ and $z := y^{-1}x$.

If H is closed, then for any $y \in G$, $y^{-1}x^\uparrow H$ is convex (by (22)). Its minimum can be found in polynomial time:

$$(43) \quad \text{Let } H \text{ be a closed set of which we can test membership in polynomial time. Then for any } x, a \in G, \mu_{a,H}(x) = \min(a^{-1}x^\uparrow H) \text{ can be found in polynomial time.}$$

Indeed, by (22), $\min(a^{-1}x^\uparrow H) = \min(\min(a^{-1}x^\uparrow)H)$. Hence (43) follows from (42) and (41).

Finally,

(44) Let H be a closed set of which we can test membership in polynomial time. Then for any $x, y, z \in G$, we can test in polynomial time if y belongs to $x^\uparrow H(z^\uparrow)^{-1}$.

Indeed, $y \in x^\uparrow H(z^\uparrow)^{-1}$ if and only if $y^{-1}x^\uparrow H \cap z^\uparrow \neq \emptyset$. This is the case if and only if $z^{-1}y^{-1}x^\uparrow H \cap z^{-1}z^\uparrow \neq \emptyset$. The latter statement is equivalent to: $\min(z^{-1}y^{-1}x^\uparrow H)$ belongs to the ideal $z^{-1}z^\uparrow$. As this minimum can be determined in polynomial time by (43) and as membership of $z^{-1}z^\uparrow$ can be tested in polynomial time, we have proved (44).

3. The cohomology feasibility problem

Let $D = (V, E)$ be a directed graph and let G be a group. Two functions $\varphi, \psi : E \rightarrow G$ are called *cohomologous* if there exists a function $f : V \rightarrow G$ such that $\psi(e) = f(u)^{-1}\varphi(e)f(w)$ for each edge $e = (u, w)$. One directly checks that this gives an equivalence relation.

We give a polynomial-time algorithm for the following *cohomology feasibility problem* for graph groups:

(45) *given:* a directed graph $D = (V, E)$, an undirected graph $([k], F)$, a function $\varphi : E \rightarrow G_F$, and for each edge e a closed set $H(e) \subseteq G_F$,
find: a function $\psi : E \rightarrow G_F$ such that ψ is cohomologous to φ and such that $\psi(e) \in H(e)$ for each $e \in E$.

The running time of the algorithm for this problem is bounded by a polynomial in $n := |V|$, $m := |E|$, $\sigma := \max\{|\varphi(e)| \mid e \in E\}$, and the maximum time needed to test if any word of polynomial size belongs to $H(e)$ (over all edges e). (The number k of generators can be bounded by $m\sigma$, since we may assume that all generators occur among the $\varphi(e)$.) More precisely, there exist polynomials p_1 and p_2 such that the problem takes time $p_1(n, m, \sigma)\tau_H(p_2(n, m, \sigma))$, where $\tau_H(x)$ is the time needed to test membership of the $H(e)$ for words of size at most x .

Note that, by the definition of cohomologous, equivalent to finding a function ψ as in (45), is finding a function $f : V \rightarrow G_F$ satisfying:

(46) $f(u)^{-1}\varphi(e)f(w) \in H(e)$ for each edge $e = (u, w)$.

We call such a function f *feasible*.

We can assume that

(47) $|\varphi(e)| \leq 1$ for each edge e .

Indeed, if $\varphi(e) = xy$ for edge $e = (u, w)$, we can split the edge into two edges $(u, v), (v, w)$, where v is a new vertex, and define $\varphi(u, v) := x$, $\varphi(v, w) := y$, $H(u, v) := H(e)$, and $H(v, w) := \{1\}$. The new problem is equivalent to the original problem: if f is a solution to the original problem, we can set $f(v) := yf(w)$, and obtain a solution for the new problem;

conversely, if f is a solution to the original problem, forgetting the value of f on v , we obtain a solution to the original problem.

3.1. Pre-feasible functions

Given the input of the cohomology feasibility problem (45), we call a function $f : V \rightarrow G_F$ *pre-feasible* if for each edge $e = (u, w)$ of D there exist $x \geq f(u)$ and $z \leq f(w)$ such that $x^{-1}\varphi(e)z \in H(e)$. Clearly, each feasible function is pre-feasible. There is a trivial pre-feasible function f , defined by $f(v) := 1$ for each $v \in V$. Note that f is pre-feasible if and only if

$$(48) \quad \mu_{\varphi(e), H(e)}(f(u)) \leq f(w) \text{ for each edge } e = (u, w).$$

The collection G_F^V of all functions $f : V \rightarrow G_F$ can be partially ordered by: $f \leq g$ if and only if $f(v) \leq g(v)$ for each $v \in V$. Then G_F^V forms a lattice if we add an element ∞ at infinity. Let \wedge and \vee denote meet and join. Then (36)(ii) and (iii) directly give:

$$(49) \quad \text{Let } f_1 \text{ and } f_2 \text{ be pre-feasible functions. Then } f_1 \wedge f_2 \text{ and, if } f_1 \vee f_2 < \infty, f_1 \vee f_2 \text{ are pre-feasible again.}$$

It follows that for each function $f : V \rightarrow G_F$ there is a unique smallest pre-feasible function $\bar{f} \geq f$, provided that there exists at least one pre-feasible function $g \geq f$. If no such g exists we set $\bar{f} := \infty$. By (49), $\overline{f \vee g} = \bar{f} \vee \bar{g}$ for any two functions f, g with $f \vee g$ finite.

3.2. A subroutine finding \bar{f}

Condition (48) suggests a ‘constraint satisfaction’ algorithm to find \bar{f} for a given function f . Let input $D = (V, E), F, \varphi, H$ for the cohomology feasibility problem be given. For any edge e , we can determine $\mu_{\varphi(e), H(e)}(f(u))$ in polynomial time, by (43).

Subroutine to find \bar{f} : Find an edge $e = (u, w)$ for which $m_e := \mu_{\varphi(e), H(e)}(f(u)) \not\leq f(w)$. If $m_e \vee f(w)$ is finite, reset $f(w) := m_e \vee f(w)$ and iterate. If $m_e \vee f(w) = \infty$, output $\bar{f} := \infty$. If no such edge e exists, output $\bar{f} := f$.

Then the output of the subroutine (if any) is correct. For let f' be the reset function. If \bar{f} is finite, then $f \leq f' \leq \bar{f}$, since $f'(w) = \mu_{\varphi(e), H(e)}(f(u)) \vee f(w) \leq \mu_{\varphi(e), H(e)}(f(u)) \vee \bar{f}(w) = \bar{f}(w)$, since \bar{f} is pre-feasible. So in this case $\bar{f}' = \bar{f}$. This moreover implies that if $m_e \vee f(w) = \infty$ then $\bar{f} = \infty$.

3.3. Running time of the subroutine

For each walk $P = e_1 e_2 \dots e_t$, where e_1, \dots, e_t are consecutive edges of D , we set $\varphi(P) := \varphi(e_1)\varphi(e_2)\dots\varphi(e_t)$ and $H(P) := H(e_1)H(e_2)\dots H(e_t)$.

We will study the running time of the subroutine under the condition that the cohomology feasibility problem has a solution, or more weakly, that

$$(50) \quad \text{for each closed walk } C, H(C) \text{ contains a conjugate of } \varphi(C).$$

For any function $f : V \rightarrow G_F$, let $|f| := \max\{|f(v)| \mid v \in V\}$.

Proposition 10. *If (47) and (50) hold, then for any $f : V \rightarrow G_F$ with \bar{f} finite:*

$$(51) \quad |\bar{f}| \leq k|f| + 220k(2nk)^9.$$

Proof. Define $c := 2nk$ and $m := 3c^2$, and suppose $|\bar{f}| > k|f| + 220kc^9$. Consider the (infinite) directed graph \mathcal{D} with vertex set $\mathcal{V} := \{(v, p) \mid v \in V, p \text{ peak}\}$ and edge set all pairs $((v, p), (w, q)) \in \mathcal{V} \times \mathcal{V}$ with $e = (v, w) \in E$ and $q \leq \mu_{\varphi(e), H(e)}(p)$. Let \mathcal{S} be the set of all vertices (v, p) of \mathcal{D} with $p \leq f(v)$. Then for each $v \in V$ and peak p :

$$(52) \quad \bar{f}(v) = \bigvee \{p \mid \text{there exists a walk in } \mathcal{D} \text{ from } \mathcal{S} \text{ to } (v, p)\}.$$

This follows with (25) and Proposition 9(iii). Hence:

$$(53) \quad \text{if } p \leq \bar{f}(v), \text{ then there exists a walk in } \mathcal{D} \text{ from } \mathcal{S} \text{ to } (v, p).$$

Since $|\bar{f}| > k|f| + 220kc^9$, there exists $w \in V$ with $|\bar{f}(w)| > k|f| + 220kc^9$. Hence there exists a peak $q \leq \bar{f}(w)$ with $|q| > |f| + 220c^9$ (since $\bar{f}(w)$ is a join of at most k peaks). By (53), there is a walk \mathcal{P} in \mathcal{D} from \mathcal{S} to (w, q) . Choose a shortest such walk \mathcal{P} ; let it have length ℓ . Since $|\varphi(e)| \leq 1$ by assumption, $|p| \leq |\varphi(e)^{-1}p'| \leq |p'| + 1$ for each edge $((v, p), (v', p'))$ of \mathcal{D} , where $e = (v, v')$. Hence $\ell \geq |q| - |f| > 220c^9$.

Let \mathcal{P} traverse vertices $(v_0, q_0), (v_1, q_1) \dots, (v_\ell, q_\ell)$ of \mathcal{D} , in this order. So $v_\ell = w$ and $q_\ell = q$. For each $u \in V$ and each symbol α let $I_{u, \alpha}$ denote the set of indices $j \in \{\ell - 3c^3, \dots, \ell\}$ such that $v_j = u$ and q_j is an α -peak. Then there exist $u \in V$ and a symbol α such that $|I_{u, \alpha}| > 3c^3/c = m$. Choose $j_0 < j_1 < \dots < j_m$ in $I_{u, \alpha}$. Set $p_i := q_{j_i}$ for $i = 0, 1, \dots, m$. So each p_i is an α -peak and $p_0 < p_1 < \dots < p_m$. We will apply Proposition 8 to $p := p_0$.

Let C_i be the $u - u$ walk $v_{j_{i-1}}, v_{j_{i-1}+1}, \dots, v_{j_i-1}, v_{j_i}$ in D , and let $C := C_1 C_2 \dots C_m$ (the concatenation of C_1, \dots, C_m). Then $|C| \leq 3c^3$ and so there exists $i \in \{1, \dots, m\}$ with $|C_i| \leq |C|/m \leq 3c^3/m = c$.

Let $a := \varphi(C)^{-1}$ and $a'' := \varphi(C_i)^{-1}$. Then $|a| = |\varphi(C)| \leq |C| \leq 3c^3$ and $|a''| = |\varphi(C_i)| \leq |C_i| \leq c$. This gives

$$(54) \quad |p_0| \geq |q| - 3c^3 \geq 220c^9 - 3c^3 \geq 216c^9 = 8(3c^3)^3 \geq 8|a|^3.$$

Moreover, $p_0 \leq p_m \leq \mu_{\varphi(C), H(C)}(p_0) \leq ap_0$ and similarly $p_i \leq p_{i+1} \leq a''p_i$. As α occurs in $p_i^{-1}p_{i+1}$, α also occurs in $p_i^{-1}a''p_i$. By (26), $p_0 \leq a'p_0$ for some a' with $|a'| \leq |a''|$ and $|p_0^{-1}a'p_0|_\alpha = |p_i^{-1}a''p_i|_\alpha$.

As we assume that (50) holds, some conjugate x of $\varphi(C)$ belongs to $H(C)$. So x^{-1} is a conjugate of a . Hence, by Proposition 8, x^{-1} has a segment s such that $|p_0^{-1}ap_0s^{-1}|_\alpha \leq 2|a'|^2 \leq 2c^2 < 3c^2 = m$. Now $p_m \leq \min(\varphi(C)^{-1}p_0^\uparrow H(C)) \leq ap_0s^{-1}$, since $s^{-1} \in H(C)$, as $H(C)$ is closed and $x \in H(C)$. So $|p_0^{-1}p_m|_\alpha < m$, contradicting the fact that p_m contains at least m α 's more than p_0 . \blacksquare

This implies, where $n := |V(D)|$, and where, for any $\sigma \in \mathbb{Z}_+$, $\tau_H(\sigma)$ is the maximum time needed to test if any word of size $\leq \sigma$ belongs to $H(e)$, for any given edge e .

Theorem 1. *There exist polynomials p_1 and p_2 such that, if (47) and (50) hold, then the*

running time of the subroutine is bounded by $p_1(n, k, |f|)\tau_H(p_2(n, k, |f|))$.

Proof. At each iteration, we increase $|f(v)|$ for some vertex v . Hence Proposition 10 implies that, if \bar{f} is finite, the number of iterations is bounded by some polynomial p_1 in $n, k, |f|$. If the subroutine exceeds this number of iterations, we conclude that $\bar{f} = \infty$.

Since in each iteration, the reset f' satisfies $|f'| \leq |\bar{f}|$, and since $|\bar{f}|$ is bounded by a polynomial p_2 in $n, k, |f|$, in each iteration we only need to test membership of words of size at most $p_2(n, k, |f|)$. \blacksquare

3.4. A polynomial-time algorithm for the cohomology feasibility problem for graph groups

We now describe the algorithm for the cohomology feasibility problem for graph groups. Let input $D = (V, E)$, F , φ , H of (45) be given.

Let \mathcal{F} be the collection of all functions $f : V \rightarrow G_F$ such that for each edge $e = (u, w)$ of D there exist $x \geq f(u)$ and $z \geq f(w)$ satisfying $x^{-1}\varphi(e)z \in H(e)$; equivalently:

$$(55) \quad \varphi(e) \in f(u)^\uparrow H(e)(f(w)^\uparrow)^{-1}.$$

This can be tested in polynomial time by (44). So for any given function f one can check in polynomial time whether f belongs to \mathcal{F} . Trivially, if $f \in \mathcal{F}$ and $g \leq f$ then $g \in \mathcal{F}$. Moreover:

$$(56) \quad \text{Let } f_1, \dots, f_t \text{ be functions such that } f_i \vee f_j \in \mathcal{F} \text{ for all } i, j. \text{ Then } f := f_1 \vee \dots \vee f_t \in \mathcal{F}.$$

Proof. We must show that for each edge $e = (u, w)$, $\varphi(e)$ belongs to $f(u)^\uparrow H(e)(f(w)^\uparrow)^{-1}$. Since $f_i \vee f_j \in \mathcal{F}$ for all i, j , we know

$$(57) \quad \varphi(e) \in f_i(u)^\uparrow H(e)f_j(w)^\uparrow^{-1}$$

for all i, j . Hence by (19),

$$(58) \quad \varphi(e) \in \bigcap_i \bigcap_j f_i(u)^\uparrow H(e)(f_j(w)^\uparrow)^{-1} = \left(\bigcap_i f_i(u)^\uparrow \right) H(e) \left(\bigcap_j (f_j(w)^\uparrow)^{-1} \right) = f(u)^\uparrow H(e)(f(w)^\uparrow)^{-1}.$$

Here i and j range over $1, \dots, t$. \blacksquare

In the following theorem, ‘solvable in polynomial time’ means as before that there exist polynomials p_1 and p_2 such that the problem is solvable in time $p_1(n + m, k, \rho)\tau_H(p_2(n + m, k, \rho))$, where $n := |V(D)|$, $m := |E(D)|$, ρ is the maximum of $|\varphi(e)|$ over all $e \in E$, and where $\tau_H(\sigma)$ again is the maximum time needed to test if any word of size $\leq \sigma$ belongs to $H(e)$, for any given edge e .

Theorem 2. *The cohomology feasibility problem for graph groups is solvable in polynomial time.*

Proof. We can assume again that $|\varphi(e)| \leq 1$ for each edge e . Moreover, we can assume that with each edge $e = (u, w)$ also $e^{-1} = (w, u)$ is an edge, with $\varphi(e^{-1}) = \varphi(e)^{-1}$ and $H(e^{-1}) = H(e)^{-1}$.

For any $e = (u, w) \in E$, let f_e be the function defined by

$$(59) \quad f_e(v) := \begin{cases} \varphi(e) & \text{if } v = u, \\ 1 & \text{if } v \neq u. \end{cases}$$

Let L be the set of pairs $\{e, e^{-1}\}$ from E such that $\varphi(e) \notin H(e)$. Let N be the collection of all pairs $\{e, d\}$ from E such that the function $\bar{f}_e \vee \bar{f}_d = \infty$, or is finite and does not belong to \mathcal{F} (possibly $e = d$).

Choose a subset B of E such that B intersects each pair in L and such that B contains no pair in N . This is a special case of the 2-satisfiability problem, and hence can be solved in polynomial time. Assuming that there exists a feasible function f , then B exists, as $B := \{e = (u, v) \in E \mid \varphi(e) \leq f(u)\}$ would have the required properties.

If we find B , define f by:

$$(60) \quad f(v) := \bigvee_{e \in B} \bar{f}_e.$$

We are done by proving that f is feasible. Since $\bar{f}_e \vee \bar{f}_b < \infty$ for each pair $\{e, d\} \subseteq B$, we know $f < \infty$. Moreover, f is the join of a finite number of pre-feasible functions, and hence f is pre-feasible. So by (24) it suffices to prove that for each edge $e = (u, w)$:

- (61) (i) there exist $x \geq f(u)$ and $z \geq f(w)$ such that $x^{-1}\varphi(e)z \in H(e)$,
(ii) there exist $x \leq f(u)$ and $z \leq f(w)$ such that $x^{-1}\varphi(e)z \in H(e)$.

To prove (61)(i), note that it is equivalent to: $f \in \mathcal{F}$. As $\bar{f}_e \vee \bar{f}_b \in \mathcal{F}$ for all $a, b \in B$, (56) gives $f \in \mathcal{F}$.

To prove (61)(ii), if it does not hold then $\varphi(e) \notin H(e)$, hence $\{e, e^{-1}\} \in L$. So e or e^{-1} belongs to B . By symmetry, we can assume that $e \in B$. So $f_e \leq f$, and therefore $\varphi(e) \leq f(u)$. So we can take $x := \varphi(e)$ and $z := 1$ in (61)(ii). \blacksquare

An analysis of this algorithm shows that the cohomology feasibility problem has a solution if and only if for each vertex u and each pair C, C' of (undirected) $u - u$ walks in D there exists $x \in G$ such that $x^{-1}\varphi(C)x \in H(C)$ and $x^{-1}\varphi(C')x \in H(C')$. This condition is trivially necessary.

4. Planar graphs

We repeat the *partially disjoint paths problem* for directed planar graphs:

- (62) *given:* a directed planar graph $D = (V, E)$, vertices $r_1, s_1, \dots, r_k, s_k$ of D , and a set F of pairs $\{i, j\}$ with $i, j \in [k]$,
find: a k -tuple $\mathcal{P} = (P_1, \dots, P_k)$, where P_i is a directed $r_i - s_i$ path P_i , for $i = 1, \dots, k$, such that P_i and P_j are disjoint whenever $\{i, j\} \in F$.

We can assume without loss of generality:

$$(63) \quad r_1, s_1, \dots, r_k, s_k \text{ are distinct, each } r_i \text{ has outdegree 1 and indegree 0, and each } s_i \text{ has indegree 1 and outdegree 0.}$$

Again, let G_F be the graph group generated by g_1, \dots, g_k and relations $g_i g_j = g_j g_i$ whenever $\{i, j\} \notin F$. For each solution \mathcal{P} of (62), let $\chi_{\mathcal{P}} : E \rightarrow G_F$ be defined by:

$$(64) \quad \chi_{\mathcal{P}}(e) := \prod_{\substack{i \\ P_i \text{ traverses } e}} g_i.$$

The order in which we take this product is irrelevant, since if both P_i and P_j traverse e , then $\{i, j\} \notin F$ and hence $g_i g_j = g_j g_i$.

Let \mathcal{F} be the collection of faces of D . Call $\varphi, \psi : E \rightarrow G_F$ *homologous* if there exists $f : \mathcal{F} \rightarrow G_F$ such that for each edge e : $\psi(e) = f(F)^{-1} \varphi(e) f(F')$, where F and F' are the left-hand and the right-hand face at e , respectively (seen when traversing e in forward direction).

4.1. Finding partially disjoint paths of prescribed homology

We first consider the homology version of the partially disjoint paths problem:

$$(65) \quad \begin{aligned} &\textit{given:} \text{ a directed planar graph } D = (V, E), \text{ vertices } r_1, s_1, \dots, r_k, s_k \text{ of } D \text{ satisfying} \\ &\quad (63), \text{ a set } F \text{ of pairs } \{i, j\} \text{ with } i, j \in [k], \text{ and a function } \varphi : E \rightarrow G_F, \\ &\textit{find:} \text{ a solution } \mathcal{P} \text{ of (62) such that } \chi_{\mathcal{P}} \text{ is homologous to } \varphi. \end{aligned}$$

Proposition 11. *Problem (65) is solvable in polynomial time.*

Proof. We can assume that problem (65) has a solution — that is, φ is homologous to $\chi_{\mathcal{P}}$ for some solution \mathcal{P} of (62).

Let \mathcal{F} be the collection of faces of D . Consider the dual directed graph $D^* = (\mathcal{F}, E^*)$, where for each edge e of D there is a directed edge $e^* \in E^*$ from the face at the left-hand side of e to the face at the right-hand side of e . We define $\hat{\varphi}(e^*) := \varphi(e)$ and

$$(66) \quad H(e^*) := \{\prod_{i \in I} g_i \mid I \subseteq [k], I \text{ stable set in } ([k], F)\},$$

where I is *stable* if it contains no pair in F as subset. Note that $H(e^*)$ is a closed subset of G_F .

We extend the planar graph D^* by a number of further ‘nonplanar’ edges, as follows. Consider any vertex $v \notin \{r_1, s_1, \dots, r_k, s_k\}$ of D and two faces F and F' of D incident with v . Let e_1, \dots, e_t be the edges incident with v that are crossed when going clockwise from F to F' around v . Then add to D^* an edge $e_{v, F, F'}$ from F to F' , and define

$$(67) \quad \hat{\varphi}(e_{v, F, F'}) := \varphi(e_1)^{\sigma_1} \dots \varphi(e_t)^{\sigma_t},$$

where, for $j = 1, \dots, t$, $\sigma_j := 1$ if e_j is oriented away from v and $\sigma_j := -1$ if e_j is oriented towards v . Note that, as by assumption φ is homologous to $\chi_{\mathcal{P}}$ for some solution \mathcal{P} of (62),

we necessarily have $\hat{\varphi}(e_{v,F',F}) = \hat{\varphi}(e_{v,F,F'})^{-1}$.

Moreover, define

$$(68) \quad H(e_{v,F,F'}) := \{\prod_{i \in I} g_i^{\tau(i)} \mid I \subseteq [k], I \text{ stable set in } ([k], F), \tau : I \rightarrow \{+1, -1\}\}.$$

Also $H(e_{v,F,F'})$ is a closed subset of G_F .

Let \hat{D} be the extended directed graph, and consider the cohomology feasibility problem Π for $(\hat{D}, \hat{\varphi})$, in which we require that the output is only weakly allowed on the edges in E' . As, by assumption, φ is homologous to $\chi_{\mathcal{P}}$ for some \mathcal{P} , problem Π has a solution, namely $\chi_{\mathcal{P}}$. Conversely, let $\psi : E(\hat{D}) \rightarrow G_F$ be any solution of Π . Define $\check{\psi} : E(D) \rightarrow G_F$ by $\check{\psi}(e) := \psi(e^*)$ for $e \in E(D)$. Then $\check{\psi}$ is equal to $\chi_{\mathcal{P}}$ for some \mathcal{P} . This because, by our assumption, φ is homologous to $\chi_{\mathcal{P}}$ for some \mathcal{P} . Hence, for the edge e incident with r_i , $\varphi(e)$ is conjugate to g_i (as by (63) e is incident at both sides to the same face), therefore $\check{\psi}(e)$ is conjugate to g_i . Since $\check{\psi}(e)$ is allowed, it follows that in fact $\check{\psi}(e) = g_i$. So only P_i traverses e , and in the forward direction. Similarly for the edge incident with s_i .

Therefore, the proposition follows from Theorem 2. ■

4.2. Enumerating homologies of disjoint paths

We finally describe an algorithm that finds, for any input of (62), a collection Φ of functions $\varphi : E \rightarrow G_F$ with the property that

$$(69) \quad \text{for each solution } \mathcal{P} \text{ of (62), } \chi_{\mathcal{P}} \text{ is homologous to at least one } \varphi \in \Phi.$$

So, although there exist infinitely many homology classes (if $F \neq \emptyset$), in our algorithm we can restrict ourselves to a number of homology classes that, for fixed k , is bounded by a polynomial in the size of the graph.

Proposition 12. *Fixing k , a collection Φ satisfying (69) can be found in polynomial time.*

Proof. Again, we can assume (63). Moreover, we can assume that D is weakly connected and that (for the convenience of the exposition) each $i \in [k]$ is contained in at least one pair $\{i, j\}$ in F (otherwise we can easily reduce the problem). We also can assume that each vertex $v \neq r_1, s_1, \dots, r_k, s_k$ has total degree $\deg(v)$ equal to 3: replace v by a directed circuit of length $\deg(v)$ and attach the edges incident with v to the different vertices of the circuit (in a planar manner of course). Any φ found for the modified graph can be ‘shrunk’ to the smaller graph.

Choose a spanning tree T in D . We will consider graphs T' obtained from T by replacing each edge e by a number (possibly 0) of parallel edges. These edges form a *parallel class*, denoted by π_e . Each such graph T' is trivially planar, by drawing the edges properly parallel in the plane.

We moreover consider undirected walks in such graphs T' . (An *undirected walk* may traverse edges in any direction.) Call undirected walks W and W' in T' *crossing* if there is a vertex v and distinct edges e_1, e_2, e_3, e_4 of T' incident with v , in clockwise or counter-clockwise order, such that W traverses e_1 and e_3 consecutively, and W' traverses e_2 and e_4 consecutively. If $W = W'$, we say that W is *self-crossing*.

In particular, we consider k -tuples $\mathcal{W} = (W_1, \dots, W_k)$ of undirected walks in T' such that

- (70) (i) W_i runs from r_i to s_i and is not self-crossing, for each $i = 1, \dots, k$,
(ii) W_i and W_j are not crossing, for each $\{i, j\} \in F$,
(iii) each edge of T' is traversed by precisely one W_i .

The last condition implies that T' is determined by W_1, \dots, W_k .

For each k -tuple \mathcal{W} satisfying (70), define $\varphi_{\mathcal{W}} : E \rightarrow G_F$ as follows. If e is an edge of D not in T , set $\varphi_{\mathcal{W}}(e) := 1$. If $e = (u, w)$ is an edge of T , let e_1, \dots, e_t be the edges in π_e , from left to right with respect to the orientation (u, w) of e . Let $\alpha_j := g_i$ if e_j is traversed by W_i in the direction from u to w , and let $\alpha_j := g_i^{-1}$ if e_j is traversed by W_i in the direction from w to u . Define $\varphi_{\mathcal{W}}(e) := \alpha_1 \dots \alpha_t$. Then:

- (71) For each solution \mathcal{P} of (62), there exists T' and $\mathcal{W} = (W_1, \dots, W_k)$ satisfying (70) such that $\chi_{\mathcal{P}}$ and $\varphi_{\mathcal{W}}$ are homologous and such that, for each $i = 1, \dots, k$, each parallel class in T' is traversed at most $2|E|$ times by W_i .

To see this, reroute P_1, \dots, P_k along T as follows. For each e in $E(D) \setminus E(T)$, let Q_e be the path in T connecting the ends of e . Order the edges in $E(D) \setminus E(T)$ as e_1, e_2, \dots, e_m such that if Q_{e_j} is longer than Q_{e_i} then $j > i$. Then for $j = 1, \dots, m$, if P_i traverses e_j , reroute P_i along Q_{e_j} ; that is, add edges parallel to the edges in Q_{e_j} , in the disk enclosed by e_j and Q_{e_j} , and replace e_j in P_i by the new edges (in order). This gives T' and W_1, \dots, W_k as required, proving (71).

So to cover all homology classes of solutions of problem (62), it suffices to enumerate all T' and W_1, \dots, W_k satisfying (70).

In fact, we can assume that each W_i is *non-returning* in the following sense. Let W_i traverse edges e , vertex v , and edges e' consecutively.

- (72) (i) If $v \notin \{r_1, s_1, \dots, r_k, s_k\}$, then e and e' belong to different parallel classes incident with v .
(ii) If $v \in \{r_j, s_j\}$ for some $j \in [k]$, then e and e' enclose the starting or ending edge of W_j .

This can be attained as follows. Suppose W_i, e, v, e' violate (72). Fixing v , choose W_i, e, e' such that the number of edges inbetween of (that is, enclosed by) e and e' is as small as possible. Then each edge inbetween of e and e' is traversed by some W_j with $j \neq i$ (as W_i is not self-crossing) and $\{i, j\} \notin F$ (as W_i and W_j cross). So deleting e and e' from W_i and from T' , gives a walk system \mathcal{W}' again satisfying (70), with $\varphi_{\mathcal{W}'} = \varphi_{\mathcal{W}}$, and with a smaller total length. Iterating this, we end up with each W_i non-returning.

This implies that if $v \in V \setminus \{r_1, s_1, \dots, r_k, s_k\}$ has degree 1 in T , it is incident with no edges in T' . Delete such vertices from T repeatedly. Let T_0 be the final graph. It is a tree with maximum degree 3 and with $2k$ vertices of degree 1 (namely, $r_1, s_1, \dots, r_k, s_k$). Hence T_0 has $2k - 2$ vertices of degree 3. The vertices of degree 1 and 3 are connected by $4k - 3$ internally vertex-disjoint paths, together forming T_0 .

Let $\mathcal{W} = (W_1, \dots, W_k)$ be a k -tuple of walks satisfying (70) and (72). Consider a vertex v of degree 2 in T_0 , say incident with edges e and e' of T_0 . By (72), W_i traverses edges in π_e as often as it traverses edges in $\pi_{e'}$.

For each $i = 1, \dots, k$, define $h_i : E(T_0) \rightarrow \{0, 1, \dots, 2|E|\}$ by: $h_i(e)$ is the number of times that W_i traverses π_e , in any direction (for $e \in E(T_0)$). Then we can derive $\varphi_{\mathcal{W}}$ from h_1, \dots, h_k , without knowing \mathcal{W} :

Claim. Given h_1, \dots, h_k , one can find $\varphi_{\mathcal{W}}$ in polynomial time.

Proof. Consider any $\{i, j\} \in F$. Let T'' be the subgraph of T' consisting of the edges traversed by W_i and W_j . We know T'' since we know h_i and h_j . We determine an undirected graph H with vertex set $E(T'')$, calling two edges in $E(T'')$ *associated* if they form an edge of H .

First, consider any vertex v of T_0 of degree 2. Let e and e' be the edges of T_0 incident with v , and consider the parallel classes π_e and $\pi_{e'}$ in T'' . As $|\pi_e| = h_i(e) + h_j(e) = h_i(e') + h_j(e') = |\pi_{e'}|$, we can order the edges in π_e as e_1, \dots, e_m from left to right when going towards v , and similarly, the edges in $\pi_{e'}$ as e'_1, \dots, e'_m , from left to right when going away from v . For each $t = 1, \dots, m$, we ‘associate’ e_t and e'_t .

Next, consider any vertex v of T_0 of degree 3. Let e , e' , and e'' be the edges of T_0 incident with v . Consider the parallel classes π_e , $\pi_{e'}$, and $\pi_{e''}$ in T'' . As W_i and W_j are non-returning (that is, satisfy (72)(i)), we know that there exist nonnegative integers a , b , and c such that $|\pi_e| = b + c$, $|\pi_{e'}| = a + c$, and $|\pi_{e''}| = a + b$. These numbers are unique and can be directly calculated from $|\pi(e)|$, $|\pi(e')|$, and $|\pi(e'')|$. This implies that the edges in $\pi_e \cup \pi_{e'} \cup \pi_{e''}$ can uniquely be pairwise ‘associated’ in such a fashion that any two associated pairs of edges are noncrossing at v and such that no two edges in the same parallel class are associated.

Finally, consider any vertex v of T_0 of degree 1. So v belongs to $\{r_1, s_1, \dots, r_k, s_k\}$. Let e be the edge of T_0 incident with v . Let e_1, \dots, e_t be the edges in the parallel class π_e of T'' , in order. ‘Associate’ e_i with e_{t+1-i} for each $i = 1, \dots, \lfloor \frac{1}{2}t \rfloor$. So if t is odd (which is the case if and only if $v \in \{r_i, s_i, r_j, s_j\}$), one edge in π_e remains unassociated at v (namely the middle edge).

Then the graph H with $E(T'')$ as vertex set and all pairs of associated edges of T'' as edges of H , consists of two paths, corresponding to W_i and W_j in T' . These sets of edges form two walks that we can orient, one from r_i to s_i , the other from r_j to s_j . Then for each edge e of T_0 we know the order, from left to right, in which W_i and W_j traverse the parallel class π_e of T'' , and we can derive the direction. Concluding, we can derive the subword of $\varphi_{\mathcal{W}}(e)$ made up by the symbols g_i , g_i^{-1} , g_j , and g_j^{-1} . (It is important here that we know that H comes from an $r_i - s_i$ walk W_i and an $r_j - s_j$ walk W_j . So H contains no circuit, for which we would not know whether it belongs to W_i or to W_j .)

As we can do this for each $\{i, j\} \in F$, we can derive $\varphi_{\mathcal{W}}(e)$. This follows from the fact that for any word w with symbols $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$, if we know for each $\{i, j\} \in F$ the subword $w_{i,j}$ of w made up by $g_i, g_i^{-1}, g_j, g_j^{-1}$, we can determine w as word up to transposition of commuting symbols (but without cancellation): Start by finding an $i \in [k]$ and $\alpha \in \{g_i, g_i^{-1}\}$ that occurs first in $w_{i,j}$ for each j with $\{i, j\} \in F$. By transposition we can assume that α is the first symbol of w . Then delete the first α from each $w_{i,j}$ with

$\{i, j\} \in F$, and iterate.

Thus, temporarily, we do not cancel g_i with g_i^{-1} or g_j with g_j^{-1} , but work in the semi-group generated by $g_1, g_1^{-1}, \dots, g_k, g_k^{-1}$ with relations $g_i g_j = g_j g_i$, $g_i g_j^{-1} = g_j^{-1} g_i$, and $g_i^{-1} g_j^{-1} = g_j^{-1} g_i^{-1}$ for all distinct i, j with $\{i, j\} \notin F$. At the end we factor out to the group G_F .

Concluding, we can find the element of G_F represented by word w . (Here we use the assumption that each i is contained in some pair in F .) \square

We finally describe the required algorithm. Enumerate all k -tuples of functions $h_1, \dots, h_k : E(T_0) \rightarrow \{0, 1, \dots, 2|E|\}$ with the property that if e and e' are the edges of T_0 incident with a vertex of T_0 of degree 2, then $h_i(e) = h_i(e')$ for each i . Determine, if possible, $\varphi_{\mathcal{W}}$. All such $\varphi_{\mathcal{W}}$ form Φ .

Since T_0 consists of vertices of degree 1 and 3 together with $4k - 3$ internally vertex-disjoint paths connecting these vertices, there are $((2|E| + 1)^{4k-3})^k$ such k -tuples h_1, \dots, h_k . For fixed k , this is polynomially bounded. \blacksquare

4.3. Finding partially disjoint paths

Concluding, we have:

Theorem 3. *For each fixed k , the partially disjoint paths problem in directed planar graphs is solvable in polynomial time.*

Proof. Directly from Propositions 11 and 12. \blacksquare

5. Some extensions and open questions

The theorem can be extended to the case where for each edge e of D a subset K_e of $[k]$ is given, prescribing that e may be traversed only by paths P_i with $i \in K_e$. This amounts to restricting I in (66) to subsets of K_e . Instead of requiring disjointness of certain pairs of paths, one may relax this to requiring that certain pairs of paths are noncrossing: so they are allowed to ‘touch’ each other in a vertex, but not cross. This amounts to deleting the ‘nonplanar’ edges $e_{v, F, F'}$.

One may impose further conditions of the following kind. Choose an (undirected) path Q in the dual graph D^* , connecting two faces F and F' of D . Then one may restrict the total ‘flow’ of paths P_i in D that intersect Q : as long as the restriction can be described by a closed subset of G_F , this requirement translates into an extra nonplanar edge added to the dual graph D^* , like before we did for paths in D^* connecting two faces incident with a vertex v .

Moreover, the theorem extends to directed graphs D on any fixed compact surface instead of planar graphs. Then, instead of considering the spanning tree T in Section 4.2, one considers a minimal connected spanning subgraph T that is cellularly embedded, i.e., each face is a disk (assuming without loss of generality that D is cellularly embedded). Fixing the surface, the number of edges in T is only a fixed amount more than in a spanning tree, and the enumeration of homology classes can be bounded accordingly.

We finally mention some open questions. The running time of the algorithm above is bounded by a polynomial with exponent depending on k (in fact, $O(k^2)$). This raises the question if the problem is ‘fixed parameter tractable’; that is, can the degree of the polynomial be fixed independently of k , while the dependence of k is only in the coefficient. As mentioned, this question was answered confirmatively by Cygan, Marx, Pilipczuk, and Pilipczuk [3] for the k *fully* disjoint paths problem in directed planar graphs.

Another open question is if the condition of fixing k can be relaxed to fixing other parameters of the graph $\Gamma = ([k], F)$. One may think of fixing the maximum degree of Γ , or (more weakly) fixing the chromatic number of Γ , or (even more weakly) fixing the clique number of Γ . A different open question is if instead of fixing k , it suffices to fix the number of faces that can cover all terminals (by the face boundaries).

A closely related open question is the complexity of the edge-disjoint version of the problem. Even for the following problem it is not known whether it is polynomial-time solvable or NP-complete: given a directed planar graph $D = (V, E)$ and vertices r and s , find a directed $r - s$ path P and a directed $s - r$ path Q such that P and Q are edge-disjoint. The corresponding problem for the undirected case is polynomial-time solvable (for any fixed number of paths), even for general nonplanar graphs, by Robertson and Seymour [14].

Let us finally question whether the polynomial-time solvability of the cohomology feasibility problem for graph groups (polynomial-time even for *unfixed* k) has other applications, for instance to free partially commutative semigroups as studied for inhomogeneous sorting and scheduling of concurrent processes (cf. Anisimov and Knuth [1], Diekert [4,5]).

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