Orthogonal Representations and Connectivity of Graphs

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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ABSTRACT

It is proved that a graph on n nodes is k-connected if and only if its nodes can be represented by real vectors in dimension $n-k$ such that (a) nonadjacent nodes are

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represented by orthogonal vectors and (b) any $n - k$ of them are linearly independent. We show that the closure of the set of all representations with properties (a) and (b) is irreducible as an algebraic variety, and study the question of irreducibility of the variety of all representations with property (a).

0. INTRODUCTION

Let G be a graph and $d \geqslant 1$, an integer. We want to represent each node of G in $\mathbb{R}^d$ by a vector in such a way that nonadjacent nodes are represented by orthogonal vectors. Such an assignment of vectors is called an *orthogonal representation* of G.

Orthogonal representations of graphs were introduced by Lovász (1979) in the study of the Shannon capacity of a graph. Grötschel, Lovász, and Schrijver (1986) showed that they are intimately related to the vertex packing polytope, and used them (1984) to design polynomial-time algorithms for finding maximum cliques and optimum colorings in perfect graphs. In these studies, metric properties of orthogonal representations play the main role. We shall be concerned with an even more immediate question: what is the minimum dimension of the space in which orthogonal representations with certain nondegeneracy properties exist?

It is trivial that orthogonal representations exist for each G and d: e.g., we can represent each node by the 0 vector. To exclude such degeneracies, we study orthogonal representations in which any d representing vectors are linearly independent (we call these *general-position orthogonal representations*). Our main result in Section 1 says that G has a general-position orthogonal representation in $\mathbb{R}^d$ if and only if G is $(n - d)$-connected. The "only if" part of this result is easy; what is trickier is to construct a general position orthogonal representation for each $(n - d)$-connected graph. There is, in fact, a trivial algorithm to construct this representation: we select the representing vectors one by one, obeying the orthogonality conditions imposed by the graph. This always yields an orthogonal representation; we have to do the selection so as to avoid unnecessary degeneracies. This is simply achieved by selecting each vector at random from among all candidates (from a suitable distribution). We shall show that the distribution of the resulting random orthogonal representation is independent of the order in which the selection was made, and that with probability 1, it will be in general position. We shall also prove a related result for directed graphs.

Thus it seems that orthogonal representations have connections with several basic invariants of graphs. This justifies a study of their structure for its own sake. In Section 2, we investigate the set of all orthogonal representations and the set of all general-position orthogonal representations of a graph as algebraic geometric varieties. In the latter case we show that the variety is always irreducible. It is easy to see that the variety of all orthogonal representations is irreducible only if G is $(n-d)$-connected; however, contrary to the results in Section 1, this is not a necessary and sufficient condition. We show that for $d \leqslant 3$, it is necessary and sufficient, and the same holds if $d = 4$ and if the complement of G is connected and nonbipartite. We give an example (the complement of the cube) showing that one cannot extend this result to complements of bipartite graphs. A complete characterization of the irreducibility of the variety of all orthogonal representations of the graph G (and the study of other properties of this variety) remains open.

This paper was also motivated by results of Linial, Lovász, and Wigderson (1986a, 1986b), who gave various geometric and linear-algebraic conditions for the k-connectivity of a graph, and used these conditions to design efficient randomized connectivity tests. The connectivity conditions given in this paper could be used in an algorithmic fashion in the same way, and we shall briefly sketch these applications in Section 3.

1. GENERAL-POSITION ORTHOGONAL REPRESENTATIONS

Let G be a graph. An *orthogonal representation* of G in $\mathbb{R}^d$ is an assignment $f: V(G) \to \mathbb{R}^d$ such that $f(u)$ and $f(v)$ are orthogonal for every pair of distinct nonadjacent nodes u and v. An *orthonormal representation* is an orthogonal representation such that $\|f(u)\| = 1$ for every $u \in V(G)$. We say that the orthogonal representation is *in general position* if every set of d representing vectors is linearly independent. If f is a general-position orthogonal representation, then $f(u) \neq 0$ and so $f(u)/\|f(u)\|$ is a general-position orthonormal representation in the same dimension.

Another natural "nondegeneracy" property of an orthogonal representation is to be *faithful*: this means that $f(u)$ and $f(v)$ are orthogonal if and only if u and v are nonadjacent.

The assignment $f \equiv 0$ is a trivial orthogonal representation for any graph. In dimension $n = |V(G)|$, every graph G has a general-position orthonormal representation (using n mutually orthogonal unit vectors). It is easy to give a faithful representation in this same dimension. It seems to be difficult to find the smallest dimension in which a given graph G has an orthonormal representation. If we consider general-position representations, however, then the least possible dimension is given by the following theorem, which is one of the main results in the paper.

THEOREM 1.1. *A graph G with n nodes has a general-position orthogonal representation in $\mathbb{R}^d$ if and only if G is $(n-d)$-connected.*

The condition that the given set of representing vectors is in general position is not easy to check. Therefore, it is worthwhile to formulate another version of the condition:

THEOREM 1.1′. *If G is a graph with n nodes, then the following are equivalent*:

(i) *G is $(n-d)$-connected;*

(ii) *G has a general-position orthogonal representation in $\mathbb{R}^d$;*

(iii) *G has an orthonormal representation in $\mathbb{R}^d$ such that for each node v, the vectors representing the nodes nonadjacent to v are linearly independent.*

Proof. First we show (ii) $\Rightarrow$ (iii). Assume f is a general-position orthonormal representation of G in $\mathbb{R}^d$. Let v be any vertex. Suppose v has a set W of d nonneighbors. Then by the definition of a general-position representation, the vectors representing W are linearly independent and hence span $\mathbb{R}^d$. This contradicts the fact that these vectors are orthogonal to $f(v)$. Hence v has at most $d-1$ nonneighbors and, by the definition of a general-position representation, the vectors representing them are linearly independent, proving (iii).

That (iii) $\Rightarrow$ (i) is also easy: assume that G is not $(n-d)$-connected; then it has a set A of $n-d-1$ nodes separating it into two components with vertex sets B and C with $|B|+|C|=d+1$. For any orthogonal representation f of G every vector in $f(B)$ is orthogonal to every vector in $f(C)$, and thus $\dim f(B)+\dim f(C)\leqslant d$. It follows that either $\dim f(B)<b$ or $\dim f(C)<c$, i.e., either the vectors representing B or the vectors representing C are not linearly independent. Assume for example that B is represented by linearly dependent vectors; then any node $v\in C$ violates (iii).

The difficult part of the theorem is (i) $\Rightarrow$ (ii), i.e., to demonstrate the existence of a general-position orthogonal (or orthonormal) representation for $(n-d)$-connected graphs. Actually, the construction is almost trivial; the difficulty is the proof of its validity. Let $(v_1,\dots,v_n)$ be any ordering of the nodes of G. Let us choose $f(v_1)$, $f(v_2),\dots$ consecutively as follows. $f(v_1)$ is any vector of unit length. Suppose that $f(v_i)$ $(1\leqslant i\leqslant j)$ are already chosen. Then we choose $f(v_{j+1})$ from the unit sphere, subject to the constraints that it has to be orthogonal to certain previous vectors $f(v_i)$. These orthogonality constraints restrict $f(v_{j+1})$ to a linear subspace L_{j+1}. Note that if G is $(n-d)$-connected, then every node of it has degree at least $n-d$, and hence

$$\dim L_{j+1} \geqslant d - \#\{i : i \leqslant j,\ v_i v_{j+1} \notin E(G)\} \geqslant d - (d-1) = 1,$$

and so $f(v_{j+1})$ can always be chosen. Of course, it may happen that the orthonormal representation constructed this way is not in general position. To avoid such degeneracies, we choose the vectors at random. More exactly, we choose $f(v_{j+1})$ from the uniform distribution over the unit sphere of L_{j+1}.

This way we get a random mapping $f: V(G) \to \mathbb{R}^d$. We call f the *random sequential orthogonal representation* of G [associated with the ordering $(v_1, \ldots, v_n)$]. The following is a sharpening of Theorem 1.1:

THEOREM 1.2. *Let G be any graph, and fix any ordering of its vertices. Let f be the random sequential orthogonal representation of G. Then*

(a) *if G is not $(n-d)$-connected, then f is not in general position;*

(b) *if G is $(n-d)$-connected, then with probability* 1, *f is in general position.*

We have proved (a) already. The crucial step in the proof of (b) is the following lemma.

MAIN LEMMA 1.3. *If G is $(n-d)$-connected, then the distribution of the random sequential orthogonal representation is independent of the ordering of the nodes.*

Knowing this Main Lemma, the proof of (b) in Theorem 1.2 is easy. Let W be any d-element subset of $V(G)$. If we order the nodes so that the first d nodes are the elements of W, then with probability 1 the vectors $f(w)$ $(w \in W)$ will be chosen linearly independent. Since the distribution of f is independent of the ordering, these d vectors will be linearly independent whatever initial ordering we take. This holds for every d-tuple, and hence, with probability 1, the orthonormal representation f will be in general position.

Proof of the Main Lemma. It suffices to prove that if we swap two consecutive nodes v_j and v_{j+1} in the ordering, the distribution of f does not change. In fact, it suffices to show that the distribution of the restriction f' of f to $\{v_1, \ldots, v_{j+1}\}$ does not change, since then the choice of the rest is the same.

We prove this by induction on j. For $j = 1$ the assertion is obvious.

First assume that there is a path in G connecting v_j and v_{j+1} and containing only nodes v_i with $i \leqslant j+1$. Let P be a shortest such path and t

its length. We also use induction on t (for j fixed). If $t = 1$, then v_j and v_{j+1} are adjacent in G and the assertion is obvious, since their representative vectors are chosen independently. So suppose that v_j and v_{j+1} are nonadjacent in G. Let v_i be any internal node of P. So in the ordering we have $\dots v_i \dots v_j v_{j+1} \dots$ Now the distribution of f' does not change if we

(1) swap v_i and v_j (by the induction hypothesis on j, since this can be achieved by swapping consecutive nodes earlier in the order than v_{j+1}): we obtain $\dots v_j \dots v_i v_{j+1} \dots$;

(2) then swap v_i and v_{j+1} (by the induction hypothesis on t): we obtain $\dots v_j \dots v_{j+1} v_i \dots$;

(3) then swap v_j and v_{j+1} [same as (1)]: we obtain $\dots v_{j+1} \dots v_j v_i \dots$;

(4) then swap v_j and v_i [same as (2)]: we obtain $\dots v_{j+1} \dots v_i v_j \dots$;

(5) finally, swap v_{j+1} and v_i [same as (1) again]: we obtain $\dots v_i \dots v_{j+1} v_j \dots$.

So the distribution of f' does not change if v_j and v_{j+1} are swapped.

Second, assume that there is no path connecting v_j to v_{j+1} containing only nodes v_i with $i \leqslant j+1$. This means that $\{v_1, \dots, v_{j+1}\}$ can be partitioned into two sets A and B such that $v_j \in A$ and $v_{j+1} \in B$, and no edge connects A to B. It also follows that $\{v_{j+2}, \dots, v_n\}$ separates G, and hence $n - (j+1) \geqslant n - d$, i.e., $|A| + |B| = j + 1 \leqslant d$. Let A' be the set of nodes in $A - \{v_j\}$ nonadjacent to v_j, and B' be the set of nodes in $B - \{v_{j+1}\}$ nonadjacent to v_{j+1}.

Assume now that $f(v_1), \dots, f(v_{j-1})$ have been selected. Let L_A denote the subspace generated by $f(A - \{v_j\})$, and L'_A the subspace generated by $f(A')$. Let L''_A be the orthogonal complement of L'_A in L_A. Let L_B, L'_B, and L''_B be defined analogously. Clearly, L_A and L_B are orthogonal. Finally, let M be the orthogonal complement of $L_A \cup L_B$ in $\mathbb{R}^d$. The computation above shows that $\dim M \geqslant 2$.

Every unit vector orthogonal to $f(B \cup A')$ [i.e., every possible choice for $f(v_j)$] is of the form $a + m$, where $a \in L''_A$ and $m \in M$, and a similar description can be given for $f(v_{j+1})$. So we may choose $f(v_j)$ and $f(v_{j+1})$ as follows:

(a) we select nonnegative real numbers α and μ such that $\alpha^2 + \mu^2 = 1$ from an appropriate distribution (these will be $\|a\|$ and $\|m\|$; it does not matter for our argument what this distribution is),

(b) select nonnegative real numbers β and ν such that $\beta^2 + \nu^2 = 1$ from an appropriate distribution,

(c) select any unit vectors $x \in L''_A$ and $y \in L''_B$ from a uniform distribution,

(d) select any unit vector p from M from a uniform distribution,

(e) select any unit vector q from $M \cap \{p\}^{\perp}$ from a uniform distribution,
(f) form $f(v_j) = \alpha x + \mu p$ and $f(v_{j+1}) = \beta y + \nu q$.

Now if v_j and v_{j+1} are swapped, then only the roles of p and q in (d) and (e) are interchanged. If $\dim M \geqslant 2$, then selecting in either order yields the same distribution on pairs of orthogonal unit vectors from M. This proves the Main Lemma. ■

REMARK. Note that the proof of the "easy" part of Theorems 1.1 and 1.2 gives, in fact, more: it follows that if G is not $(n-d)$-connected, then for every orthogonal representation f of G in $\mathbb{R}^d$, there is a node v whose nonneighbors are represented by linearly dependent vectors. This remark is important in algorithmic applications, since recognizing whether a set of the vectors is in general position seems to be a hard problem. (As far as we know, its complexity is open.) Cf. also Section 3.

We do not know how to determine the minimum dimension of a *faithful* orthogonal representation. It was proved by Maehara (1987) that if the maximum degree of the complementary graph $\overline{G}$ of a graph G is D, then G has a faithful orthogonal representation in D^3 dimensions. He conjectured that this result can be improved to $D+1$. Rödl (1987) proved that the bound D^3 can be improved to $2D+1$. [This result is implicit in work of Erdös and Simonovits (1980).] Note that the condition that the maximum degree of $\overline{G}$ is D is equivalent to saying that the minimum degree of G is $n-D-1$ (where n is the number of nodes). It will follow from our results above that Maehara's conjecture is true if we strengthen its assumption by requiring that G is $(n-D-1)$-connected. [Note that this implies Rödl's result, since a graph with minimum degree $n-D-1$ is at least $(n-2D)$-connected.]

COROLLARY 1.4. *Every $(n-d)$-connected graph on n nodes has a faithful orthogonal representation in $\mathbb{R}^d$.*

Proof. It suffices to show that in a random sequential orthogonal representation, the probability of the event that two given adjacent nodes are represented by orthogonal vectors is 0. By the Main Lemma, we may define the representation from an ordering starting with these two nodes. But then the assertion is obvious. ■

Assume that the complement $\overline{G}$ of G is a bipartite graph with color classes A and B, and $\overline{G}$ does not contain a complete bipartite subgraph with more than d nodes. Then G is $(n-d)$-connected. Hence we can construct an orthonormal representation of G by selecting the vectors representing A

first, and then the vectors representing B. In both stages, the selections can be done independently (and so even in parallel). So for bipartite graphs, the algorithm constructing the representation is even simpler. (For a discussion of algorithmic applications, see Section 3.)

We can use the above observation to derive analogues of Theorems 1.1 and 1.2 for directed graphs. Let D be any directed graph with n nodes. We define an *orthogonal birepresentation* of G as a pair (g, h) of mappings $g, h: V(D) \to \mathbb{R}^d$ such that for each non-arc uv, $(g(u), h(v)) = 0$. We say that this birepresentation is *in general position* if any d of the vectors $g(u)$ and $h(u)$ are linearly independent.

Corollary 1.5. *A digraph D with n nodes has a general-position orthogonal birepresentation in $\mathbb{R}^d$ if and only if it is strongly $(n-d)$-connected.*

The reduction to Theorem 1.1 goes as follows. Construct an auxiliary undirected graph G by representing each node v of D by two nodes v' and v'', connecting every v' to every w', every v'' to every w'', and every v' to the corresponding v'', and connecting v' to w'' iff vw is an arc in D. Thus $\overline{G}$ is bipartite. Now observe that D is strongly k-connected if and only if G is $(n+k)$-connected, and also that an orthogonal representation of G corresponds to an orthogonal birepresentation of D. Thus Theorem 1.1 implies Corollary 1.5.

Next, observe that the construction of an orthogonal birepresentation is even easier than the construction of an orthogonal representation in the undirected case. Assume that every node of D has indegree at least $n-d$. First, choose $g: V(D) \to \mathbb{R}^d$, $\|g\| \equiv 1$, at random. Then choose another mapping $h: V(D) \to \mathbb{R}^d$ as follows: for each $v \in V(D)$, $h(v)$ is a random unit vector orthogonal to all vectors $g(w)$ such that wv is not an arc in D. Call the random variable (g, h) the *random orthonormal birepresentation* of D. Clearly g and h form a random sequential orthonormal representation of G. Applying Theorem 1.2, we obtain:

Corollary 1.6. *Let D be any digraph, and (g, h) its random orthogonal birepresentation in dimension d.*

(a) *If D is not $(n-d)$-connected, then (g, h) is not in general position.*

(b) *If G is $(n-d)$-connected, then with probability* 1, *(g, h) is in general position.*

Remark. Again, we can sharpen (a) and assert that if D is not $(n-d)$-connected, then there exists a node v of G such that the vectors $h(u)$

representing the nodes not reachable from v on an arc are linearly dependent.

2. THE VARIETY OF ORTHOGONAL REPRESENTATIONS

Let G be a k-connected graph with n nodes, and set $d = n - k$. Then we know that G has a general position orthogonal representation in $\mathbb{R}^d$. One may suspect that more is true: every orthogonal representation in $\mathbb{R}^d$ is the limit of general position orthogonal representations, i.e., the set $\mathrm{GOR}^d(G)$ of general-position orthogonal representations is everywhere dense in the set $\mathrm{OR}^d(G)$ of all orthogonal representations of G. We shall see that this is not true in general, but can be proved under additional hypotheses about the graph G.

One reason for asking this question is the following. The set $\mathrm{OR}^d(G)$ is an algebraic variety in $\mathbb{R}^{nd}$, and it is a natural question whether it is irreducible. (A set $A \subset \mathbb{R}^N$ is *irreducible* if whenever the product $p \cdot q$ of two polynomials in N variables vanishes on A, then either p or q vanishes on A; equivalently, the polynomial ideal $\{p : p \text{ vanishes on } A\}$ is a prime ideal.) Let us begin with the question of irreducibility of the set $\mathrm{GOR}^d(G)$ of general position orthogonal representations of G. This can be settled quite easily.

THEOREM 2.1. *Let G be any graph and $d \geqslant 1$. Then $\mathrm{GOR}^d(G)$ is irreducible.*

Proof. Let G have n nodes. We may assume that G is $(n-d)$-connected, else $\mathrm{GOR}^d(G)$ is empty and the assertion is vacuously true.

First we show that there exist vectors $\phi_v = \phi_v(\mathbf{X}) \in \mathbb{R}^d$ $[v \in V(G)]$ whose entries are multivariate polynomials with real coefficients in variables $\mathbf{X}$ (the number of these variables does not matter) such that whenever u and v are nonadjacent, $\phi_u \cdot \phi_v$ is identically 0, and such that every general-position representation of G arises from ϕ by substituting for the variables appropriately. We do this by induction on n.

Let $v \in V(G)$. Suppose that the vectors of polynomials $\phi_u(\mathbf{X}')$ of length d exist for all $u \in V(G) - \{v\}$ satisfying the requirements above for the graph $G - v$ [since $G - v$ has $n - 1$ nodes and is $(n - d - 1)$-connected, this is indeed the right induction hypothesis]. Let $\phi' = (\phi_u : u \in V(G) - \{v\})$. Let $u_1, \dots, u_m$ be the nodes in $V(G) - \{v\}$ nonadjacent to v; clearly $m \leqslant d - 1$. Let $x_1, \dots, x_{d-1-m}$ be vectors of length d composed of new variables, and let

y be another new variable; $\mathbf{X}$ will consist of $\mathbf{X}'$ and these new variables. Consider the $d \times (d-1)$ matrix

$$F = \left(\phi_{u_1}, \ldots, \phi_{u_m}, x_1, \ldots, x_{d-1-m}\right),$$

and let p_j be the determinant of the submatrix obtained by dropping the jth row. Then we define $\phi_v = y(p_1, \ldots, p_d)^T$.

It is obvious from the construction and elementary linear algebra that ϕ_v is orthogonal to every vector ϕ_u for which u and v are nonadjacent. We show that every general-position orthogonal representation of G can be obtained from ϕ by substitution. In fact, let f be a general-position orthogonal representation of G. Then $f' = f|_{V(G)-\{v\}}$ is a general-position orthogonal representation of $G - v$ in $\mathbb{R}^d$, and hence by the induction hypothesis, f' can be obtained from ϕ' by substituting for the variables $\mathbf{X}'$. The vectors $f(u_1), \ldots, f(u_m)$ are linearly independent and orthogonal to $f(v)$; let $a_1, \ldots, a_{d-1-m}$ be vectors completing the system $\{f(u_1), \ldots, f(u_m)\}$ to a basis of $f(v)^{\perp}$. Substitute $x_i = a_i$. Then the vector $(p_1, \ldots, p_d)^T$ will become a nonzero vector parallel to $f(v)$, and hence y can be chosen so that ϕ_v will be equal to $f(v)$.

Note that from the fact that $\phi(\mathbf{X})$ is in general position for some substitution it follows that the set of substitutions for which it is in general position is everywhere dense.

From here the proof is quite easy. Let p and q be two polynomials such that $p \cdot q$ vanishes on $\text{GOR}^d(G)$. Then $p(\phi(\mathbf{X})) \cdot q(\phi(\mathbf{X}))$ vanishes on an everywhere dense set of substitutions for $\mathbf{X}$, and hence it vanishes identically. So either $p(\phi(\mathbf{X}))$ or $q(\phi(\mathbf{X}))$ vanishes identically; say the first occurs. Since every general-position orthogonal representation of G arises from ϕ by substitution, it follows that p vanishes on $\text{GOR}^d(G)$. ∎

REMARK. The proof in fact shows that every orthogonal representation of G with the property that the nonneighbors of every node are represented by linearly independent vectors can be obtained from ϕ by substitution. Hence the set of all such representations is also irreducible, and $\text{GOR}^d(G)$ is dense in this set.

Now back to the (perhaps) more natural question of irreducibility of $\text{OR}^d(G)$. It is easy to see that $\text{OR}^d(G)$ is not irreducible if G is not $(n-d)$-connected. On the other hand, Theorem 2.1 implies the following.

LEMMA 2.2. *If* $\text{GOR}^d(G)$ *is dense in* $\text{OR}^d(G)$, *then* $\text{OR}^d(G)$ *is irreducible.*

We shall see that $\mathrm{OR}^d(G)$ is irreducible if G is $(n-d)$-connected and $d \leqslant 3$. On the other hand, we shall see that for $d = 4$, not every $(n-d)$-connected graph gives an irreducible variety, but those graphs whose complement is connected and nonbipartite do. The general description of all graphs with $\mathrm{OR}^d(G)$ irreducible is open.

Sometimes it will be more convenient to keep the complement $\overline{G}$ of the graph G in mind. Note that G is $(n-d)$-connected if and only if $\overline{G}$ does not contain any complete bipartite graph with more than d nodes (with nonempty color classes) as a subgraph. In particular, G is $(n-2)$-connected iff $\overline{G}$ has maximum degree $\leqslant 1$; G is $(n-3)$-connected iff $\overline{G}$ has maximum degree $\leqslant 2$ and contains no $K_{2,2}$; G is $(n-4)$-connected if and only if $\overline{G}$ has maximum degree $\leqslant 3$ and contains no $K_{2,3}$. Note that $(n-4)$-connected graphs are quite rich and interesting.

Let us say that an orthogonal representation of a graph G in $\mathbb{R}^d$ is *special* (or *d-special*) if it does not belong to the topological closure of $\mathrm{GOR}^d(G)$. We say that the graph G on n nodes is *d-special* if it is $(n-d)$-connected and has a d-special orthogonal representation. A graph G is *d-critical* if it is d-special but no induced subgraph of it is. (Observe that the induced subgraphs of a non-d-special graph are non-d-special.) It is easy to see that if G is d-critical then $\overline{G}$ is connected. A graph G is d-special if and only if at least one of the complements of connected components of its complement is d-special.

LEMMA 2.3. *Let G be a d-critical graph, and f a special orthogonal representation of G in $\mathbb{R}^d$. Then the vectors representing the nonneighbors of any node v are linearly dependent.*

Proof. Call a node "good" if its nonneighbors are represented by linearly independent vectors. Suppose that a node v *is* "good," i.e., $f(u_1), \dots, f(u_m)$ are linearly independent, where $u_1, \dots, u_m$ are the nonneighbors of v. We construct a general-position orthogonal representation of G in an ε-neighborhood of f for any given $\varepsilon > 0$. [The distance of two representations $f, g: V(G) \to \mathbb{R}^d$ is defined by

$$\|f - g\| = \max_{u \in V(G)} \|f(u) - g(u)\|,$$

where $\|f(u) - g(u)\|$ is the euclidean distance of $f(u)$ and $g(u)$.] Extend $f(u_1), \dots, f(u_m)$ by arbitrary vectors $a_1, \dots, a_{d-1-m}$ to a basis of $f(v)^{\perp}$. Let, say,

$$\det(f(u_1), \dots, f(u_m), a_1, \dots, a_{d-1-m}, f(v)) = 1.$$

Now $f' = f|_{V(G)-\{v\}}$ is an orthogonal representation of $G - v$, and hence by the criticality of G, there exists a general position orthogonal representation f_1 of $G - v$ in $\mathbb{R}^d$ in the ε'-neighborhood of f', where ε' is a sufficiently small positive number (in particular, ε' must be smaller than $\varepsilon/4$). We extend f_1 to an orthogonal representation of G as follows. Clearly if ε' is small enough, the vectors $f_1(u_1), \ldots, f_1(u_m), a_1, \ldots, a_{d-1-m}$ are linearly independent, and hence they uniquely determine a vector $f_1(v)$ orthogonal to all of them such that

$$\det(f_1(u_1), \ldots, f_1(u_m), a_1, \ldots, a_{d-1-m}, f_1(v)) = 1.$$

It is obvious that f_1 is an orthogonal representation of G and that if ε' is small enough, then $\|f(v) - f_1(v)\| < \varepsilon/4$ and so f_1 is in the $\varepsilon/4$-neighborhood of f. Unfortunately, it does not follow in general that this extended f_1 is in general position; but at least every "good" node remains "good" if ε' is small enough. Moreover, we know that any d vectors representing nodes different from v are linearly independent; in particular, every node adjacent to v is "good." Now if w is any other "good" node, then we can repeat the same argument and find an orthogonal representation f_2 closer to f_1 than $\varepsilon/8$ in which every node previously good remains good, and in addition all the neighbors of w become good.

Since G is connected, by repeating this argument at most n times we obtain an orthogonal representation f_0 of G in the $\varepsilon/2$-neighborhood of f in which every node is "good," i.e., the nonneighbors of every node are represented by linearly independent vectors. By the remark following the proof of Theorem 2.1, such a representation is in the closure of $\mathrm{FOR}^d(G)$, and hence we find in its $\varepsilon/2$-neighborhood a general position orthogonal representation f^* of G. Clearly $\|f^* - f\| < \varepsilon$, and this proves the lemma. ∎

If f is a d-special representation of G, then, by the definition of d-special, there exists an $\varepsilon > 0$ such that if g is another orthogonal representation of G in $\mathbb{R}^d$ and $\|f - g\| < \varepsilon$, then g is also d-special. There must be linear dependencies among the vectors $f(v)$; if ε is small enough, then there will be no new dependencies among the vectors $g(v)$. We say that a d-special orthogonal representation f is *very special* if there exists an $\varepsilon > 0$ such that for every orthogonal representation g with $\|f - g\| < \varepsilon$, and every subset $U \subset V(G)$, $f(U)$ is linearly dependent iff $g(U)$ is. Roughly speaking, a very special orthogonal representation is one which is locally freest. Clearly every d-special graph has a very special representation.

LEMMA 2.4. *In every very special representation of a d-critical graph, any two representing vectors are linearly independent.*

Proof. Let f be a very special orthogonal representation of the given graph G, and ε the positive threshold in the definition of "very special." First assume that $f(v)$ is the 0-vector for some v. Since the number of nonneighbors of v is at most $d-1$, we can replace $f(v)$ by a nonzero vector orthogonal to all vectors $f(u)$ where u is a nonneighbor of v, and shorter than ε. This is clearly another d-special orthogonal representation with fewer 0-vectors, a contradiction.

Second, let v and w be two nodes with $f(v)$ and $f(w)$ parallel. By Lemma 2.3, the set of vectors $f(u)$, where u is a nonneighbor of v, is linearly dependent, and hence these vectors span a linear subspace L of dimension at most $d-2$. Thus there exists a vector $a \in L^{\perp}$ not parallel to $f(v)$. We can replace $f(v)$ by $f(v) + \delta a/\|a\|$, and obtain another orthogonal representation of G. If $0 < \delta < \varepsilon$, then this new representation is also d-special, and if δ is small enough, then it has fewer pairs of parallel representing vectors than f, a contradiction again. ∎

COROLLARY 2.5. *If G is a d-critical graph with n nodes, then every node has degree at most $n-4$.*

Proof. Let f be a very special orthogonal representation of G in $\mathbb{R}^d$. For any vertex v, the set $\{f(u) \mid u \text{ not adjacent to } v\}$ is linearly dependent by Lemma 2.3, so by Lemma 2.4, v has at least three nonneighbors. ∎

COROLLARY 2.6. *If $d \leqslant 3$ and G is an $(n-d)$-connected graph with n nodes, then $\mathrm{GOR}^d(G)$ is everywhere dense in $\mathrm{OR}^d(G)$.*

Let f be an orthogonal representation of G and $v \in V(G)$. Let A_v be the linear span of the nonneighbors of v, and B_v its orthogonal complement. So $f(v) \in B_v$.

LEMMA 2.7. *Let G be a d-critical graph, and f a very special representation of G. Let $v \in V(G)$, and u be a nonneighbor of v such that $f(u)$ is linearly dependent on the vectors representing the other nonneighbors. Then $B_u \subset A_v$.*

Proof. Suppose not; then B_u contains a vector b arbitrarily close to $f(u)$ but not in A_v. Then replacing $f(u)$ by b, we obtain another orthogonal representation f' of G. Moreover, b does not depend linearly on the vectors representing the other nonneighbors of v, which contradicts the definition of "very special" representations. ∎

Next we turn to the case $d = 4$.

THEOREM 2.8. *If G is a 4-critical graph, then $\overline{G}$ is 3-regular and bipartite.*

Proof. Let G have n nodes. Then Corollary 2.5 implies that it is regular of degree $n-4$, i.e., $\overline{G}$ is 3-regular. Consider the subspaces A_v and B_v defined above. Lemma 2.3 implies that $\dim A_v \leqslant 2$, and Lemma 2.4 implies that $\dim A_v \geqslant 2$, so $\dim A_v = 2$, and hence also $\dim B_v = 2$. Thus Lemma 2.7 implies that for any two nonadjacent nodes u and v, $A_u = B_v$. So, fixing any node v, the rest of the nodes fall into two classes: those with $A_u = A_v$ and those with $A_u = B_v$. Moreover, any edge in $\overline{G}$ connects nodes in different classes. Hence $\overline{G}$ is bipartite as claimed. ∎

COROLLARY 2.9. *If $\overline{G}$ is a connected nonbipartite graph, then G is not 4-special.*

We conclude this section with an example of a 4-special (in fact, 4-critical) graph. Let Q denote the graph of the (ordinary) 3-dimensional cube, and let $G = \overline{Q}$. Note that Q is bipartite; let $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$ be its color classes. The indices are chosen so that u_i is adjacent to v_i in G. Consider an orthogonal representation f of G in which the elements of U are represented by vectors in a 2-dimensional subspace L of $\mathbb{R}^4$ and the elements of V are represented by vectors in the orthogonal complement $L^\perp$ of L. Clearly, this is an orthogonal representation for any choice of the representing vectors in these subspaces. On the other hand, we claim that such a representation is the limit of general-position orthogonal representations only if the cross ratio of $f(u_1), f(u_2), f(u_3), f(u_4)$ is equal to the cross ratio of $f(v_1), f(v_2), f(v_3), f(v_4)$. [The cross ratio $(x_1x_2x_3x_4)$ of four vectors x_1, x_2, x_3, x_4 in a 2-dimensional subspace can be defined as follows: we write $x_3 = \lambda_1 x_1 + \lambda_2 x_2$ and $x_4 = \mu_1 x_1 + \mu_2 x_2$, and take $(x_1x_2x_3x_4) = (\lambda_1\mu_2)/(\lambda_2\mu_1)$. This number is invariant under linear transformations.]

To prove this claim, we consider any general-position orthogonal representation g of G. Let M be the linear subspace spanned by the vectors $g(u_1)$ and $g(u_2)$. Then its orthogonal complement $M^\perp$ is spanned by $g(v_3)$ and $g(v_4)$. Let b_i be the orthogonal projection of $g(u_i)$ onto M $(i = 1,2,3,4)$, and c_i the orthogonal projection of $g(v_i)$ onto $M^\perp$ $(i = 1,2,3,4)$. If we show that

$(b_1 b_2 b_3 b_4) = (c_1 c_2 c_3 c_4)$, then the claim follows, since $g \to f$ and hence $b_i \to f(u_i)$ and $c_i \to f(v_i)$.

The proof of $(b_1 b_2 b_3 b_4) = (c_1 c_2 c_3 c_4)$ is an exercise in linear algebra (or projective geometry) and is left to the reader.

We do not know if there is any other 4-critical graph. An analysis of the cases $d \geq 5$ seems even harder.

3. ALGORITHMIC APPLICATIONS

The conditions given in Theorem 1.1 and Corollary 1.5 can be used to design efficient randomized algorithms to test k-connectivity. (As remarked in the introduction, an analogous application of other geometric connectivity conditions by Linial, Lovász, and Wigderson (1986a, 1986b) initiated our work on the topic of this paper.) More exactly, we can use Theorem 1.2 and Corollary 1.6. Assume that a graph G and a number $k \geq 1$ is given, and we want to test if G is k-connected. We construct a random orthogonal representation of G in $n - k$ dimensions and check whether or not the nonneighbors of each node are linearly independent. If the answer is yes, we conclude that G is k-connected. If the answer is no, we can conclude with large probability that G is not k-connected.

Similarly, when using Corollary 1.6, we choose the representation g of the nodes of the given digraph at random, and then compute the representation h for each node again at random. Then we check for each node v that the vectors $h(u)$ representing the nonneighbors u of v are linearly independent. (This procedure, of course, also applies to undirected graphs.) An advantage of this second algorithm is that it is easily parallelizable, while the first algorithm is genuinely sequential.

Several implementation details have to be filled in. All calculations are done modulo a reasonably large prime, and Schwartz's lemma (1980) can be used to estimate the probability of obtaining the wrong answer. To compute a basis in the orthogonal complement of given subspaces, one can use fast matrix inversion techniques, of which the best currently known is due to Coppersmith and Winograd (1987). Since the details are essentially identical with the methods of Linial, Lovász, and Wigderson (1986a), we do not go into them.

For any k-connected graph G, the above algorithm generates a random orthogonal representation of G in $\mathbb{R}^{n-k}$ which for a k-connected graph is almost surely in general position. We do not know an efficient deterministic algorithm that constructs a general-position orthogonal representation in $\mathbb{R}^{n-k}$ for any k-connected graph.

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