Nullspace embeddings for outerplanar graphs

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Contents

Abstract

We study relations between geometric embeddings of graphs and the spectrum of associated matrices, focusing on outerplanar embeddings of graphs. For a simple connected graph $G = (V, E)$, we define a

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"good" G-matrix as a $V \times V$ matrix with negative entries corresponding to adjacent nodes, zero entries corresponding to distinct nonadjacent nodes, and exactly one negative eigenvalue. We give an algorithmic proof of the fact that it G is a 2-connected graph, then either the nullspace representation defined by any "good" G-matrix with corank 2 is an outerplanar embedding of G, or else there exists a "good" G-matrix with corank 3.

1 Introduction

We study relations between geometric embeddings of graphs, the spectrum of associated matrices and their signature, and topological properties of associated cell complexes. We focus in particular on 1-dimensional and 2-dimensional embeddings of graphs, in the hope that the techniques can be extended to higher dimensions.

Spectral parameters of graphs. The basic connection between graphs, matrices, and geometric embeddings considered in this paper can be described as follows. We define a *G-matrix* for an undirected graph $G = (V, E)$ as a symmetric real-valued $V \times V$ matrix that has a zero in position (i, j) if i and j are distinct nonadjacent nodes. The matrix is *well-signed* if $M_{ij} < 0$ if i and j are distinct adjacent nodes. (There is no condition on the diagonal entries.) If, in addition, M has exactly one negative eigenvalue, then let us call it *good* (for the purposes of this introduction). Let $\kappa(G)$ denote the largest d for which there exists a good G-matrix.

The parameter κ is closely tied to certain topological properties of the graph. Combining results of [1], [2], [8], [5] and [9], one gets the following facts:

If G is connected, then $\kappa(G) \leq 1 \Leftrightarrow G$ is a path,
If G is 2-connected, then $\kappa(G) \leq 2 \Leftrightarrow G$ is outerplanar,
If G is 3-connected, then $\kappa(G) \leq 3 \Leftrightarrow G$ is planar,
If G is 4-connected, then $\kappa(G) \leq 4 \Leftrightarrow G$ is linklessly embeddable.

We study algorithmic aspects of the first two facts. Let us discuss here the second, which says that if G is a 2-connected graph, then either it has an embedding in the plane as an outerplanar map, or else there exists a good G-matrix with corank 3 (and so the graph is not outerplanar). To construct an outerplanar embedding, we use the nullspace of any good G-matrix with corank 2.

Nullspace representations. To describe this construction, suppose that a $V \times V$ matrix M has corank d. Let $U \in \mathbb{R}^{d \times n}$ be a matrix whose rows form a basis of the nullspace of M. This matrix satisfies the equation

$$UM = 0, \tag{1}$$

where U is a $d \times n$ matrix of rank d and M is a G-matrix with corank d. Let u_i be the column of U corresponding to node $i \in V$. The mapping $u: V \to \mathbb{R}^d$ is called the *nullspace representation of V defined by M*. It is unique up to linear transformations of $\mathbb{R}^d$. (For the purist: the map $V \to \ker(M)^*$ is canonically defined; choosing the basis in $\ker(M)$ just identifies $\ker(M)$ with $\mathbb{R}^d$.)

If $G = (V, E)$ is a graph and $u: V \to \mathbb{R}^d$ is any map, we can extend it to the edges by mapping the edge ij to the straight line segment between u_i and u_j. If u is the nullspace representation of V defined by M, then this extension gives the *nullspace representation of G defined by M*.

In this paper we give algorithmic proofs of two facts:

(1) If G is connected graph with $\kappa(G) = 1$, then the nullspace representation defined by any well-signed G-matrix M with one negative eigenvalue and with corank 1 yields an embedding of G in the line.

(2) If G is 2-connected and $\kappa(G) = 2$, then the nullspace representation defined by any well-signed G-matrix M with one negative eigenvalue and with corank 2 yields an outerplanar embedding of G.

The proofs are algorithmic in the sense that (say, in the case of (2)) for every 2-connected graph we either construct an outerplanar embedding or a good G-matrix with corank 3 in polynomial time. The alternative proof that can be derived from the results of [6] uses the minor-monotonicity of the Colin de Verdière parameter (see below), and this way it involves repeated reference to the Implicit Function Theorem, and does not seem to be implementable in polynomial time. The word "yields" above hides some issues concerning normalization, to be discussed later.

Paper [6] also contains the analogous result for planar graphs, which was extended in [7]:

(3) If G is 3-connected and $\kappa(G) = 3$, then the nullspace representation defined by any well-signed G-matrix with one negative eigenvalue and with corank 2 yields a representation of G as the skeleton of a convex 3-polytope.

Again, the proof uses the minor-monotonicity of the Colin de Verdière parameter and the Implicit Function Theorem, and thus it is not algorithmic.

It would be interesting to see whether our approach can be extended to an algorithmic proof for dimension 3. (While we focus on the case $\kappa(G) = 2$, some of our results do bear upon higher dimensions, in particular the results in Section 2.2 below.)

A further extension to dimension 4 would be particularly interesting, since 4-connected linklessly embeddable graphs are characterized by the property that $\kappa(G) \leq 4$, but it is not known whether the nullspace representation obtained from a good G-matrix of corank 4 yields a linkless embedding of the graph.

The Strong Arnold Hypothesis and the Colin de Verdière number. We conclude this introduction with a discussion of the connection between the parameter $\kappa(G)$ and the graph parameter $\mu(G)$ introduced by Colin de Verdière [1]. This latter is defined similarly to κ as the maximum corank of a good G-matrix M, where it is required, in addition, that M has a nondegeneracy property called the *Strong Arnold Property.* There are several equivalent forms of this property; let us formulate one that is related to our considerations in the sense that it uses any nullspace representation u defined by M: if a symmetric $d \times d$ matrix N satisfies $u_i^\mathsf{T} N u_i = 0$ for all $i \in V$ and $u_i^\mathsf{T} N u_j = 0$ for each edge ij of G, then $N = 0$. In more geometric terms this means that the nullspace representation of the graph defined by M is not contained in any nontrivial homogeneous quadric.

The relationship between μ and κ is not completely clarified. Trivially $\mu(G) \leq \kappa(G)$. Equality does not hold in general: consider the graph $G_{l,m}$ made from an $(l+m)$-clique by removing the edges of an m-clique. If $l \geq 1$ and $m \geq 3$, then $\mu(G_{l,m}) = l + 1$ whereas $\kappa(G_{l,m}) = l + m - 2$. (Note that $G_{l,m}$ is not $l + 1$-connected.)

Colin de Verdière's parameter has several advantages over κ. First, it is minor-monotone, while $\kappa(G)$ is not minor-monotone, not even subgraph-monotone: any path P satisfies $\kappa(P) \leq 1$, but a disjoint union of paths can have arbitrarily large $\kappa(G)$. Furthermore, the connection with topological properties of graphs holds for μ without connectivity conditions:

$\mu(G) \leq 1 \Leftrightarrow G$ is a disjoint union of paths,
$\mu(G) \leq 2 \Leftrightarrow G$ is outerplanar,
$\mu(G) \leq 3 \Leftrightarrow G$ is planar,
$\mu(G) \leq 4 \Leftrightarrow G$ is linklessly embeddable in $\mathbb{R}^3$.

Our use of κ is motivated by its easier definition and by the (slightly) stronger, algorithmic results.

We see from the facts above that by requiring that G is $\mu(G)$-connected, we have $\mu(G) = \kappa(G)$ for $\mu(G) \leq 4$. In fact, it was shown by Van der

Holst [3] that if G is 2-connected outerplanar or 3-connected planar, then *every* good G-matrix has the Strong Arnold Property. This also holds true for 4-connected linklessly embeddable graphs [9]. One may wonder whether this remains true for $\mu(G)$-connected graphs with larger $\mu(G)$. This would imply that $\mu(G) = \kappa(G)$ for every $\mu(G)$-connected graph.

2 G-matrices

2.1 Nullspace representations

Let us fix a connected graph $G = (V, E)$ on node set $V = [n]$, and an integer $d \geq 1$. We denote by $\mathcal{W} = \mathcal{W}(G, d)$ the set of well-signed G-matrices with corank at least d, and by $\mathcal{W}_0 = \mathcal{W}_0(G, d)$, the set of well-signed G-matrices with corank exactly d. We define $\mathcal{W}' = \mathcal{W}'(G, d)$ as the set of G-matrices in $\mathcal{W}(G, d)$ with exactly one negative eigenvalue (of multiplicity 1). We denote by $\mathcal{M}_u$ the linear space of G-matrices M with $UM = 0$, by $\mathcal{W}_u$, the set of well-signed G-matrices in $\mathcal{M}_u$, and by $\mathcal{W}'_u$, the set of matrices in $\mathcal{W}_u$ with exactly one negative eigenvalue.

We can always perform a linear transformation of $\mathbb{R}^d$, i.e., replace U by AU, where A is any nonsingular $d \times d$ matrix. In the case when $\text{corank}(M) = d$ (which will be the important case for us), the matrix U is determined by M up to such a linear transformation of $\mathbb{R}^d$.

Another simple transformation we use is "node scaling": replacing U by $U' = UD$ and M by $M' = D^{-1}MD^{-1}$, where D is a nonsingular diagonal matrix. Then M' is a G-matrix and $U'M' = 0$. Through this transformation, we may assume that every nonzero vector u_i has unit length. We call such a representation *normalized*.

One of our main tools will be to describe more explicit solutions of the basic equation (1) in dimensions 1 and 2. More precisely, given a graph $G = (V, E)$ and a representation $U : V \to \mathbb{R}^2$, our goal is to describe all G-matrices M with $UM = 0$. Note that it suffices to find the off-diagonal entries: if M_{ij} is given for $ij \in E$ in such a way that

$$\sum_{j \in N(i)} M_{ij} u_j \parallel u_i, \tag{2}$$

then there is a unique choice of diagonal entries M_{ii} that gives a matrix with $UM = 0$:

$$M_{ii} = -\sum_j M_{ij} \frac{u_j^\mathsf{T} u_i}{u_i^\mathsf{T} u_i}. \tag{3}$$

2.2 G-matrices and eigenvalues

In this section we consider eigenvalues of well-signed G-matrices; we consider the connected graph G and the dimension parameter d fixed. We start with a couple of simple observations.

Lemma 1 *Let M be a well-signed G-matrix with corank $d \geq 1$ and let $U \in \mathbb{R}^{d\times n}$ such that $UM = 0$ and $\mathrm{rank}(U) = d$.*

(a) *If M is positive semidefinite, then $d = 1$, and all entries of U are nonzero and have the same sign.*

(b) *If M has a negative eigenvalue, then the origin is an interior point of the convex hull of the columns of U.*

Proof. Let λ be the smallest eigenvalue of M. As G is connected, λ has multiplicity one by the Perron–Frobenius theorem, and M has a positive eigenvector v belonging to λ. If $\lambda = 0$, then this multiplicity is $d = 1$, and U consists of a single row parallel to v. If $\lambda < 0$, then every row of U, being in the nullspace of M, is orthogonal to v. Thus the entries of v provide a representation of 0 as a convex combination of the columns of U with positive coefficients. □

Lemma 2 *If $d \geq 2$, then the set $\mathcal{W}'$ is relatively closed in $\mathcal{W}$, and $\mathcal{W}' \cap \mathcal{W}_0$ is relatively open in $\mathcal{W}$.*

Proof. Let $\lambda_i(M)$ denote the i-th smallest eigenvalue of the matrix M. We claim that for any $M \in \mathcal{W}$,

$$M \in \mathcal{W}' \iff \lambda_2(M) \geq 0. \tag{4}$$

Indeed, if $M \in \mathcal{W}'$, then trivially $\lambda_2(M) \geq 0$. Conversely, if $\lambda_2(M) \geq 0$, then M has at most one negative eigenvalue. By Lemma 1(a), it has exactly one, that is, $M \in \mathcal{W}'$. This proves (4). Since $\lambda_2(M)$ is a continuous function of M, the first assertion of the lemma follows.

We claim that if $d \geq 2$, for any $M \in \mathcal{W}$,

$$M \in \mathcal{W}' \cap \mathcal{W}_0 \iff \lambda_{d+2}(M) > 0. \tag{5}$$

Indeed, if $M \in \mathcal{W}' \cap \mathcal{W}_0$, then M has one negative eigenvalue and exactly d zero eigenvalues, and so $\lambda_{d+2}(M) > 0$. Conversely, assume that $\lambda_{d+2}(M) > 0$. Since M has at least d zero eigenvalues and at least one negative eigenvalue (by Lemma 1(a)), we must have equality in both bounds, which means that $M \in \mathcal{W}' \cap \mathcal{W}_0$. This proves (5). Continuity of $\lambda_{d+2}(M)$ implies the second assertion. $\square$

This last proposition implies that each nonempty connected subset of $\mathcal{W}_0$ is either contained in $\mathcal{W}'$ or is disjoint from $\mathcal{W}'$. We formulate several consequences of this fact.

Lemma 3 *Suppose that G is 2-connected, and let M be a well-signed G-matrix with one negative eigenvalue and with corank $d = \kappa(G)$. Let u be the nullspace representation defined by M, let $v \in \mathbb{R}^d$, and let $J := \{i :\ u_i = v\}$. If $|J| \geq 2$, then the origin 0 belongs to the convex hull of $u(V \setminus J)$.*

Proof. For $i \in V$, let e_i be the i-th unit basis vector, and for $i, j \in V$, let D^{ij} be the matrix $(e_i - e_j)(e_i - e_j)^\mathsf{T}$. Define

$$M^\alpha := M + \alpha \sum_{\substack{ij \in E \\ i,j \in J}} M_{ij} D^{ij} \qquad (\alpha \in [0,1]).$$

The definition of J implies that $\ker(M) \subseteq \ker(D^{ij})$ for all $i, j \in J$, and hence $\ker(M) \subseteq \ker(M^\alpha)$ for each $\alpha \in [0,1]$. So $\text{corank}(M^\alpha) \geq \text{corank}(M) = \kappa(G)$ for each $\alpha \in [0,1]$. Moreover, M^α is a well-signed G-matrix for each $\alpha \in [0,1)$. Since $M = M^0 \in \mathcal{W}'$, Lemma 2 implies that $M^\alpha \in \mathcal{W}'$ for each $\alpha \in [0,1)$. By the continuity of eigenvalues, M^1 has at most one negative eigenvalue. Note that $M^1_{ij} = 0$ for any two distinct $i, j \in J$.

Assume that 0 does not belong to the convex hull of $\{u_i :\ i \notin J\}$. Then there exists $c \in \mathbb{R}^{\kappa(G)}$ such that $u_i^\mathsf{T} c < 0$ for each $i \notin J$. As 0 belongs to interior of the convex hull of $u(V)$ by Lemma 1(b), this implies that $u_i^\mathsf{T} c = v^\mathsf{T} c > 0$ for each $i \in J$.

As $|J| \geq 2$, the 2-connectivity of G implies that J contains two distinct nodes, say nodes 1 and 2, that have neighbors outside J. Since $\ker(M) \subseteq \ker(M^1)$, we have $\sum_j M^1_{1j} u_j = 0$, and hence

$$M^1_{11} u_1^\mathsf{T} c = -\sum_{j \neq 1} M^1_{1j} u_j^\mathsf{T} c = -\sum_{j \notin J} M^1_{1j} u_j^\mathsf{T} c.$$

As $u_1^\mathsf{T} c > 0$ and $u_j^\mathsf{T} c < 0$ for $j \notin J$, and as $M^1_{1j} \leq 0$ for all $j \notin J$, and $M^1_{1j} < 0$ for at least one $j \notin J$, this implies $M^1_{11} < 0$. Similarly, $M^1_{22} < 0$. As $M^1_{12} = 0$, the first two rows and columns of M' induce a negative definite 2×2 submatrix of M^1. This contradicts the fact that M^1 has at most one negative eigenvalue. $\square$

For the next step we need a simple lemma from linear algebra.

Lemma 4 *Let A and M be symmetric $n \times n$ matrices. Assume that A is 0 outside a $k \times k$ principal submatrix, and let M_0 be the complementary $(n-k) \times (n-k)$ principal submatrix of M. Let a and b denote the number of negative eigenvalues of A and M_0, respectively. Then for some $s > 0$, the matrix $sM + A$ has at least $a + b$ negative eigenvalues.*

Proof. We may assume $A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} M_1 & M_2^\mathsf{T} \\ M_2 & M_0 \end{pmatrix}$, with A_0 and M_1 having order $k \times k$. By scaling the last $n-k$ rows and columns of $sM+A$ by $1/\sqrt{s}$, we get the matrix $\begin{pmatrix} sM_1 + A_0 & \sqrt{s}M_2^\mathsf{T} \\ \sqrt{s}M_2 & M_0 \end{pmatrix}$. Letting $s \to 0$, this tends to $B = \begin{pmatrix} A_0 & 0 \\ 0 & M_0 \end{pmatrix}$. Clearly, B has $a + b$ negative eigenvalues, and by the continuity of eigenvalues, the lemma follows. □

Lemma 5 *Let M be a well-signed G-matrix with one negative eigenvalue and with corank $d = \kappa(G)$, let u be the nullspace representation defined by M, and let C be a clique in G of size at most $\kappa(G)$ such that the origin belongs to the convex hull of $u(C)$. Then $G - C$ is disconnected.*

Proof. We can write $0 = \sum_i a_i u_i$ with $a_i \geq 0$, $\sum_i a_i = 1$, and $a_i = 0$ if $i \notin C$. Let A be the matrix $-aa^\mathsf{T}$. Since a is nonzero, A has a negative eigenvalue.

Since $\sum_i a_i u_i = 0$, we have $\ker(M) \subseteq \ker(M + sA)$ for each s. This implies that $\operatorname{corank}(M + sA) \geq \operatorname{corank}(M)$ for each s. Moreover, $M + sA$ is a well-signed G-matrix for $s \geq 0$. Hence, as $M \in \mathcal{W}'$, we know by Lemma 2 that $M + sA \in \mathcal{W}'$ for every $s \geq 0$. In other words, $M + sA$ has one negative eigenvalue for every $s \geq 0$.

Let M_0 be the matrix obtained from M by deleting the rows and columns with index in C. Note that M_0 has no negative eigenvalue: otherwise by Lemma 4, $M + sA$ has at least two negative eigenvalues for some $s > 0$, a contradiction.

Now suppose that $G - C$ is connected. As $u(C)$ is linearly dependent and $|C| \leq \operatorname{corank}(M)$, $\ker(M)$ contains a nonzero vector x with $x_i = 0$ for all $i \in C$. Then by the Perron–Frobenius theorem, $\operatorname{corank}(M_0) = 1$ and $\ker(M_0)$ is spanned by a positive vector y. As G is connected, x is orthogonal to the positive eigenvector belonging to the negative eigenvalue of M. So x has both positive and negative entries. On the other hand, $x|_{V \setminus C} \in \ker(M_0)$, and so $x|_{V \setminus C}$ must be a multiple of y, a contradiction. □

Taking C a singleton, we derive:

Corollary 6 *Let G be a 2-connected graph, let $M \in \mathcal{W}'$ have corank $\kappa(G)$, and let u be the nullspace representation defined by M. Then $u_i \neq 0$ for all i. Equivalently, the nullspace representation defined by M can be normalized by node scaling.*

3 1-dimensional nullspace representations

As a warmup, let us settle the case $d = 1$. For every connected graph $G = (V, E)$, it is easy to construct a singular G-matrix with exactly one negative eigenvalue: start with any G-matrix, and subtract an appropriate constant from the main diagonal. Our goal is to show that unless the graph is a path and the nullspace representation is a monotone embedding in the line, we can modify the matrix to get a G-matrix with one negative eigenvalue and with corank at least 2.

3.1 Nullspace and neighborhoods

We start with noticing that given vector $u \in \mathbb{R}^V$, it is easy to describe the matrices in $\mathcal{W}_u$. Indeed, consider any matrix $M \in \mathcal{M}_u$. Then for every node i with $u_i = 0$, we have

$$\sum_{j \in N(i)} M_{ij} u_j = \sum_j M_{ij} u_j = 0. \tag{6}$$

Furthermore, for every node i with $u_i \neq 0$, we have

$$M_{ii} = -\frac{1}{u_i} \sum_{j \in N(i)} M_{ij} u_j. \tag{7}$$

Conversely, if we specify the off-diagonal entries of a G-matrix M so that (6) is satisfied, then we can define M_{ii} for nodes $i \in \mathrm{supp}(u)$ according to (7), and for nodes i with with $u_i = 0$ arbitrarily, we get a matrix in $\mathcal{M}_u$.

As an application of this construction, we prove the following lemma.

Lemma 7 *Let $u \in \mathbb{R}^V$. Then $\mathcal{W}_u \neq \emptyset$ if and only if for every node i with $u_i = 0$, either all its neighbors satisfy $u_j = 0$, or it has neighbors both with $u_j < 0$ and $u_j > 0$.*

Proof. By the remark above, it suffices to specify negative numbers M_{ij} for the edges ij so that (6) is satisfied for each i with $u_i = 0$. The edges between two nodes with $u_i = 0$ play no role, and so the conditions (6) can be considered separately. For a fixed i, the single linear equation for the M_{ij} can be satisfied by negative numbers if and only if the condition in the lemma holds. □

We need the following fact about the neighbors of the other nodes.

Lemma 8 *Let $u \in \mathbb{R}^V$, $M \in \mathcal{W}_u$, and suppose that M has a negative eigenvalue $\lambda < 0$, with eigenvector $\pi > 0$. Then every node i with $u_i > 0$ has a neighbor j for which $u_j/\pi_j < u_i/\pi_i$.*

Proof. Suppose not. Then $u_j \geq \pi_j u_i/\pi_i$ for every $j \in N(i)$, and so

$$0 = \sum_j M_{ij} u_j \leq M_{ii} u_i + \sum_{j \in N(i)} M_{ij} \frac{\pi_j}{\pi_i} u_i = \frac{u_i}{\pi_i} \Big(\sum_j M_{ij} \pi_j \Big) = \lambda u_i < 0,$$

a contradiction. □

3.2 Auxiliary algorithms

No we turn to the algorithmic part, starting with some auxiliary algorithms.

Algorithm 1 (Interpolation)
Input: a vector $u \in \mathbb{R}^V$ and two matrices $M \in \mathcal{W}'_u$ and $M' \in \mathcal{W}_u \setminus \mathcal{W}'_u$.
Output: a matrix $M'' \in \mathcal{W}'_u$ with corank at least 2.

Consider the well-signed G-matrices $M^t = tM' + (1-t)M$ $(0 \leq t \leq 1)$. As $\mathcal{W}' \cap \mathcal{W}_0$ is open and closed in $\mathcal{W}_0$, there must be points $t \in [0,1]$ where $\operatorname{corank}(M^t) > 1$. We can find these values t by considering any nonsingular $(n-1) \times (n-1)$ submatrix of M, and the corresponding submatrix B^t of M^t. Then every value of t with $\operatorname{corank}(M^t) > 1$ is a root of the algebraic equation $\det(B^t) = 0$, so only these have to be inspected. The first such point will give a matrix $M^t \in \mathcal{W}'_u$ with $\operatorname{corank}(M^t) > 1$.

Algorithm 2 (Double node)
Input: a vector $u \in \mathbb{R}^V$, two nodes i and j with $u_i = u_j = 0$, and a matrix $M \in \mathcal{W}_u$.
Output: a matrix $M' \in \mathcal{W}_u$ with at least two negative eigenvalues.

Subtract $t > 0$ from both diagonal entries M_{ii} and M_{jj}, to get a matrix M'. Trivially $M' \in \mathcal{W}_u$. Furthermore, if $t > 2\max\{|M_{ii}|, |M_{jj}|, |M_{ij}|\}$, then the submatrix of M' formed by rows and columns i and j has negative trace and positive determinant, and so it has two negative eigenvalues. This implies that M' has at least two negative eigenvalues.

Algorithm 3 (Double cover)

Input: a vector $u \in \mathbb{R}^V$, two edges ab and cd with $u_a, u_c < 0$ and $u_b, u_d > 0$, and a matrix $M \in \mathcal{W}_u$.

Output: a matrix $M' \in \mathcal{W}_u$ with at least two negative eigenvalues.

Assume that $b \neq d$ (the case when $a \neq c$ can be treated similarly). Define the symmetric matrix $N^{ab} \in \mathbb{R}^{V \times V}$ by

$$(N^{ab})_{ij} = \begin{cases} u_b/u_a, & \text{if } \{i,j\} = \{a,b\}, \\ -u_b^2/u_a^2, & \text{if } i = j = a, \\ -1, & \text{if } i = j = b, \\ 0, & \text{otherwise,} \end{cases}$$

and define N^{cd} analogously. Then $N^{ab}u = N^{cd}u = 0$, and so $M' = M + tN^{ab} + tN^{cd} \in \mathcal{W}_u$ for every $t > 0$. Furthermore, if $t > 2\max\{|M_{bb}|, |M_{dd}|, |M_{bd}|\}$, then M' has at least two negative eigenvalues by the same argument as in Algorithm 2.

3.3 Embedding in the line

Now we come to the main algorithm for dimension 1.

Algorithm 4

Input: A connected graph $G = (V, E)$.

Output: Either an embedding $u : V \to \mathbb{R}$ of G (then G is a path), or a well-signed G-matrix with one negative eigenvalue and corank at least 2.

Preparation. We find a matrix $M \in \mathcal{W}'(G)$. This is easy by creating any well-signed G-matrix and subtracting its second smallest eigenvalue from the diagonal. We may assume that $\text{corank}(M) = 1$, else we are done.

Let $u \neq 0$ be a vector in the nullspace of M, and let π be an eigenvector belonging to its negative eigenvalue. We apply node-scaling, and get that the matrix $M' = \text{diag}(\pi) M \text{diag}(\pi)$ is in $\mathcal{W}'(G)$ and the vector $w = (u_i/\pi_i : i \in V)$ is in its nullspace. By Lemma 8, this means that if we replace M by M' and u by w, then we get a vector $u \in \mathbb{R}^n$ and a matrix $M \in \mathcal{W}'_u$ such that every node i with $u_i > 0$ has a neighbor j with $u_j < u_i$, and every node i with $u_i < 0$ has a neighbor j with $u_j > u_i$.

Let us define a *cell* as an open interval between two consecutive points u_i. If every cell is covered by only one edge, then G is a path and u defines an embedding of G in the line, and we are done. Else, let us find a cell (a, b)

covered by at least two edges that is nearest the origin. Replacing u by $-u$ if necessary, we may assume that $b > 0$.

Main step. Below, we are going to maintain the following conditions. We have a vector $u \in \mathbb{R}^V$ and a matrix $M \in \mathcal{W}'_u$; every node i with $u_i > 0$ has a neighbor j with $u_j < u_i$; there is a cell (a, b) with $b > 0$ that is doubly covered, and that is nearest the origin among such cells.

We have to distinguish some cases.

Case 1. If $a < 0$, then we use the Double Cover Algorithm 3 to find a matrix $M' \in \mathcal{W}_u$ with two negative eigenvalues, and the Interpolation Algorithm 1 returns a matrix with the desired properties.

Case 2. If $a \geq 0$, then let u_p be the smallest nonnegative entry of u.

Case 2.1. Assume that $u_p = 0$. If there is a node $j \neq p$ with $u_j = 0$, then run the Double Node algorithm 2 to get a matrix in $\mathcal{W}_u$ with at least two negative eigenvalues, and we can finish by the Interpolation Algorithm 1 again. So we may assume that $u_j \neq 0$ for $j \neq p$.

Let $(0, c)$ be the cell incident with 0 ($c > 0$), and let M' be obtained from M by replacing the (p, p) diagonal entry by 0, then $M' \in \mathcal{W}_u$. It follows by Lemma 1 that M' is not positive semidefinite. If M' has more than one negative eigenvalue, then we can run the Interpolation Algorithm 1. So we may assume that $M' \in \mathcal{W}'_u$.

For $t \in (0, c)$, consider the G-matrices A^t defined for edges ij by

$$A^t_{ij} = A^t_{ji} = \begin{cases} M_{ij}, & \text{if } i, j \neq p, \\ \dfrac{u_j}{u_j - t} M_{pj}, & \text{if } i = p, \end{cases}$$

and on the diagonal by

$$A^t_{ii} = -\frac{1}{u_i - t} \sum_{j \in N(i)} A^t_{ij}(u_j - t).$$

Clearly, A^t is a well-signed G-matrix and $A^t(u - t) = 0$. This means that $A^t \in \mathcal{W}_{u-t}$. Lemma 1 implies that A^t has at least one negative eigenvalue. Furthermore, if $t \to 0$, then $A^t_{ij} \to M_{ij}$; this is trivial except for $i = j = p$, when, using that $\sum_{j \in N(p)} M_{pj} u_j = -M_{pp} u_p = 0$, we have

$$A^t_{pp} = \frac{1}{t} \sum_{j \in N(p)} M_{pj} u_j = 0.$$

Thus $A^t \to M'$ as $t \to 0$.

If the matrix $A^{c/2}$ has one negative eigenvalue, then replace M by $A^{c/2}$ and u by $u - c/2$, and return to the Main Step. Note that the number of nodes with $u_i \geq 0$ has decreased, while those with $u_i > 0$ did not change.

If it has more than one, then consider the points $t \in (0, c/2]$ where $\text{corank}(A^t) > 1$ (such a value of t exists by Lemma 2). These values of t can be found like in the Interpolation Algorithm 1. The smallest such value of t gives $A^t \in \mathcal{W}'(G)$ and $\text{corank}(A^t) > 1$, and we are done.

Case 2.2. Assume that $u_p > 0$. Let σ and τ denote the cells to the left and to the right of u_p (so $0 \in \sigma$). There is no other node q with $u_q = u_p$ (since from both nodes, an edge would start to the left, whereas 0 is covered only once). From u_p, there is an edge starting to the left, and also one to the right (since by connectivity, there is an edge covering τ, and this must start at p, since σ is covered only once). Therefore, $\mathcal{W}_{u-u_p} \neq \emptyset$ by Lemma 7. Following the proof of this Lemma, we can construct a matrix $B \in \mathcal{W}_{u-u_p}$.

For $t \in [0, u_p)$, consider the G-matrices B^t defined for edges ij by

$$B^t_{ij} = B^t_{ji} = \begin{cases} B_{ij}, & \text{if } i, j \neq p, \\ \dfrac{u_j - u_p}{u_j - t} B_{pj}, & \text{if } i = p, \end{cases}$$

and on the diagonal by

$$B^t_{ii} = -\frac{1}{u_i - t} \sum_{j \in N(i)} B^t_{ij}(u_j - t).$$

Clearly, B^t is a well-signed G-matrix and $B^t(u - t) = 0$. Furthermore, $B^t \to B$ if $t \to u_p$.

If B has one negative eigenvalue, then replace M by B and u by $u - u_p$, and go to the Main Step. Note that the number of nodes with $u_i > 0$ has decreased, while those with $u_i \geq 0$ did not change.

If B^0 has more than one negative eigenvalue, then we call the Interpolation Algorithm 1, to get a matrix in $\mathcal{W}'_u$ with corank at least 2. Finally, if B has more than one negative eigenvalue and B^0 has only one, then there must be values of t such that $\text{corank}(B^t) > 1$. We can find these values just as in the Interpolation Algorithm 1. For the smallest such value of t we have $B^t \in \mathcal{W}'(G)$ and $\text{corank}(B) > 1$, and we are done.

4 2-dimensional nullspace representations

4.1 G-matrices and circulations

Our goal in this section is to provide a characterization of G-matrices and their nullspace representations in dimension 2.

A *circulation* on an undirected simple graph G is a real function $f:\ V\times V$ such that is supported on adjacent pairs, is skew symmetric and satisfies the flow conditions:

$$f(i,j)=0\ (ij\notin E),\quad f(i,j)=-f(j,i)\ (ij\in E),\quad \sum_j f(i,j)=0\ (i\in V).$$

If we fix an orientation of the graph, then it suffices to specify the values of f on the oriented edges; the values on the reversed edges follow by skew symmetry. A *positive circulation* on an oriented graph (V,A) is a circulation on the underlying undirected graph that takes positive values on the arcs in A.

For any representation $u:\ V\to\mathbb{R}^k$, we define its *area-matrix* as the (skew-symmetric) matrix $T=T(u)$ by $T_{ij}:=\det(u_i,u_j)$. This number is the signed area of the parallelogram spanned by u_i and u_j, and it can also be described as $T_{ij}=u_i^\top u_j'$, where u_j' is the vector obtained by rotating u_j counterclockwise over 90°.

Given a graph G and a representation $u:\ V\to\mathbb{R}^2$ by nonzero vectors, we define a directed graph (V,A_u) and an undirected graph (V,E_u) by

$$\begin{aligned}A_u &:=\{(i,j)\in V\times V\mid ij\in E, T(u)_{ij}>0\}\\ E_u &:=\{ij\in E\mid T(u)_{ij}=0\}.\end{aligned}$$

So E is partitioned into A_u and E_u, where (V,A_u) is an oriented graph in which each edge is oriented counterclockwise as seen from the origin. The graph (V,E_u) consists of edges that are contained in a line through the origin.

Given a representation $u:\ V\to\mathbb{R}^2$, a circulation f on (V,A_u) and a function $g:\ E_u\to\mathbb{R}$, we define a G-matrix $M(u,f,g)$ by

$$M(u,f,g)_{ij}=\begin{cases}-f_{ij}/T_{ij}, & \text{if } ij\in A_u,\\ g(ij), & \text{if } ij\in E_u.\end{cases}$$

We define the diagonal entries by (3), and let the other entries be 0.

Lemma 9 *Let $G = (V, E)$ be a graph, let $u : V \to \mathbb{R}^2$ be a representation of V by nonzero vectors. Then*

$$\mathcal{M}_u = \big\{M(u, f, g) : \ f \text{ is a circulation on } (V, A_u) \text{ and } g : \ E_u \to \mathbb{R}\big\}.$$

Proof. First, we prove that $M(u, f, g) \in \mathcal{M}_u$ for every circulation on (V, A_u) and every $g : E_u \to \mathbb{R}$. Using that $M(u, f, g) = M(u, f, 0) + M(u, 0, g)$, it suffices to prove that $M(u, f, g) \in \mathcal{M}_u$ if either $g = 0$ or $h = 0$. If $M = M(u, f, 0)$, then using that f is a circulation, we have

$$\Big(\sum_j M_{ij} u_j\Big)^{\mathsf{T}} u'_i = \sum_j f_{ij} = 0.$$

This means that $\sum_j M_{ij} u_j^{\mathsf{T}}$ is orthogonal to u'_i, and so parallel to u_i. As remarked above, this means that $M(u, f, 0) \in \mathcal{M}_u$. If $M = M(u, 0, g)$, then for every $i \in V$,

$$\sum_{j \in N(i)} M_{ij} u_j = \sum_{j:\, ij \in E_u} g(ij) u_j$$

This vector is clearly parallel to u_i, proving that $M(u, 0, g) \in \mathcal{M}_u$.

Second, given a matrix $M \in \mathcal{M}_u$, define $f_{ij} = -T_{ij} M_{ij}$ for $ij \in A_u$ and $g_{ij} = M_{ij}$ for $ij \in E_u$. Then f is a circulation. Indeed, for $i \in V$,

$$\sum_{ij \in A_u} f_{ij} = -\sum_{ij \in A_u} M_{ij} u_j^{\mathsf{T}} u'_i = -\sum_{j \in V} M_{ij} u_j^{\mathsf{T}} u'_i = \Big(-\sum_{j \in V} M_{ij} u_j\Big)^{\mathsf{T}} u'_i = 0.$$

Furthermore, $M(u, f, g) = M$ by simple computation. □

Note that the G-matrix $M(u, f, g)$ is well-signed if and only if f is a positive circulation on (V, A_u) and $g < 0$. Thus,

Corollary 10 *Let $G = (V, E)$ be a graph, let $u : V \to \mathbb{R}^2$ be a representation of V by nonzero vectors. Then*

$$\mathcal{W}_u = \big\{M(u, f, g) : \ f \text{ is a positive circulation on } (V, A_u), \\ g : \ E_u \to \mathbb{R}, \ g < 0\big\}.$$

In particular, it follows that $\mathcal{W}_u \neq \emptyset$ if and only if A_u carries a positive circulation. This happens if and only if each arc in A_u is contained in a directed cycle in A_u; that is, if and only if each component of the directed graph (V, A_u) is strongly connected.

The signature of eigenvalues of $M(u, f, g)$ is a more difficult question, but we can say something about $M(u, 0, g)$ if $g < 0$. Let H be a connected component of the graph (V, E_u), and let M_H be the submatrix of $M(u, 0, g)$ formed by the rows and columns whose index belongs to $V(H)$. Then M_H is a well-signed H-matrix. The vectors u_i representing nodes $i \in V(H)$ are contained in a single line through the origin. Lemma 1 implies that M_H has at least one negative eigenvalue unless $u(V(H))$ is contained in a semiline starting at the origin. Let us call such a component *degenerate*. Then we can state:

Lemma 11 *Let $u : V \to \mathbb{R}^2$ be a representation of V with nonzero vectors, and let $g : E_u \to \mathbb{R}$ be a function with negative values. Then the number of negative eigenvalues of $M(u, 0, g)$ is at least the number of non-degenerate components of (V, E_u).*

4.2 Shifting the origin

For a representation $(u_1, \ldots, u_n)$ in $\mathbb{R}^2$ and $p \in \mathbb{R}^2$, let us write $u - p$ for the representation $(u_1 - p, \ldots, u_n - p)$.

Consider the cell complex made by the (two-way infinite) lines through distinct points u_i and u_j with $ij \in E$. The 1- and 2-dimensional cells are called 1-*cells* and 2-*cells*, respectively. Two cells c and d are *incident* if $d \subseteq \overline{c} \setminus c$ or $c \subseteq \overline{d} \setminus d$.

Two points p and q belong to the same cell if and only if $A_{u-p} = A_{u-q}$ and $E_{u-p} = E_{u-q}$. Hence, for any cell c, we can write A_c and E_c for A_{u-p} and E_{u-p}, where p is an arbitrary element of c. For any cell c, set $\mathcal{W}_c := \bigcup_{p \in c} \mathcal{W}_{u-p}$. It follows by Lemma 9 that if $\mathcal{W}_c \neq \emptyset$, then $\mathcal{W}_{u-p} \neq \emptyset$ for every $p \in c$. It also follows that $\mathcal{W}_c$ is connected for each cell c, as it is the range of the continuous function $M(u - p, f, g)$ on the connected topological space of triples (p, f, g) where $p \in c$, f is a positive circulation on A_c, and g is a negative function on E_c.

The following lemma is an essential tool in the proof.

Lemma 12 *Let c be a cell with $\mathcal{W}_c \neq \emptyset$ and let $q \in \overline{c}$. Then $M(u - q, 0, g) \in \overline{\mathcal{W}_c}$ for some negative function g on E_{u-q}.*

Proof. Choose any $p \in c$. Note that $q \in \overline{c}$ implies that $E_{u-p} \subseteq E_{u-q}$. Let $M \in \mathcal{W}_{u-p}$, then by Lemma 9 we can write $M = M(u - p, h, g')$ with some positive circulation h on A_{u-p} and negative function g' on E_{u-p}. Define $g(ij) = M_{ij}$ for $ij \in E_{u-q}$. For $\alpha \in (0, 1]$, define $p_\alpha = (1 - \alpha)q + \alpha p$, and consider the G-matrices $M_\alpha = M(u - p_\alpha, \alpha h, g')$. Clearly $M_\alpha \in \mathcal{W}_c$. It suffices to prove that

$$M_\alpha \to M(u - q, 0, g) \qquad (\alpha \to 0). \tag{8}$$

Consider any position (i, j) with $i \neq j$. If $ij \in E_{u-p}$, then the (i, j) matrix entries in M_α and $M(u - q, 0, g)$ are both equal to $g'(ij)$, independently of α. If $ij \notin E_{u-p}$, then for each $\alpha \in (0, 1]$ we have $ij \notin E_{u-\alpha p}$, and

$$(M_\alpha)_{ij} = \frac{-\alpha h_{ij}}{T(u - p_\alpha)_{ij}}. \tag{9}$$

If $ij \in E_{u-q} \setminus E_{u-p}$, then there is a line through u_i, u_j, and q. Hence $T(u - p_\alpha)_{ij} = \alpha T(u - p)_{ij}$ for each $\alpha \in (0, 1]$, and so

$$(M_\alpha)_{ij} = \frac{-h_{ij}}{T(u - p)_{ij}} = M_{ij}.$$

If $ij \notin E_{u-q}$, then (9) implies that $(M_\alpha)_{ij} \to 0$ as $\alpha \to 0$, since $\lim_{\alpha\to 0} T(u - p_\alpha)_{ij} = T(u - q)_{ij} \neq 1$.

So (8) holds on all off-diagonal positions. By (3), it holds for the diagonal entries as well. □

Corollary 13 *Let c be a cell with $\mathcal{W}_c \neq \emptyset$ and $q \in \overline{c}$. Then for every matrix $M \in \mathcal{W}_{u-q}$ there is a matrix $M' \in \mathcal{W}_{u-q} \cap \overline{\mathcal{W}_c}$ that differs from M only on entries corresponding to edges in E_{u-q} and on the diagonal entries.*

Proof. By Lemma 9 we can write $M = M(u - q, f, g)$ with some positive circulation f on A_{u-q} and negative function g on E_{u-q}. By Lemma 12, there is a negative function g' on E_{u-q} such that $M(u - q, 0, g') \in \overline{\mathcal{W}_c}$. There are points $p_k \in c$ and matrices $M_k \in \mathcal{W}_{u-p_k}$ such that $M_k \to M(u - q, 0, g')$ as $k \to \infty$. Then $M_k + M(u - p_k, f, 0)$ belongs to $\mathcal{W}_{u-p_k}$ and $M_k + M(u - p_k, f, 0) \to M(u - q, 0, g') + M(u - q, f, 0) = M(u - q, f, g')$ as $k \to \infty$, showing that $M' = M(u - q, f, g')$ belongs to $\overline{\mathcal{W}_c}$. Furthermore, $M - M' = M(u - q, 0, g - g')$ is nonzero on entries in E_{u-q} and on the diagonal entries only. □

Corollary 14 *If c and d are incident cells, then $\mathcal{W}_c \cup \mathcal{W}_d$ is connected.*

Proof. We may assume that $d \subseteq \overline{c} \setminus c$, and that both $\mathcal{W}_c$ and $\mathcal{W}_d$ are nonempty (otherwise the assertion follows from the connectivity of $\mathcal{W}_c$ and $\mathcal{W}_d$).

Choose $q \in d$. Since $\mathcal{W}_d \neq \emptyset$, Corollary 13 implies that $\mathcal{W}_d$ and $\overline{\mathcal{W}_c}$ intersect, and by the connectivity of $\mathcal{W}_c$ and $\mathcal{W}_d$, this implies that $\mathcal{W}_c \cup \mathcal{W}_d$ is connected. □

Call a segment σ in the plane *separating*, if σ connects points u_a and u_b for some $a, b \in V$, with the property that $V \setminus \{a, b\}$ can be partitioned into two nonempty sets X and Y such that no edge of G connects X and Y and such that the sets $\{u_i \mid i \in X\}$ and $\{u_i \mid i \in Y\}$ are on distinct sides of the line through σ. Note that this implies that σ is a 1-cell.

Lemma 15 *Let G be a connected graph, and let σ be a separating segment connecting u_i and u_j, with incident 2-cells R and Q. If $\mathcal{W}_\sigma \cup \mathcal{W}_R \neq \emptyset$, then A_Q contains a directed circuit traversing ij.*

Proof. We may assume that σ connects u_1 and u_2, and that edge 12 of G is oriented from 1 to 2 in A_Q. Let ℓ be the line through σ, and let H and H' be the open halfplanes with boundary ℓ containing Q and R, respectively.

Choose $p \in \sigma \cup R$ with $\mathcal{W}_{u-p} \neq \emptyset$. Note that A_Q and A_{u-p} differ only for edge 12. Any edge $ij \neq 12$ has the same orientation in A_Q as in A_{u-p}.

Since H contains points u_i, since G is connected, and since ℓ crosses no $u_i u_j$ with $ij \in E$, G has an edge $1k$ or $2k$ with $u_k \in H$. By symmetry, we can assume that $2k$ is an edge. Then in A_{u-p}, edge $2k$ is oriented from 2 to k. As $\mathcal{W}_{u-p} \neq \emptyset$, A_{u-p} has a positive circulation. So A_{u-p} contains a directed circuit D containing $2k$. The edge preceding $2k$, say $j2$, must have $u_j \in H'$, as p belongs to $\sigma \cup R$. Therefore, since $\{1, 2\}$ separates nodes k and j, D traverses node 1. So the directed path in D from 2 to 1 together with the edge 12 forms the required directed circuit C in A_{u-q}. □

Corollary 16 *Let G be a connected graph, let σ be a separating segment, and let R be a 2-cell incident with σ. Then $\mathcal{W}_\sigma \neq \emptyset$ if and only if $\mathcal{W}_R \neq \emptyset$.*

Proof. Let σ connect u_1 and u_2. If $\mathcal{W}_\sigma \neq \emptyset$, then A_σ has a positive circulation f'. By Lemma 15, A_R contains a directed circuit C traversing 12. Let f be the incidence vector of C. Then $f' + f$ is a positive circulation on A_R. So $\mathcal{W}_R \neq \emptyset$.

Conversely, if $\mathcal{W}_R \neq \emptyset$, then A_R has a positive circulation f. By Lemma 15, A_R contains a directed cycle through the arc 21, which gives a directed path P from 1 to 2 not using 12. It follows that by rerouting $f(1, 2)$ over P, we obtain a positive circulation on A_σ, showing that $\mathcal{W}_\sigma \neq \emptyset$. □

4.3 Outerplanar nullspace embeddings

Let $G = (V, E)$ be a graph. A mapping $u: V \to \mathbb{R}^2$ is called *outerplanar* if its extension to the edges gives an embedding of G in the plane, and each u_i is incident with the unbounded face of this embedding.

Theorem 17 *Let G be a 2-connected graph with $\kappa(G) = 2$. Then the normalized nullspace representation defined by any well-signed G-matrix with one negative eigenvalue and with corank 2 is an outerplanar embedding of G.*

Proof. Let u be such a normalized nullspace representation (this exists by Corollary 6). Let K be the convex hull of $u(V)$. Since all u_i have unit length, each u_i is a vertex of K. We define a *diagonal* as the line segment connecting points $u_i \neq u_j$, where $ij \in E$. We don't know at this point that the points u_i are different and that diagonals do not cross; so the same diagonal may represent several edges of G, and may consist of several 1-cells.

Let P denote the set of points $p \in \mathbb{R}^2 \setminus u(V)$ with $\mathcal{W}'_{u-p} \neq \emptyset$. Clearly, the origin belongs to P. Lemma 1(b) implies that

Claim 1 *P is contained in the interior of K.*

(It will follow below that P is equal to the interior of K.)

Consider again the cell complex into which the diagonals cut K. By the connectivity of the sets $\mathcal{W}_c$ and by Lemma 2, P is a union of cells.

Claim 2 *$\overline{P}$ cannot contain a point $u_i = u_j$ for two distinct nodes i and j.*

Indeed, since $u_i = u_j$ is a vertex of the convex hull of $u(V)$, we can choose $p \in P$ close enough to v so that it is not in the convex hull of $u(V) \setminus \{v\}$. This, however, contradicts Lemma 3.

Claim 3 *No point $p \in \overline{P} \setminus u(V)$ is contained in two different diagonals.*

Indeed, consider any cell $c \subseteq P$ with $p \in \overline{c}$. Since $\mathcal{W}_c \neq \emptyset$, Lemma 12 implies that there is a negative function g on E_{u-p} such that $M(u - p, 0, g) \in \overline{\mathcal{W}_c}$. As all matrices in $\mathcal{W}_c$ have exactly one negative eigenvalue, $M(u-p, 0, g)$ has at most one negative eigenvalue. Lemma 11 implies that (V, E_{u-p}) has at most one non-degenerate component. But every diagonal containing p is contained in a non-degenerate component of (V, E_{u-p}), and these components are different for different diagonals, so p can be contained in at most one diagonal. This proves Claim 3.

It is easy to complete the proof now. Clearly, P is bounded by one or more polygons. Let p be a vertex of $\overline{P}$, and assume that $p \notin u(V)$. Then p belongs to two diagonals (defining the edges of P incident with p), contradicting Claim 3. Thus all vertices of P are contained in $u(V)$. This implies that $\overline{P}$ is a convex polygon spanned by an appropriate subset of $u(V)$.

To show that $\overline{P} = K$, assume that the boundary of P has an edge σ contained in the interior of K and let $R \subseteq P$ be a 2-cell incident with σ, and let Q be the 2-cell incident with σ on the other side. Clearly, $\mathcal{W}_R \neq \emptyset$, and by Corollary 16, $\mathcal{W}_\sigma \neq \emptyset$ and by the same Corollary, $\mathcal{W}_Q \neq \emptyset$. The sets $\mathcal{W}_\sigma \cup \mathcal{W}_R$ and $\mathcal{W}_\sigma \cup \mathcal{W}_Q$ are connected by Corollary 14, and hence so is $\mathcal{W}_\sigma \cup \mathcal{W}_R \cup \mathcal{W}_Q$. We also know that $\mathcal{W}' \cap \mathcal{W}_R \neq \emptyset$. Since $\mathcal{W}'$ is open and closed in $\mathcal{W}$ (Lemma 2, note that in this case $\mathcal{W}' = \mathcal{W}' \cap \mathcal{W}_0$ as $\kappa(G) = 2$), we conclude that $\mathcal{W}' \cap \mathcal{W}_Q \neq \emptyset$, i.e., $Q \subseteq P$. But this contradicts the definition of σ.

Thus P is equal to the interior of K. Claim 2 implies that the points u_i are all different, and Claim 3 implies that the diagonals do not cross. ∎

4.4 Algorithm

The considerations in this section give rise to a polynomial algorithm achieving the following.

Algorithm 5

Input: A 2-connected graph $G = (V, E)$.

Output: Either an outerplanar embedding $u : V \to \mathbb{R}^2$ of G, or a well-signed G-matrix with one negative eigenvalue and corank at least 3.

The algorithm progresses along the same lines as the algorithm in Section 3.3, with auxiliary algorithms analogous to those in Section 3.2. We omit the details.

Remark 18 Suppose that the input to our algorithm is a 3-connected planar graph. Then the algorithm outputs a well-signed G-matrix with one negative eigenvalue and corank at least 3. Computing the nullspace representation defined by this matrix, and performing node-scaling as described in [7], we get a representation of G as the skeleton of a 3-polytope.

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