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ON THE b-STABLE SET POLYTOPE OF GRAPHS WITHOUT BAD K_4*

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Abstract. We prove that for a graph $G = (V, E)$ without bad K_4 subdivision, and for $b \in \mathbf{Z}_+^{V \cup E}$, the b-stable set polytope is determined by the system of constraints determined by the vertices, edges, and odd circuits. We also prove that this system is totally dual integral. This relates to t-perfect graphs.

Key words. t-perfect, graph, polytope, stable set

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Let $G = (V, E)$ be a graph and let $b \in \mathbf{Z}_+^{V \cup E}$. Then a *$b$-stable set* in G is a vector $x \in \mathbf{Z}_+^V$ satisfying $x_v \leq b_v$ for every vertex v and $x_u + x_v \leq b_{uv}$ for every edge uv. The *b-stable set polytope* of G is defined as the convex hull of the b-stable sets in G.

We will use the following notation. For sets $B \subseteq A$ and a vector $x \in \mathbf{R}^A$, let χ^B be the characteristic vector of B and let $x(B) := x^T \chi^B$. For an edge $\{u, v\}$ we will use the shorthand notation uv.

The vectors in the b-stable set polytope obviously satisfy the following system of inequalities:

$$\text{(1)} \qquad \begin{aligned} &\text{(i)} \quad 0 \leq x_v \leq b_v \quad \text{for each } v \in V; \\ &\text{(ii)} \quad x_u + x_v \leq b_{uv} \quad \text{for each edge } uv \in E; \\ &\text{(iii)} \quad x(VC) \leq \lfloor \tfrac{1}{2} b(EC) \rfloor \quad \text{for each odd circuit } C. \end{aligned}$$

We call a graph G *t-perfect with respect to b* if the b-stable set polytope is determined by (1). Since each integral vector satisfying (1) is a b-stable set, the polytope determined by (1) equals the b-stable set polytope if and only if it is integral. We call a graph G *strongly t-perfect with respect to b* if system (1) is totally dual integral.

For any weight function $w \in \mathbf{Z}_+^V$ and any $b \in \mathbf{Z}_+^{V \cup E}$, denote by $\alpha(G, b, w)$ the maximum *w-weight* $w^T x$ of a b-stable set x in G. Define a *w-cover* as a family of vertices, edges, and odd circuits in G that covers each vertex v at least w_v times. The *b-cost* of a w-cover is defined as the sum of the costs of its elements, where the cost of a vertex v equals b_v, the cost of an edge e equals b_e, and the cost of an odd circuit C equals $\lfloor \frac{1}{2} b(EC) \rfloor$. Denote by $\tilde{\rho}(G, b, w)$ the minimum cost of a w-cover. Strong t-perfection can now be characterized equivalently as follows: a graph $G = (V, E)$ is strongly t-perfect with respect to b if and only if $\alpha(G, b, w) = \tilde{\rho}(G, b, w)$ for every weight function $w \in \mathbf{Z}_+^V$.

Call a subdivision of K_4 *odd* if each triangle of K_4 has become an odd circuit. An odd subdivision of K_4 is called *bad* if there are no two disjoint edges e, f of K_4 such

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that e and f are not subdivided and the other four edges have become even length paths. We say that a graph *has a bad K_4 subdivision* if it has a subgraph that is a bad K_4 subdivision.

In [4], it was proved that a graph has no bad K_4 subdivision if and only if each subgraph is t-perfect with respect to the all-one vector. Here the "if" part follows from the fact that a bad K_4 subdivision is not t-perfect with respect to the all-one vector (see [1]). In [5], it was proved that graphs without bad K_4 subdivision are also strongly t-perfect with respect to the all-one vector. In this paper we prove that graphs having no bad K_4 subdivision are strongly t-perfect with respect to every $b \in \mathbf{Z}_+^{V\cup E}$, which implies our theorem.

THEOREM. *Let $G = (V, E)$ be a graph. Then the following are equivalent:*

(i) *G has no bad K_4 subdivision.*

(ii) *G is t-perfect with respect to each $b \in \mathbf{Z}_+^{V\cup E}$.*

(iii) *G is strongly t-perfect with respect to each $b \in \mathbf{Z}_+^{V\cup E}$.*

Proof. If G satisfies (ii), then also each subgraph of G satisfies (ii). So the implication (ii) $\Longrightarrow$ (i) follows from the fact that a bad K_4 subdivision is not t-perfect with respect to the all-one vector (see [1]).

The implication (iii) $\Longrightarrow$ (ii) follows from the fact that any totally dual integral system with integral right-hand side determines an integral polyhedron.

To prove the implication (i) $\Longrightarrow$ (iii), it will be convenient to first prove the implication (i) $\Longrightarrow$ (ii). Let $G = (V, E)$ be a graph without bad K_4 subdivision, and let $b \in \mathbf{Z}_+^{V\cup E}$. We show that the polytope P determined by (1) is integral. Suppose that x is a nonintegral vertex of P. Let x' be defined by $x'_v := x_v - \lfloor x_v \rfloor$ for every vertex v, and let b' be defined by $b'_v := b_v - \lfloor x_v \rfloor$ for every vertex v and $b'_e := b_e - \lfloor x_u \rfloor - \lfloor x_v \rfloor$ for every edge $e = uv$. Then x' is a nonintegral vertex of the polytope determined by (1) with b replaced by b'. Let $G' := (V, F)$, where $F := \{e \in E |\ b'_e = 1\}$. Since G' has no bad K_4 subdivision and x' satisfies the constraints (1) for the graph G' and the all-one vector $\chi^{V\cup F}$ instead of b, x' is a convex combination of incidence vectors of stable sets in G' by [5]. Each of these incidence vectors is a b'-stable set. Hence x' is a convex combination of b'-stable sets in G, a contradiction. This proves the implication (i) $\Longrightarrow$ (ii).

The remainder of this proof consists of showing the implication (i) $\Longrightarrow$ (iii). The idea is to reduce the general statement to the case in which b is the all-one vector.

Suppose the implication (i) $\Longrightarrow$ (iii) is false. Let the graph $G = (V, E)$ and $b \in \mathbf{Z}_+^{V\cup E}$ form a counter example with (first) $|V| + |E|$ minimal and (second) $b(V)$ minimal. Let $w \in \mathbf{Z}_+^V$ be any weight function for which $\alpha(G, b, w) < \tilde{\rho}(G, b, w)$. Note that by the minimality of G, we know that G has no isolated vertices. We observe the following facts about w and b.

CLAIM 1.

$$\text{(2)} \qquad \begin{array}{ll} \text{(i)} & \alpha(G,b,w) < \alpha(G-e, b|_{G-e}, w) \quad \textit{for each edge } e \in E, \\ \text{(ii)} & b_{uv} < b_u + b_v \quad \textit{for each edge } uv \in E, \\ \text{(iii)} & 1 \le b_u \le b_{uv} \quad \textit{for each edge } uv \in E, \\ \text{(iv)} & 1 \le w_v \quad \textit{for each vertex } v \in V. \end{array}$$

Proof. By the minimality of G, we know that

$$\alpha(G, b, w) < \tilde{\rho}(G, b, w) \le \tilde{\rho}(G - e, b|_{G-e}, w) = \alpha(G - e, b|_{G-e}, w).$$

This gives (i). If for some edge uv we have $b_{uv} \ge b_u + b_v$, then every $b|_{G-uv}$-stable set in $G - uv$ is a b-stable set in G, contradicting (i). Hence we have (ii). Suppose that $b_u > b_{uv}$ for some edge uv. Let $b' := b - \chi^u$. Now we have

$$\alpha(G, b', w) = \alpha(G, b, w) < \tilde{\rho}(G, b, w) = \tilde{\rho}(G, b', w),$$

contradicting the minimality of b. Hence we have $0 \le b_{uv} - b_v < b_u \le b_{uv}$, and (iii) follows. Suppose that $w_v = 0$ for some vertex v. Let $b' := b|_{G-v}$ and $w' := w|_{G-v}$. Then

$$\alpha(G - v, b', w') = \alpha(G, b, w) < \tilde{\rho}(G, b, w) \le \tilde{\rho}(G - v, b', w'),$$

contradicting the minimality of G. Hence we have (iv). □

For the b-stable sets of maximum weight we have the following.

CLAIM 2. *Let x be a b-stable set of w-weight $w^T x = \alpha(G, b, w)$. Then $x_v \le 1$ for each $v \in V$.*

Proof. To see this, suppose that $x_v > 1$ for some vertex v. Let $x' := x - \chi^v$ and $b' := b - \chi^{\{v\} \cup \delta(v)}$. For any b'-stable set $\tilde{x}$ in G, we have

$$w^T \tilde{x} = w^T(\tilde{x} + \chi^v) - w_v \le \alpha(G, b, w) - w_v = w^T x',$$

and hence x' is a maximum w-weight b'-stable set in G. By minimality of b, there exists a w-cover F of b'-cost $\tilde{\rho}(G, b', w) = \alpha(G, b', w)$.

Since $x_v > 1$, we have $x'_v > 0$, and hence by "complementary slackness" v is covered exactly w_v times by F. This implies that F has b-cost

$$\tilde{\rho}(G, b', w) + w_v = \alpha(G, b', w) + w_v = \alpha(G, b, w),$$

a contradiction. □

CLAIM 3. *For every edge $f \in E$ we have $b_f \le 2$.*

Proof. Suppose that Claim 3 is not true and that we have $b_f \ge 3$ for some edge $f = uv$. Let $w' := w + N \cdot \chi^f$, where $N := w(V) + 1$. Then

$$\alpha(G, b, w') = \tilde{\rho}(G, b, w'),$$

since otherwise by Claim 2 applied to w' we have for any maximum w'-weight b-stable set x the inequality

$$w'^T x = w^T x + N(x_u + x_v) \le N - 1 + 2N < 3N,$$

while $x' := b_u \chi^u + (b_f - b_u)\chi^v$ is a b-stable set of w'-weight

$$w'^T x' \ge N \cdot b_f \ge 3N,$$

contradicting the optimality of x.

So we can choose w such that

$$\begin{aligned} \alpha(G, b, w) &< \tilde{\rho}(G, b, w), \\ \alpha(G, b, w + \chi^f) &= \tilde{\rho}(G, b, w + \chi^f). \end{aligned} \tag{3}$$

Let $F := \{v_1, \ldots, v_r, e_1, \ldots, e_s, C_1, \ldots, C_t\}$ be a minimum b-cost $w + \chi^f$-cover, where the v_i are vertices, the e_i are edges, and the C_i are odd circuits. Note that none of the e_i is the edge f, since otherwise $\tilde{\rho}(G, b, w) \le \tilde{\rho}(G, b, w + \chi^f) - b_f$, which would imply that $\tilde{\rho}(G, b, w) \le \alpha(G, b, w + \chi^f) - b_f \le \alpha(G, b, w)$. Let $G' := G - f$, let

$b' := b|_{G'}$, and let x' be a maximum w-weight b'-stable set in G'. Then $\alpha(G', b', w) \geq \alpha(G, b, w) + 1$ by Claim 1, and hence

$$x'(f) > b_f. \tag{4}$$

For any odd circuit C traversing f, we have

$$x'(VC) \leq \lfloor \tfrac{1}{2} b(EC) \rfloor + \tfrac{1}{2}(x'(f) - b_f + 1), \tag{5}$$

since $2x'(VC) \leq x'(f) + b(EC - f) = x'(f) + b(EC) - b_f$. Now let l be the number of circuits in F traversing f. We obtain

$$\begin{aligned}
\tilde{\rho}(G, b, w + \chi^f) &= \alpha(G, b, w + \chi^f) \leq \alpha(G, b, w) + b_f \leq \alpha(G', b', w) - 1 + b_f \\
&= w^T x' - 1 + b_f = (w + \chi^f)^T x' - (x'(f) - b_f + 1) \\
&\leq -(x'(f) - b_f + 1) + \sum_{i=1}^{r} x'(v_i) + \sum_{i=1}^{s} x'(e_i) + \sum_{i=1}^{t} x'(VC_i) \\
&\leq (\tfrac{1}{2} l - 1)(x'(f) - b_f + 1) + \sum_{i=1}^{r} b_{v_i} + \sum_{i=1}^{s} b_{e_i} + \sum_{i=1}^{t} \lfloor \tfrac{1}{2} b(EC_i) \rfloor \\
&= (\tfrac{1}{2} l - 1)(x'(f) - b_f + 1) + \tilde{\rho}(G, b, w + \chi^f).
\end{aligned} \tag{6}$$

Hence we have $(l - 2)(x'(f) - b_f + 1) \geq 0$. Since $x'(f) - b_f + 1 > 0$ by (4), we have $l \geq 2$.

We may assume that C_1 and C_2 traverse f. Decompose the cycle $EC_1 \Delta EC_2$ into circuits $C'_1, \ldots, C'_q$, where $C'_1, \ldots, C'_p$ are odd and $C'_{p+1}, \ldots, C'_q$ are even. Choose in each C'_i with $i = p+1, \ldots, q$ a perfect matching M_i with $b(M_i) \leq \frac{1}{2} b(EC'_i)$. Now the circuits C_1 and C_2 are removed from the cover F, and the circuits $C'_1, \ldots, C'_p$, the edges in the matchings $M_{p+1}, \ldots, M_q$, and the edges in $EC_1 \cap EC_2$ are added to the cover. This gives a $w + \chi^f$-cover F' of b-cost

$$\begin{aligned}
&\tilde{\rho}(G, b, w + \chi^f) - \lfloor \tfrac{1}{2} b(EC_1) \rfloor - \lfloor \tfrac{1}{2} b(EC_2) \rfloor \\
&\qquad + b(EC_1 \cap EC_2) + \sum_{i=1}^{p} \lfloor \tfrac{1}{2} b(EC'_i) \rfloor + \sum_{i=p+1}^{q} b(M_i) \\
&\leq \tilde{\rho}(G, b, w + \chi^f) - \tfrac{1}{2}(b(EC_1) + b(EC_2) - 2) + \tfrac{1}{2}(b(EC_1 \Delta EC_2)) + b(EC_1 \cap EC_2) \\
&= \tilde{\rho}(G, b, w + \chi^f) + 1.
\end{aligned} \tag{7}$$

Hence $F' - f$ is a w-cover of b-cost at most $\tilde{\rho}(G, b, w + \chi^f) + 1 - b_f$. This implies that

$$\begin{aligned}
\alpha(G, b, w) &\leq \tilde{\rho}(G, b, w) - 1 \leq \tilde{\rho}(G, b, w + \chi^f) - b_f \\
&= \alpha(G, b, w + \chi^f) - b_f \leq \alpha(G, b, w).
\end{aligned} \tag{8}$$

So we have equality throughout and, in particular, we obtain

$$\alpha(G, b, w + \chi^f) = \alpha(G, b, w) + b_f.$$

Let x be a maximum $w + \chi^f$-weight b-stable set in G. Then

$$\alpha(G, b, w) + b_f = (w + \chi^f)^T x = w^T x + x(f) \leq \alpha(G, b, w) + b_f,$$

and hence $x(f) = b_f$ and x is a maximum w-weight b-stable set. However, $x(f) = b_f \geq 3$ implies that $x_u > 1$ or $x_v > 1$, contradicting Claim 2. □

Partition the vertex set V into $V_1 := \{v \in V|\ b_v = 1\}$ and $V_2 := \{v \in V|\ b_v = 2\}$. Thus by Claim 1, we know that the edges e spanned by V_1 have $b_e = 1$ and the other edges have $b_e = 2$. We now prove the following claim.

CLAIM 4. *Either $V_1 = \emptyset$ or $V_2 = \emptyset$.*

Proof. To prove the claim, take w with $\alpha(G, b, w) < \tilde{\rho}(G, b, w)$ such that $w(V)$ is minimal. We first prove the following:

(9) If $b_v = 1$ for some vertex v, then there exists a maximum w-weight b-stable set x with $x_v = 0$.

Indeed, let $w' := w - \chi^v$. By the minimality of w, we have

$$\alpha(G, b, w') + 1 = \tilde{\rho}(G, b, w') + 1 \geq \tilde{\rho}(G, b, w) \geq \alpha(G, b, w) + 1.$$

Hence $\alpha(G, b, w') = \alpha(G, b, w)$, implying that there exists a maximum w-weight b-stable set x satisfying $x_v = 0$.

Similarly, we have the following:

(10) If $b_e = 1$ for some edge e, then there exists a maximum w-weight b-stable set x with $x(e) = 0$.

To see this, let $w' := w - \chi^e$. By the minimality of w, we have

$$\alpha(G, b, w') + 1 = \tilde{\rho}(G, b, w') + 1 \geq \tilde{\rho}(G, b, w) \geq \alpha(G, b, w) + 1.$$

Hence $\alpha(G, b, w') = \alpha(G, b, w)$, implying that there exists a maximum w-weight b-stable set x satisfying $x(e) = 0$.

Consider an edge $e = uv$ with $u \in V_1$ and $v \in V_2$. By (9), there is a maximum w-weight b-stable set x with $x_u = 0$. By Claim 2, we know that $x_v \leq 1$. Hence $x(e) \leq 1 < 2 = b_e$. So we have that

(11) for each edge $e \in \delta(V_1)$, there is a maximum w-weight b-stable set x with $x(e) < b_e$.

Next consider an odd circuit traversing an edge in $\delta(V_1)$. We have that

(12) for each odd circuit C traversing an edge in $\delta(V_1)$, there is a maximum w-weight b-stable set x with $x(VC) < \lfloor \frac{1}{2} b(EC) \rfloor$.

Indeed, let C be an odd circuit traversing an edge in $\delta(V_1)$ and suppose that C does not traverse an edge spanned by V_1. Let $u \in V_1$ be a vertex traversed by C. By (9), there is a maximum w-weight b-stable set x with $x_u = 0$. By Claim 2 we have $x(VC) \leq |VC| - 1 < |VC| = \lfloor \frac{1}{2} b(EC) \rfloor$.

Thus we may assume that C traverses an edge spanned by U_1. Then C has three consecutive vertices t, u, and v with $t, u \in V_1$, and $v \in V_2$. By (10) there is a maximum w-weight b-stable set x with $x(tu) = 0 = b_{tu} - 1$. By Claim 2 we have $x_v \leq 1$, and hence $x(uv) \leq 1 \leq b_{uv} - 1$. Thus $2x(VC) \leq b(EC) - 2$, and hence $x(VC) < \lfloor \frac{1}{2} b(EC) \rfloor$.

Now suppose that V_1 and V_2 are nonempty. By minimality of G, we know that there is at least one edge $e \in \delta(V_1)$. Let $G' := G - e$, let $b' := b|_{G'}$, and let x' be a maximum w-weight b'-stable set in G'. Let x maximize $w^T x$ over the b-stable set polytope of G such that x is in general position on the face of optimal solutions. Then by (11) and (12), $x(e) < b_e$ and $x(VC) < \lfloor \frac{1}{2} b(EC) \rfloor$ for each odd circuit traversing e. Hence there is a $0 < \lambda \leq 1$ such that $\tilde{x} := (1 - \lambda)x + \lambda x'$ satisfies the system of constraints (1). By the implication (i) $\Longrightarrow$ (ii), $\tilde{x}$ belongs to the b-stable set polytope of G. However, $w^T \tilde{x} > w^T x$, since $w^T x' = \alpha(G', b', w) > \alpha(G, b, w) = w^T x$ by Claim 1. This contradicts the optimality of x. So either V_1 or V_2 is empty. □

If b is the all-one vector, the total dual integrality of (1) follows from [5]. So V_1 is empty, and hence $b_e = 2$ for every edge e and $b_v = 2$ for every vertex v. Denote by a $2w$-edge cover a vector $y \in \mathbf{Z}_+^E$ with $y(\delta(v)) \geq 2w_v$ for every vertex $v \in V$. It is easy to see that for any $2w$-edge cover y and any 2-stable set x, we have $w^T x \leq \sum_{e \in E} y_e \frac{1}{2} \sum_{v \in e} x(v) \leq y(E)$. By a theorem of Gallai (see [2]), G has a 2-stable set x and a $2w$-edge cover such that $w^T x = y(E)$. Denote by U_y the set of vertices v for which $y(\delta(v))$ is odd. Let x be a 2-stable set and let y be a $2w$-edge cover such that $w^T x = y(E)$ and $|U_y|$ is minimal.

If $U_y \neq \emptyset$, then there is a simple path P connecting two vertices in U_y with $y_e \geq 1$ for each $e \in EP$. Let M be a maximum size matching in P. Then $y' := y + \chi^{EP} - 2\chi^M$ is a $2w$-edge cover with $y'(E) \leq y(E)$ and $|U_{y'}| = |U_y| - 2$, a contradiction. So $y(\delta(v))$ is even for every vertex v and we can write $y = \chi^{EC_1} + \cdots + \chi^{EC_r} + \chi^{EC'_1} + \cdots + \chi^{EC'_s}$ for odd circuits $C_1, \ldots, C_r$ and even circuits $C'_1, \ldots, C'_s$. Let M_i be a perfect matching in C'_i for $i = 1, \ldots, s$. Then $C_1, \ldots, C_r$ together with the edges in the matchings $M_1, \ldots, M_s$ give a w-cover of b-cost $y(E) = w^T x$. Since x is a b-stable set, this implies that $\tilde{\rho}(G, b, w) \leq \alpha(G, b, w)$, contradicting the choice of w. This concludes the proof of the theorem. □

Remark. Let $G = (V, E)$ be a graph with $E \times V$ incidence matrix M. In [3] it was proved that the matrix

$$\begin{pmatrix} I \\ -I \\ M \\ -M \end{pmatrix}$$

has Chvátal rank at most 1 if and only if G has no odd K_4 subdivision. The equivalence of (i) and (ii) of the theorem above has the following reformulation in terms of the Chvátal rank: the matrix

$$\begin{pmatrix} I \\ -I \\ M \end{pmatrix}$$

has Chvátal rank at most 1 if and only if G has no *bad* K_4 subdivision.

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