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Blocking nonorientability of a surface

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Abstract

Let $\mathbb{S}$ be a nonorientable surface. A collection of pairwise noncrossing simple closed curves in $\mathbb{S}$ is a *blockage* if every one-sided simple closed curve in $\mathbb{S}$ crosses at least one of them. Robertson and Thomas [9] conjectured that the orientable genus of any graph G embedded in $\mathbb{S}$ with sufficiently large face-width is "roughly" equal to one-half of the minimum number of intersections of a blockage with the graph. The conjecture was disproved by Mohar (Discrete Math. 182 (1998) 245) and replaced by a similar one. In this paper, it is proved that the conjectures in Mohar (1998) and Robertson and Thomas (J. Graph Theory 15 (1991) 407) hold up to a constant error term: For any graph G embedded in $\mathbb{S}$, the orientable genus of G differs from the conjectured value at most by $O(g^2)$, where g is the genus of $\mathbb{S}$.

1. Introduction

We follow standard graph theory terminology [2]. By a *surface* we mean a compact connected PL 2-manifold without boundary. The *genus* g(G) of a graph G is the smallest integer g such that G has an embedding in the orientable surface $\mathbb{S}_g$ of genus g. The nonorientable surface of genus g will be denoted by $\mathbb{N}_g$. So, $\mathbb{N}_1$ is the projective plane and $\mathbb{N}_2$ is the Klein bottle. The *nonorientable genus* of G is the smallest g such that G admits an embedding in $\mathbb{N}_g$.

All embeddings of graphs in surfaces considered in this paper are *2-cell embeddings* in which every face is homeomorphic to an open disk in the plane. If Π is an

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embedding of a connected graph G in some surface, the *Euler genus* of Π is defined as the number $\mathbf{eg}(G,\Pi) = 2 - |V(G)| + |E(G)| - f$, where f is the number of Π-facial walks. We refer to [8] for additional information on embeddings of graphs in surfaces.

A *closed curve* on a surface $\mathbb{S}$ is a continuous PL mapping $\gamma : S^1 \to \mathbb{S}$, and we sometimes identify γ with its image $\gamma(S^1)$ in $\mathbb{S}$. If a graph G is embedded in $\mathbb{S}$, then $\mathrm{cr}(\gamma, G)$ denotes the number of points $z \in S^1$ such that $\gamma(z)$ is a point of G in $\mathbb{S}$. The curve γ is *one-sided* if every neighborhood of γ on $\mathbb{S}$ contains a Möbius strip, and *two-sided* otherwise.

2. The orientable genus of graphs with a given nonorientable embedding

Let Π be a (2-cell) embedding of a graph G into a *nonplanar* surface $\mathbb{S}$, i.e. a surface distinct from the 2-sphere. Then we define the *face-width* $\mathbf{fw}(G,\Pi)$ (also called the *representativity*) of the embedding Π as the minimum number of facial walks of G whose union contains a noncontractible curve. Alternatively, $\mathbf{fw}(G,\Pi)$ is the minimum $\mathrm{cr}(\gamma, G)$ taken over all noncontractible closed curves γ on $\mathbb{S}$.

It is easy to see that the nonorientable genus of every graph G is bounded by a linear function of the genus $\mathbf{g}(G)$. On the other hand, Auslander, Brown, and Youngs [1] proved that there are graphs embeddable in the projective plane whose orientable genus is arbitrarily large. This phenomenon is now appropriately understood after Fiedler et al. [3] proved that the genus $\mathbf{g}(G)$ of a graph G that is Π-embedded in the projective plane equals

$$\mathbf{g}(G) = \lfloor \tfrac{1}{2}\mathbf{fw}(\Pi) \rfloor \tag{1}$$

if $\mathbf{fw}(\Pi) \neq 2$. If $\mathbf{fw}(\Pi) = 2$, then $\mathbf{g}(G)$ is either 0 or 1.

This result has been generalized to the Klein bottle by Robertson and Thomas [9] as follows. Let Π be an embedding of G in $\mathbb{N}_2$. Denote by $\mathrm{ord}_2(G,\Pi)$ the minimum of $\lceil \mathrm{cr}(\gamma, G)/2 \rceil$ taken over all noncontractible and nonseparating twosided simple closed curves γ. Similarly, let $\mathrm{ord}_1(G,\Pi)$ denote the minimum of $\lfloor \mathrm{cr}(\gamma_1, G)/2 \rfloor + \lfloor \mathrm{cr}(\gamma_2, G)/2 \rfloor$ taken over all pairs γ_1, γ_2 of nonhomotopic onesided simple closed curves. The latter minimum restricted to all noncrossing pairs γ_1, γ_2 of onesided simple closed curves is denoted by $\mathrm{ord}_1'(G,\Pi)$. Let

$$g = \min\{\mathrm{ord}_1(G,\Pi), \mathrm{ord}_2(G,\Pi)\} \tag{2}$$

and

$$g' = \min\{\mathrm{ord}_1'(G,\Pi), \mathrm{ord}_2(G,\Pi)\}. \tag{3}$$

Robertson and Thomas [9] proved that if $g \geqslant 4$, then $\mathbf{g}(G) = g = g'$. Eqs. (1) and (2) imply that the genus of graphs that can be embedded in the projective plane or the Klein bottle can be computed in polynomial time.

By Thomassen [10], genus testing is **NP**-complete for general graphs. Therefore, it is interesting that the classes of projective planar graphs and graphs embeddable in the Klein bottle admit a polynomial time genus testing algorithm. Very likely the genus problem for graphs with bounded nonorientable genus is solvable in polynomial time as suggested in [9].

Robertson and Thomas [9] conjectured that (1) and (2) can be generalized as follows. Suppose that $\Gamma = \{\gamma_1, \dots, \gamma_p\}$ is a set of closed curves in the surface $\mathbb{N}_k$. Then Γ is *crossing-free* if the following holds:

(a) No γ_i crosses itself.
(b) For $1 \leqslant i < j \leqslant p$, the curves γ_i and γ_j do not cross each other.

If there exist simple closed curves $\gamma'_1, \dots, \gamma'_p$ with pairwise disjoint images in $\mathbb{N}_k$ such that γ'_i is homotopic to γ_i $(i = 1, \dots, p)$ and such that every onesided closed curve in $\mathbb{N}_k$ crosses at least one of the curves $\gamma'_1, \dots, \gamma'_p$, then we say that the family Γ is a *blockage* and that Γ *blocks onesided curves* in the surface.

Suppose that a graph G is embedded in $\mathbb{N}_k$. Robertson and Thomas [9] define the *order* of a blockage $\Gamma = \{\gamma_1, \dots, \gamma_p\}$ as

$$\operatorname{ord}(\Gamma, G) = \frac{1}{2}(k - 2p + s) + \sum_{i=1}^{p} \operatorname{ord}(\gamma_i, G), \tag{4}$$

where s is the number of onesided closed curves in Γ and

$$\operatorname{ord}(\gamma_i, G) = \begin{cases} \lfloor \operatorname{cr}(\gamma_i, G)/2 \rfloor & \text{if } \gamma_i \text{ is onesided,} \\ \lceil \operatorname{cr}(\gamma_i, G)/2 \rceil & \text{if } \gamma_i \text{ is twosided.} \end{cases}$$

Let us observe that the term $\frac{1}{2}(k - 2p + s)$ in (4) is an integer and that it is equal to the genus of the (bordered) orientable surface obtained by cutting $\mathbb{N}_k$ along the curves in Γ. It is easy to prove [9]:

Lemma 2.1. *Let G be a graph embedded in $\mathbb{N}_k$, and let Γ be a blockage in $\mathbb{N}_k$. Then* $\mathbf{g}(G) \leqslant \operatorname{ord}(\Gamma, G)$.

Based on (1)–(3) and Lemma 2.1, Robertson and Thomas proposed the following.

Conjecture 2.2 (Robertson and Thomas [9]). *Suppose that G is embedded in $\mathbb{N}_k$ with sufficiently large face-width. Let g (respectively g') be the minimum order of a blockage (crossing-free blockage) in $\mathbb{N}_k$ Then* $\mathbf{g}(G) = g = g'$.

Mohar [7] disproved this conjecture and posed a related conjecture what the correct expression for $\mathbf{g}(G)$ might be (Conjecture 2.3). The value for the orientable genus of G conjectured in [7] can differ only by a constant (depending on k) from the conjectured value of Robertson and Thomas.

Suppose that G is embedded in $\mathbb{N}_k$. Consider a crossing-free blockage $\Gamma = \{\gamma_1, \dots, \gamma_p\}$ and cut the surface $\mathbb{N}_k$ along $\gamma_1, \dots, \gamma_p$. This results in a graph $\bar{G}$

embedded in an orientable surface. If a vertex $a \in V(G)$ lies on at least one of the curves γ_i $(1 \leqslant i \leqslant p)$, then a gives rise to two or more vertices in $\bar{G}$ (called *copies* of a). Add a new vertex v_a and join it to all copies of a in $\bar{G}$. Call the resulting graph G' and note that contraction of the new edges results in the original graph G. Now, the orientable embedding of $\bar{G}$ defines local rotations of all vertices of G' except for the new vertices v_a. The minimum genus of an orientable embedding of G' extending this partial embedding is called the *genus order* of the blockage Γ. It is easy to see that in the case when no vertex of G is split into more than two vertices of $\bar{G}$, the genus order coincides with (4), and that in general it is majorized by (4).

Conjecture 2.3 (Mohar [7]). *If G is embedded in a nonorientable surface with sufficiently large face-width, then the orientable genus of G is equal to the minimum genus order of a crossing-free blockage.*

In this paper it is proved that Conjectures 2.2 and 2.3 hold up to a constant error term, even without the assumption on large face-width. It is shown that for any graph G embedded in $\mathbb{N}_g$, the orientable genus of G differs from the minimum (genus) order of a crossing-free blockage by less than $(64g)^2$. See Corollary 4.8.

3. Blocking one-sided curves

Suppose that G is a graph that is Π-embedded in some surface $\mathbb{S}$. We denote by $\Gamma = \Gamma(G, \Pi)$ the corresponding *vertex-face graph*. Its vertices are the union of vertices of G and the vertices of the geometric dual G^* of G, i.e., the Π-facial walks. The edges of Γ correspond to the incidence of vertices and faces, with multiple edges if a vertex appears more than once on a Π-facial walk. The graph Γ has a natural quadrilateral embedding in $\mathbb{S}$. The geometric dual of Γ, the graph which we shall denote by $M = M(G, \Pi)$, is known as the *medial graph* of G.

A set $B \subseteq E(M)$ is an *edge-blockage* in M if every one-sided cycle of M contains an edge of B. If $B \subseteq E(M)$, let $B^* \subseteq E(\Gamma)$ be the set of dual edges, and let $\Gamma(B^*)$ be the subgraph of Γ generated by B^*.

Lemma 3.1. *Suppose that G is Π-embedded in $\mathbb{N}_g$ and that $B \subseteq E(M)$ is an edge-blockage in M that is minimal (with respect to inclusion). Then*

(a) *$\Gamma(B^*)$ is a bipartite Eulerian graph (possibly disconnected).*
(b) *The edge set B^* of $\Gamma(B^*)$ can be partitioned into a set of edge-disjoint crossing-free closed walks. Any such partition into crossing-free closed walks is a crossing-free blockage in the surface.*
(c) *$\mathbb{N}_g \backslash \Gamma(B^*)$ is connected.*

(d) *Let n_i be the number of vertices of degree $2i+2$ in $\Gamma(B^*)$. Then*

$$\sum_{i=0}^{\infty} in_i \leqslant g-1. \tag{5}$$

Proof. To prove claim (a), suppose that $\Gamma(B^*)$ contains a vertex x of odd degree d. Let $e_1, \dots, e_d$ be the edges in B dual to the edges of $\Gamma(B^*)$ that are incident with x. By the minimality of B, there exist Π-one-sided cycles $C_i \subseteq E(M)\backslash(B\backslash e_i)$, $i = 1, \dots, d$. Let C_0 be the facial walk in M that corresponds to the vertex x of Γ. It is easy to see that the symmetric difference of the edges of these cycles, $C = C_0 + C_1 + \cdots + C_d$, contains a one-sided cycle in M. This yields a contradiction since C is disjoint from B.

(b) Any partition of B^* into closed walks is obtained as follows. For each vertex $x \in V(\Gamma(B^*))$, partition the edges incident with x into pairs and then join the paired edges to form a collection $\mathscr{C}$ of closed walks in Γ (which may be viewed as closed curves in $\mathbb{N}_g$). By choosing the pairs so that they are not crossing with any other chosen pair of edges incident with the same vertex, none of the curves in $\mathscr{C}$ crosses itself and no two of them cross each other.

Suppose that there is a onesided simple closed curve γ in $\mathbb{N}_g$ that crosses no member of $\mathscr{C}$. By elementary topology, it may be assumed that γ does not intersect any edge of Γ in its internal point, i.e., γ passes through faces and vertices of Γ. Then γ is determined (up to homotopy) by a cyclic sequence $v_1 f_1 v_2 f_2 \dots v_k f_k v_1$ of vertices $v_i \in V(\Gamma)$ and faces f_i of Γ that are traversed by γ. Note that $f_1, \dots, f_k \in V(M)$. For $i = 1, \dots, k$, let S_i be a walk in M that starts with the vertex f_{i-1} of M, traverses a segment of the facial walk in M which corresponds to v_i, and ends at f_i. Clearly, the closed walk W in M which is composed of $S_1, \dots, S_k$ is homotopic to γ (in $\mathbb{N}_g$), so it is one sided. Since γ crosses no curve from $\mathscr{C}$, each S_i contains an even number of edges of B. Let $e_1, \dots, e_{2d}$ be the edges of B that are traversed by W an odd number of times and let $C_1, \dots, C_{2d}$ be as in the proof of part (a). Then $W + C_1 + \cdots + C_{2d}$ contains a one-sided cycle that is disjoint from B, a contradiction.

(c) Suppose that $\mathbb{N}_g\backslash B^*$ is disconnected. Then there is an edge $e^* \in B^*$ such that on each side of e^* there is a different component of $\mathbb{N}_g\backslash B^*$. Let $e \in B$ be the edge which is dual to e. Let C be a Π-onesided cycle in $M\backslash(B\backslash e)$. Since C contains e, it intersects two components of $\mathbb{N}_g\backslash B^*$. Therefore, C crosses B^* at least twice, a contradiction.

(d) $\Gamma(B^*)$ is a graph in $\mathbb{N}_g$ having $n = \sum_i n_i$ vertices and $m = \sum_i (i+1)n_i$ edges. It may be disconnected, and its embedding in $\mathbb{N}_g$ may not be 2-cell. But Euler's inequality still holds: $n - m + f \geqslant \chi(\mathbb{N}_g) = 2 - g$. By (c), the number f of connected components of $\mathbb{N}_g\backslash B^*$ is 1, hence $\sum_i in_i = m - n \leqslant g - 2 + f = g - 1$. □

A vertex set $U \subseteq V(G)$ of a Π-embedded graph G is a *vertex-blockage* if every Π-one-sided cycle of G contains a vertex in U. Similarly, a set $U^* \subseteq V(G^*)$ of Π-faces is a *face-blockage* if every one-sided cycle in the dual graph G^* contains a vertex in U^*.

It is easy to see that U^* is a face-blockage if and only if every one-sided closed curve which does not contain vertices of G intersects a face in U^*.

Lemma 3.2. *Suppose that G is Π-embedded in $\mathbb{N}_g$ and that $B \subseteq E(M)$ is an edge-blockage in M that is minimal* (*with respect to inclusion*). *Let*

$$U = V(\Gamma(B^*)) \cap V(G) \quad \text{and} \quad U^* = V(\Gamma(B^*)) \cap V(G^*). \tag{6}$$

Then U is a vertex-blockage, U^* is a face-blockage in G, and the following inequalities hold:

$$2|U| \leqslant |B| \leqslant 2|U| + 2g - 2, \tag{7}$$

$$2|U^*| \leqslant |B| \leqslant 2|U^*| + 2g - 2, \tag{8}$$

$$|U| + |U^*| \leqslant |B| \leqslant |U| + |U^*| + g - 1. \tag{9}$$

Proof. Let C be a Π-one-sided cycle of G. By Lemma 3.1(b), C intersects $\Gamma(B^*)$. Hence it intersects U. This proves that U is a vertex-blockage. By duality, U^* is a face-blockage.

To prove the first inequality of (7), observe that the minimum degree in $\Gamma(B^*)$ is $\geqslant 2$ (by Lemma 3.1(a)) and that $|B|$ equals the sum of degrees of vertices in U. To verify the second inequality, we shall apply Lemma 3.1(d) and denote by n_i' the number of vertices in U whose degree in $\Gamma(B^*)$ is $2i + 2$. Then

$$\begin{aligned} |B| &= \sum_{u \in U} \deg(u) = 2|U| + 2\sum_i i n_i' \\ &\leqslant 2|U| + 2\sum_i i n_i \leqslant 2|U| + 2(g-1). \end{aligned}$$

Similar proofs yield (8) and (9). □

Let $\beta = \beta(G, \Pi)$ denote the *vertex-blockage number*, i.e. the minimum number of vertices in a vertex-blockage. Similarly, let $\beta^* = \beta^*(G, \Pi)$ be the *face-blockage number* (the minimum number of faces in a face-blockage), and $\beta' = \beta'(G, \Pi)$ the *edge-blockage number* (the minimum number of edges in an edge-blockage in M).

Corollary 3.3. *Suppose that G is Π-embedded in the nonorientable surface $\mathbb{N}_g$. Then the following inequalities hold*:

$$2\beta \leqslant \beta' \leqslant 2\beta + 2g - 2, \tag{10}$$

$$2\beta^* \leqslant \beta' \leqslant 2\beta^* + 2g - 2, \tag{11}$$

$$\beta + \beta^* \leqslant \beta' \leqslant \beta + \beta^* + g - 1. \tag{12}$$

Proof. Let B be a minimum edge-blockage, i.e. $|B| = \beta'$. Then $U = \Gamma(B^*) \cap V(G)$ is a vertex-blockage by Lemma 3.2. This easily implies that $2\beta \leqslant \beta'$.

Suppose now that U is a minimum vertex-blockage in G. If $u \in U$ is a vertex of degree d, split u into d new vertices $u_1, \dots, u_d$, with u_i joined only to the ith neighbor of u ($i = 1, \dots, d$). This operation can be performed on the surface for all vertices in U simultaneously. Since U is a blockage, the resulting graph contains no one-sided cycles. We now start identifying some of the new vertices corresponding to Π-consecutive neighbors of u, say u_i and u_{i+1}. We perform such identifications on the surface as long as possible so that the resulting graph G' contains no one-sided cycles.

Let $B \subseteq E(M)$ be the set of those edges of M that correspond to those Π-consecutive pairs u_i, u_{i+1} that have not been identified. Since G' contains no one-sided cycles, B is an edge-blockage. Moreover, since every further identification gives rise to a one-sided cycle, B is a minimal edge-blockage (with respect to inclusion). It is also obvious that $V(\Gamma(B^*)) \cap V(G) \subseteq U$. By Lemma 3.2 we thus have

$$\begin{aligned} |B| &\leqslant 2|V(\Gamma(B^*)) \cap V(G)| + 2g - 2 \\ &\leqslant 2|U| + 2g - 2 = 2\beta + 2g - 2. \end{aligned}$$

This implies the second inequality in (10).

Relation (11) follows by duality, while (12) is proved analogously. □

Corollary 3.4. *Let G be a graph that is Π-embedded in $\mathbb{N}_g$. Then*

$$\mathbf{g}(G) \leqslant \frac{1}{4}\beta'(G,\Pi) + \frac{g-1}{2},$$

$$\mathbf{g}(G) \leqslant \frac{1}{2}\beta(G,\Pi) + g - 1$$

and

$$\mathbf{g}(G) \leqslant \frac{1}{2}\beta^*(G,\Pi) + g - 1.$$

Proof. Let B be a minimum edge-blockage. By Lemma 3.1(b), $\Gamma(B^*)$ defines a crossing-free blockage $\Gamma = \{\gamma_1, \dots, \gamma_p\}$. Let s be the number of one-sided curves in Γ. Clearly, $\sum_i \mathrm{cr}(\gamma_i, G) = \frac{1}{2}|B^*| = \frac{1}{2}|B| = \frac{1}{2}\beta'(G,\Pi)$. By Lemma 3.1(c), it follows that Γ contains at most $\frac{g}{2}$ twosided curves. Consequently, $\mathrm{ord}(\Gamma, G) \leqslant \frac{1}{2}(g - 2p + s) + \frac{1}{2}\sum_i \mathrm{cr}(\gamma_i, G) + \frac{1}{2}|\{i \mid \gamma_i \text{ is two sided}\}| \leqslant \frac{1}{2}(g - 2p + s) + \frac{1}{4}\beta'(G,\Pi) + \frac{1}{2}(p - s) \leqslant \frac{1}{4}\beta'(G,\Pi) + \frac{g-1}{2}$. By Lemma 2.1, $\mathbf{g}(G) \leqslant \mathrm{ord}(\Gamma, G)$. This proves the first inequality. The second and the third inequality follow from the first one by (10) and (11), respectively. □

4. Unstable faces and blockages

Let Π_0 be an embedding of a graph G. Suppose that there is a facial walk F in which some vertex v appears twice. Then there is a simple closed curve γ in the surface which is contained in the face bounded by F such that $\gamma \cap G = \{v\}$ and γ intersects F in two distinct appearances of v in F. If γ is contractible and its interior contains a vertex or an edge of G, then we delete the vertices and edges of G in the interior of γ. This operation is called an *elementary reduction of type I*.

Suppose now that there are facial walks F and F' such that there exist distinct vertices $v, v' \in V(F) \cap V(F')$. Then there is a simple closed curve γ in the surface which is composed of two segments α, β joining v and v' in the faces bounded by F and F', respectively. If γ is contractible and its interior contains at least two edges of G, then we replace all edges and vertices in its interior by a single edge joining v and v'. Such an operation is called an *elementary reduction of type II*.

The embedded graph G is *essentially 3-connected* if no elementary reductions of type I or II are possible. See also [5]. An obvious property of elementary reductions is the following:

Lemma 4.1. *Let Π be an embedding of a graph G. If the Π'-embedded graph G' is obtained from G by a sequence of elementary reductions, then $\mathbf{g}(G') = \mathbf{g}(G)$ and $\beta(G, \Pi) = \beta(G', \Pi')$, $\beta'(G, \Pi) = \beta'(G', \Pi')$, and $\beta^*(G, \Pi) = \beta^*(G', \Pi')$.*

By Lemma 4.1, we shall be able to restrict ourselves to essentially 3-connected embeddings.

Suppose now that we have two embeddings, Π and Π', of a graph G. Let $F = v_0 e_1 v_1 \ldots v_{k-1} e_k v_0$ be a Π-facial walk. A subsequence $e_i v_i e_{i+1}$ (indices modulo k), $i \in \{1, \ldots, k\}$, is called an *angle* of F. The angle $e_i v_i e_{i+1}$ is identified with the angle $e_{i+1} v_i e_i$ obtained by traversing the facial walk F in the reverse direction. The angle $e_i v_i e_{i+1}$ is *(Π, Π')-unstable* if it is not an angle of the embedding Π'. If two consecutive angles $e_i v_i e_{i+1}$ and $e_{i+1} v_{i+1} e_{i+2}$ of the facial walk F are (Π, Π')-stable but $e_i v_i e_{i+1} v_{i+1} e_{i+2}$ is not a subwalk of a Π'-facial walk, then the angles $e_i v_i e_{i+1}$ and $e_{i+1} v_{i+1} e_{i+2}$ are said to be *weakly (Π, Π')-unstable*.

Suppose that $W = \ldots e_1 v e_2 \ldots$ and $W' = \ldots e_3 v e_4 \ldots$ are walks in a Π'-embedded graph G. If the edges $e_1, \ldots, e_4$ are distinct and their Π'-clockwise order around v is $e_1 e_3 e_2 e_4$ or $e_1 e_4 e_2 e_3$, then we say that W and W' *Π'-cross* at v. Similarly we define *Π'-crossing* of two walks at a common edge e. Two walks are *Π'-crossing* if they Π'-cross at some vertex or at some edge.

Lemma 4.2. *Let Π and Π' be embeddings of a graph G.*

(a) *If evf is a (Π, Π')-unstable angle of a Π-facial walk F, then there is a Π-facial walk F' with an angle $e'vf'$ such that F and F' Π'-cross each other at v.*
(b) *Suppose that dve and euf are weakly (Π, Π')-unstable angles of a Π-facial walk F. Let $F' = \ldots d'veuf' \ldots$ be the second Π-facial walk containing the edge e. Then F*

and F' Π'-cross each other at e. Moreover, either one of the angles $d've$ or euf' is (Π,Π')-unstable, or these two angles of F' are weakly (Π,Π')-unstable.

Proof. To prove (a), consider the local Π'-clockwise ordering $e, e_1, \dots, e_s, f, f_1, \dots, f_t$ of edges around v. Since e and f are not Π'-consecutive, we have $s \geqslant 1$ and $t \geqslant 1$. It is easy to see that there are Π-consecutive edges e', f' such that $e' = e_i$ for some i $(1 \leqslant i \leqslant s)$, and $f' = f_j$ for some j $(1 \leqslant j \leqslant t)$, or vice versa. This implies (a).

Claim (b) is obvious and we leave the details for the reader. □

A collection of cycles $C_1, \dots, C_k$ is called a *collection of bouquets* if there exist vertices $x_1, \dots, x_p$ such that every cycle C_i $(1 \leqslant i \leqslant k)$ contains precisely one of these vertices and such that for any two distinct cycles C_i, C_j $(1 \leqslant i < j \leqslant k)$, the intersection $C_i \cap C_j$ is either empty, one of the vertices $x_1, \dots, x_p$, or an edge incident to one of these vertices.

Part (a) of the following lemma is proved in [4], while part (b) is easy to see (cf., e.g., [6]).

Lemma 4.3. *Let G be a graph embedded in a surface of Euler genus g, and let $C_1, \dots, C_k$ be a collection of bouquets of cycles of G.*

(a) *If $C_1, \dots, C_k$ are noncontractible and pairwise nonhomotopic then $k \leqslant 3g$.*
(b) *If no subset of $C_1, \dots, C_k$ separates the surface then $k \leqslant g$.*

The proof of the next lemma is essentially contained in [6].

Lemma 4.4. *Let G be a Π'-embedded graph and let $\{(C_i, C_i') \mid i = 1, \dots, k\}$ be a collection of pairs of closed walks of G with the following properties:*

(a) *$C_1, \dots, C_k$ are distinct cycles of G and no two of them are Π'-crossing.*
(b) *If $1 \leqslant i < j \leqslant k$ then C_i does not Π'-cross with C_j'.*
(c) *For $i = 1, \dots, k$, $C_i \cap C_i'$ is either a vertex or an edge.*
(d) *For $i = 1, \dots, k$, C_i and C_i' are Π'-crossing at their intersection.*

Then the genus $\mathbf{g}(G, \Pi')$ of Π' is at least k.

Lemma 4.5. *Let $B_U \subseteq E(M)$ be the set of the edges of the medial graph $M(G,\Pi)$ which correspond to the (Π,Π')-unstable and to the weakly (Π,Π')-unstable angles. If Π' is an orientable embedding, then B_U is an edge-blockage for Π.*

Proof. Let C be a one-sided cycle in M. An open (normal) neighborhood of C is homeomorphic to the Möbius band. If $E(C) \cap B_U = \emptyset$, then it is easy to see that the same neighborhood would be a neighborhood of C in the embedding Π'. Since Π' is orientable, we conclude that $E(C) \cap B_U \neq \emptyset$. □

Suppose that the set $\{1, 2, \ldots, 2p\}$ is partitioned into pairs $A_i = \{a_i, b_i\}$, where $a_i < b_i$, $i = 1, \ldots, p$. Suppose that $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant p$ and that $b_i \geqslant a_j$ and $b_j \geqslant a_i$. Then the pair A_i, A_j is called a *canonical pair*. An integer $l \in \{1, 2, \ldots, 2p\}$ is *covered* by this canonical pair if either

(a) $i = j$ and $a_i \leqslant l \leqslant b_i$, or
(b) $i \neq j$ and l is either between a_i and a_j or between b_i and b_j (or both).

Lemma 4.6. *Under the assumptions given above, there is a set of at least $\lceil \sqrt{p/20} \rceil$ canonical pairs such that every $l \in \{1, 2, \ldots, 2p\}$ is covered by at most one of these pairs.*

Proof. The proof is by induction on p. The proof is obvious for $p \leqslant 20$ and easy for $21 \leqslant p \leqslant 80$ (where we need only two canonical pairs).

Suppose now that $p \geqslant 81$. Let $q = \lfloor p/2 \rfloor$. Let us first consider the case when at least $\lceil p/3 \rceil$ pairs A_i satisfy $a_i \leqslant 2q$ and $b_i > 2q$. Let Z be the set of all such pairs. Define a partial order $\preccurlyeq$ on Z by $A_i \preccurlyeq A_j$ if $a_i \leqslant a_j$ and $b_i \leqslant b_j$. By the Dilworth Theorem, this partial order either contains a chain or an antichain of cardinality $z = \lceil \sqrt{|Z|} \rceil \geqslant \lceil \sqrt{p/3} \rceil$. If $A_{i_1}, \ldots, A_{i_z}$ is a chain or an antichain, where $a_{i_1} < a_{i_2} < \cdots < a_{i_z}$, then consecutive pairs in this order are canonical pairs that cover pairwise disjoint subsets of $\{1, \ldots, 2p\}$. This gives rise to at least $\lfloor z/2 \rfloor$ canonical pairs. Since $p \geqslant 81$, $\lfloor z/2 \rfloor \geqslant \frac{1}{2}\sqrt{p/3} - \frac{1}{2} \geqslant \sqrt{p/20}$. This completes the proof in this case.

Suppose now that there are less than $\lceil p/3 \rceil$ such pairs. The remaining subset of at least $\lceil 2p/3 \rceil$ pairs A_i gives rise to two subsets containing p_1 and p_2 pairs, respectively, such that the pairs in the first set are contained in $\{1, \ldots, 2q\}$, and the pairs from the second set are contained in $\{2q+1, \ldots, 2p\}$. Note that $p_1 + p_2 \geqslant \lceil 2p/3 \rceil$ and that $p_1 \leqslant \lfloor p/2 \rfloor$ and $p_2 \leqslant \lceil p/2 \rceil$. In fact, we may assume that $p_1, p_2 \leqslant p/2$. (If $p_2 > p/2$, then we take $q = \lceil p/2 \rceil$ and repeat the above proof.)

By the induction hypothesis, these sets of pairs contain at least $\rho = \lceil \sqrt{p_1/20} \rceil + \lceil \sqrt{p_2/20} \rceil$ canonical pairs that cover disjoint sets. The above conditions on p_1, p_2 imply that $\rho \geqslant \sqrt{(p/2)/20} + \sqrt{(p/6)/20} > \sqrt{p/20}$. This completes the proof. $\square$

Theorem 4.7. *Let G be a graph that is Π-embedded in the nonorientable surface $\mathbb{N}_g$. Then*

$$\tfrac{1}{2}\beta^*(G, \Pi) - (64g)^2 \leqslant \mathbf{g}(G) \leqslant \tfrac{1}{2}\beta^*(G, \Pi) + g - 1. \tag{13}$$

Proof. The second inequality in (13) holds by Corollary 3.4. To prove the first one, it suffices to verify that the bound $\mathbf{g}(G) \geqslant \frac{1}{2}|\mathcal{F}| - (64g)^2$ holds for some face-blockage $\mathcal{F}$ (not necessarily a minimum one). By Lemma 4.1, we may assume that the Π-embedded graph G is essentially 3-connected.

Let Π' be an orientable embedding of G with genus $\mathrm{g}(G)$. Let $B_U \subseteq E(M)$ be the set of those edges of the medial graph $M(G, \Pi)$ which correspond to the (Π, Π')-unstable and to the weakly (Π, Π')-unstable angles. By Lemma 4.5, the set B_U is an edge-blockage.

If a vertex v appears more than once on a facial walk F, then we say that the angles of F at the appearances of v are 1-*singular*. If there are distinct facial walks F, F' such that there exist distinct vertices $v, v' \in V(F) \cap V(F')$ which are not consecutive on (at least) one of these facial walks, then we say that the angles of F and of F' at v and v' are 2-*singular*. For $i = 1, 2$, let $B_i \subseteq E(M)$ be the set of the edges which correspond to the i-singular angles. Since G is essentially 3-connected, the edges in B_i^* correspond to the edges in noncontractible cycles of length $2i$ in the vertex-face graph Γ.

Let B be an edge-blockage contained in $B_U \cup B_1 \cup B_2$ of minimum cardinality. Let $\Lambda = \Gamma(B^*)$ be the subgraph of Γ generated by the edges dual to B.

Consider the connected components of Λ which are cycles. On each of these cycles, select a vertex, and let A_0 be the set of all selected vertices. By Lemma 3.1(c), no subset of these cycles separates the surface and hence, by Lemma 4.3(b), $|A_0| \leqslant g$.

Denote by A_3 the set of vertices of Λ containing A_0 and all vertices of degree >2 in Λ. Let A_4 be the set of all vertices of Λ whose distance in Λ from A_3 is 1 or 2. By (5) we have

$$|A_3 \cup A_4| \leqslant 5|A_0| + \sum_{i \geqslant 1} (1 + 2 \cdot (2i + 2))n_i \leqslant 5g + 9 \sum_{i \geqslant 1} in_i \leqslant 14g - 9.$$

Similar arguments as used above imply that the graph $\Lambda - (A_3 \cup A_4)$ is the union of $r \leqslant 3g - 2$ disjoint paths $P_1, \ldots, P_r$. Choose arbitrarily an orientation of each of the paths $P_1, \ldots, P_r$. If C is a Π-facial walk corresponding to a vertex of P_i $(1 \leqslant i \leqslant r)$, let $v_C \in V(G)$ be the vertex of G that follows C in Λ in the chosen direction of P_i. If the edge of Λ joining C and v_C belongs to B_U^*, then Lemma 4.2 implies that there is a Π-facial walk C' such that C and C' Π'-cross at v_C or Π'-cross at a common edge incident with v_C. We say that C' is a *mate* of C. If the edge joining C and v_C is in B_2^*, then we let the *mate* C' of C be a face such that C and C' intersect at v_C and at another vertex that is not adjacent with v_C.

Let A_1 be the set of vertices of Λ which correspond to Π-facial walks that are not cycles of G. For $x \in A_1$, let F be the corresponding facial walk, and let $v \in V(F)$ be a vertex of G that appears twice in F. Since G is essentially 3-connected, v and F determine a noncontractible cycle of length 2 in Γ. (Possibly, the edges of that cycle are not contained in Λ.) Choose one such 2-cycle for every $x \in A_1$, and let $C_1, \ldots, C_k$ $(k = |A_1|)$ be the resulting collection of cycles of Γ. Clearly, $C_1, \ldots, C_k$ form a bouquet collection in Γ. If $k > 9g$, then Lemma 4.3(a) implies that four of the cycles are homotopic to each other, say Q_1, Q_2, Q_3, Q_4. These cycles of length 2 in Γ may intersect but their vertices corresponding to faces of G are distinct. We may assume that C_1 and C_4 bound a cylinder (or a disk) that contains C_2 and C_3. Now, we add to B^* the edges of Q_1 and Q_4. This gives rise to a new edge-blockage contained in $B_U \cup B_1 \cup B_2$ whose cardinality is $\leqslant |B| + 4$. Since Q_2 and Q_3 are contained in the cylinder (disk) bounded by Q_1 and Q_4, we may remove the edges of B^* incident with the vertices of A_1 that are on Q_2 and Q_3 and also remove the edges of Q_4 and still have a blockage $B' \subset B^* \cup E(C_1)$. Clearly, $|B'| < |B^*|$, a contradiction. Consequently, $|A_1| \leqslant 9g$.

Let A_2 be the set of vertices of $P_1, \ldots, P_r$ that are not in A_1 and correspond to Π-facial cycles which intersect their mate in more than just a vertex or an edge. Let C be such facial cycle, and let C' be its mate. Since G is essentially 3-connected, there is a noncontractible 4-cycle Q in Γ whose vertices are C, C', v_C and another vertex $y \in V(C \cap C')$. Let Z be the set of all such 4-cycles of Γ. For $Q \in Z$, we denote its vertices by $C(Q)$, $C'(Q)$, $v_C(Q)$, and $y(Q)$.

It is a simple exercise to prove that there is a subset $Z_1 \subseteq Z$ of cardinality $\geqslant \frac{1}{9}|A_2|$ such that for any $Q_1, Q_2 \in Z_1$, $v_C(Q_1) \neq y(Q_2)$ and $C(Q_1) \neq C'(Q_2)$. (Hint: Consider the directed graph on all v_C and y-vertices, with an edge from $v_C(Q)$ to $y(Q)$ for each $Q \in Z$, and observe that the outdegree of this digraph is at most 1.) Clearly, $V(Q_1) \cap V(Q_2) \subseteq \{C'(Q_1), y(Q_1)\}$. If Q_1 and Q_2 intersect in two vertices, then we may assume that their intersection is the edge $C'(Q_1)y(Q_1) = C'(Q_2)y(Q_2)$.

Let $z = \sqrt{|Z_1|}$. If there is a vertex y such that $y = y(Q)$ for at least z members of Z_1, then those 4-cycles in Z_1 that contain y form a collection of bouquets of cardinality at least z. Otherwise, there is a subset of Z_1 of cardinality $\geqslant z$ such that no two cycles in this subset have their y-vertex in common. Again, this subset forms a collection of bouquets. If $z > 9g$, then four of the cycles in that collection of bouquets are homotopic, and a proof similar to the above proof of the fact that $|A_1| \leqslant 9g$ yields a contradiction to the minimality of B. This shows that $z \leqslant 9g$ and, therefore, $|A_2| \leqslant 729g^2$.

Let $F_1, \ldots, F_N$ be the facial cycles corresponding to the vertices on $P_1, \ldots, P_r$ which are not in $A_1 \cup A_2$, enumerated in the order of the paths $P_1, \ldots, P_r$ and with respect to their selected orientation. Let $F'_1, \ldots, F'_N$ be their mates. Since the facial cycles corresponding to the vertices in A form a face-blockage, we have

$$\begin{aligned} N &\geqslant \beta^*(G, \Pi) - |A_1 \cup A_2| - |A_3 \cup A_4| \\ &\geqslant \beta^*(G, \Pi) - (27g + 1)^2. \end{aligned} \tag{14}$$

If $i, j \in \{1, \ldots, N\}$ and $j - i \geqslant 2$, then we say that $\{i, j\}$ is a *bad pair* if either F_i and F_j, or F_i and F'_j intersect and Π'-cross each other. Let M be a set of bad pairs of maximum cardinality such that no two members of M have an element in common. Our goal is to prove that $|M| = O(g^2)$.

Each bad pair $\{i, j\}$ determines a path Q_{ij} joining two vertices of A: If F_i and F_j Π'-cross at vertex x, then Q_{ij} is the path of length 2 connecting F_i and F_j through x. If F_i and F'_j Π'-cross at vertex x, then Q_{ij} is the path of length 3 connecting F_i and a vertex in $F_j \cap F'_j$ through x. Clearly, $E(Q_{ij}) \subseteq B^*_C$.

If $\{i, j\}$ is a bad pair and F_i and F_j are in the same path P_a $(1 \leqslant a \leqslant r)$, then Q_{ij} and the edges of P_a determine a cycle R_{ij} which is called the *canonical cycle* of the bad pair $\{i, j\}$. Observe that

$$|E(Q_{ij})| < \tfrac{1}{2}|E(R_{ij})|. \tag{15}$$

Suppose that $\{i,j\}$ and $\{i',j'\}$ $(i<j, i'<j')$ are disjoint bad pairs such that $i'<j$ and $i<j'$. If F_j and $F_{j'}$ are in the same path P_a $(1\leqslant a\leqslant r)$ and F_i and $F_{i'}$ are in the same path P_b $(1\leqslant b\leqslant a)$, then there is a cycle $R_{ij,i'j'}$ in A that is composed of Q_{ij}, $Q_{i'j'}$ and two paths $P_{jj'}\subseteq P_a$ and $P_{ii'}\subseteq P_b$ joining the "upper" and "lower" ends of Q_{ij} and $Q_{i'j'}$, respectively. The cycle $R_{ij,i'j'}$ is called the *canonical cycle* of bad pairs $\{i,j\}$ and $\{i',j'\}$. We shall need an analogy of (15). That is not automatic, but if $|j-j'|\geqslant 4$, then the length of the segment $P_{jj'}$ is at least 7. Consequently,

$$|E(Q_{ij})| + |E(Q_{i'j'})| < |E(P_{jj'})|. \tag{16}$$

We can view $P_1\cup\cdots\cup P_r$ as being a single path by adding auxiliary edges joining the end of P_l with the beginning of P_{l+1}, $l=1,\ldots,r-1$. Then we can define canonical cycles for bad pairs (or pairs of bad pairs) also when the ends of Q_{ij} (and $Q_{i'j'}$) are not in the same path(s) P_a (and P_b). The canonical cycles that use the auxiliary edges are called *fake canonical cycles*; the others are said to be *genuine*.

In order to meet the condition $|j-j'|\geqslant 4$ needed for (16), we order the bad pairs in M according to their larger members j, and let M_1 be the subset consisting of every fourth bad pair in this order.

Lemma 4.6 shows that there is a set of at least $\lceil\sqrt{|M_1|/20}\,\rceil$ canonical cycles (using only bad pairs in M_1) whose intersections with $P_1\cup\cdots\cup P_r$ are pairwise disjoint. Since $r\leqslant 3g-2$, at most $3g-3$ of these canonical cycles are fake. Let $R_1,\ldots,R_s$ $(s\geqslant\lceil\sqrt{|M_1|/20}\,\rceil-3g+3)$ be the subset of genuine canonical cycles. These canonical cycles are disjoint except that they may have a vertex in common if the mate F_j' of F_j is the same as the mate F_l' of F_l, and F_j, F_l are in distinct canonical cycles. Therefore, $R_1,\ldots,R_s$ form a collection of bouquets.

If one of these cycles, say $R_l=R_{ij}$ (or $R_l=R_{ij,i'j'}$) would be contractible, then the replacement in B^* of $E(R_l)\cap P_a$ (or $E(P_{jj'})$) with Q_{ij} (or $Q_{ij}\cup Q_{i'j'}$) would give rise to another blockage. By (15) (or (16)), this blockage would contradict minimality of B. Therefore, R_l is noncontractible. Similar conclusion holds if two of these genuine canonical cycles are homotopic (in which case we can add to B^* the missing edges of one of them and remove the edges of the second one). Lemma 4.3 implies that $s\leqslant 3g$. Consequently, $|M|\leqslant 4|M_1|\leqslant 3240g^2$.

Let A be the set of facial cycles F_l such that l is contained in some bad pair in M. As proved above, $|A|\leqslant 2\cdot 3240g^2$. Let $C_1,\ldots,C_t$ be a maximum subsequence of $F_1,\ldots,F_N$ such that none of C_i is in A and such that, for $i=1,\ldots,t-1$, if $C_i=F_j$, then $C_{i+1}\neq F_{j+1}$. Clearly,

$$t\geqslant\tfrac{1}{2}(N-|A|)\geqslant\tfrac{1}{2}N-3240g^2. \tag{17}$$

For $j=1,\ldots,t$, let C_j' be the mate of C_j. Let us consider the collection of pairs

$$\mathscr{C}=\{(C_j,C_j')\mid j=1,\ldots,t\}.$$

We claim that $\mathscr{C}$ satisfies conditions (a)–(d) of Lemma 4.4. No facial walk C_i is in $A_1\cup A_2$. Therefore, every C_i is a cycle and (c) holds. Since the cycles in A do not participate in the sequence $C_1,\ldots,C_t$, the pairs in $\mathscr{C}$ satisfy (a) and (b). Clearly, (d) is also satisfied.

By Lemma 4.4 and inequalities (14) and (17),

$$\mathbf{g}(G) = \mathbf{g}(G,\Pi') \geqslant |\mathscr{C}| \geqslant \tfrac{1}{2}N - 3240g^2 \geqslant \tfrac{1}{2}\beta^*(G,\Pi) - (64g)^2.$$

The proof is complete. □

The "error" term $(64g)^2$ in (13) is not best possible. There are examples which show that such term of order $\Omega(g)$ is necessary, and we conjecture that (13) can be improved to

$$\mathbf{g}(G) \geqslant \tfrac{1}{2}\beta^*(G,\Pi) - O(g).$$

Corollary 4.8. *Let G be a graph embedded in $\mathbb{N}_g$, and let r be the minimum order of a crossing-free blockage. Then*

$$r - (64g)^2 - g + 1 \leqslant \mathbf{g}(G) \leqslant r.$$

Proof. The second inequality is obvious by Lemma 2.1. To verify the first one, let B be a minimum edge-blockage, and let Γ be a crossing-free blockage corresponding to $\Gamma(B^*)$ by Lemma 3.1(b). The proof of Corollary 3.4 actually gives the following inequality:

$$r \leqslant \operatorname{ord}(\Gamma, G) \leqslant \tfrac{1}{2}\beta^*(G,\Pi) + g - 1. \tag{18}$$

By Theorem 4.7, $\mathbf{g}(G) \geqslant \frac{1}{2}\beta^*(G,\Pi) - (64g)^2$. This inequality combined with (18) completes the proof. □

Finally, let us observe that Corollary 3.3 implies that $\beta(G,\Pi)$ and $\beta^*(G,\Pi)$ cannot differ too much. Therefore, $\mathbf{g}(G)$ is also approximately equal to $\beta(G,\Pi)$, up to a term which depends on g only.

It is not clear if there is an efficient algorithm for finding a minimum (crossing-free) blockage or its approximation for a graph embedded in $\mathbb{N}_g$. For every fixed g, this task is solvable in polynomial time since there is only a bounded number of possibilities for homotopies of curves in an optimum crossing-free blockage. However, this approach seems complicated, and we refrain from describing further details. The case when $g = 2$ is described in [9].

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