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Circuits in graphs embedded on the torus

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Abstract

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We give a survey of some recent results on circuits in graphs embedded on the torus. Especially we focus on methods relating graphs embedded on the torus to integer polygons in the Euclidean plane.

1. Introduction

We give a survey of some recent results on graphs embeded on the torus. In particular we are interested in finding large amounts of pairwise disjoint circuits of given 'homotopies' in the graph. As a tool we use a certain relation between graphs embedded on the torus and polygons in the Euclidean plane $\mathbb{R}^2$. There turns out to exist a relation between circuits in graphs on the torus and lattice points in such polygons. The methods and results link with convexity theory, geometry of numbers, and graph minors.

In this survey we discuss: a min–max result for the maximum number of pairwise disjoint circuits of a given 'free homotopy type' in a graph G on the torus (Theorem 1 [14]); a lower bound on the maximum number of pairwise disjoint noncontractible circuits in G, in terms of the 'face width' of G (Theorem 3 [17]); necessary and sufficient conditions for the existence of pairwise edge-disjoint closed walks of prescribed free homotopy types, provided a certain parity condition holds (Theorem 13 [5]); and a theorem on the existence of large 'grid' minors, in terms of the face width of the graph (Theorem 29 [6]).

In this paper, a graph is undirected, and a *circuit* is a *simple* closed walk, that is, no vertex is traversed more than once.

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2. Curves on the torus and their intersections

We first note some well-known facts on closed curves on the torus. Let T be the torus, which we identify with the cartesian product $S^1 \times S^1$ of two copies of the unit circle S^1 in the complex plane $\mathbb{C}$. A *closed curve* on T is a continuous function $C: S^1 \to T$. It is *simple* if C is one-to-one.

Two closed curves $C, D: S^1 \to T$ are called *freely homotopic* if there exists a continuous function $\Phi: [0, 1] \times S^1 \to T$ such that $\Phi(0, x) = C(x)$ and $\Phi(1, x) = D(x)$ for each $x \in S^1$. Free homotopy gives an equivalence relation between closed curves on T, denoted by $\sim$.

The following closed curves $C_{m,n}$ form a system of representatives for the free homotopy classes (cf. Baer [1] and Stillwell [21, Section 6.2.2]). For $m, n \in \mathbb{Z}$, let $C_{m,n}: S' \to T$ be defined by

$$C_{m,n}(x) := (x^m, x^n). \tag{1}$$

So $C_{m,n}$ goes m times around the torus in one direction, and n times in the 'orthogonal' direction. Now for each closed curve C on T there exists a unique $(m, n) \in \mathbb{Z}^2$ such that $C \sim C_{m,n}$. There exists a simple closed curve $C \sim C_{m,n}$ if and only if m and n are relatively prime.

For any two closed curves C and D on T we define

$$\begin{aligned} &\mathrm{cr}(C, D) := \text{number of intersections of } C \text{ and } D; \\ &\mathrm{mincr}(C, D) := \min\{\mathrm{cr}(\tilde{C}, \tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}. \end{aligned} \tag{2}$$

(We count multiplicities.)

It is not difficult to express $\mathrm{mincr}(C, D)$ in terms of m, n, m', n' when $C \sim C_{m,n}$ and $D \sim C_{m',n'}$:

$$\mathrm{mincr}(C_{m,n}, C_{m',n'}) = |mn' - m'n|. \tag{3}$$

3. Disjoint circuits in graphs on the torus

The first result we discuss concerns the existence of disjoint circuits in G of a given free homotopy type. Theorem 1 is a special case of a theorem proved in [14] for general compact surfaces. Here we give a direct proof for the torus.

A necessary condition for the existence of k pairwise disjoint circuits of some prescribed free homotopy type in a given graph, clearly is that each closed curve D on the torus should intersect the graph 'often enough'. This is a "cut condition" which turns out to be sufficient. To be more precise, define for any graph G embedded on the torus T and any closed curve D on T,

$$\begin{aligned} &\mathrm{cr}(G, D) := \text{number of intersections of } G \text{ and } D; \\ &\mathrm{mincr}(G, D) := \min\{\mathrm{cr}(G, \tilde{D}) \mid \tilde{D} \sim D\}. \end{aligned} \tag{4}$$

(Again we count multiplicities.)

Theorem 1. *Let G be an undirected graph embedded on the torus T, and let C be a simple closed curve on T. Then G contains k pairwise disjoint circuits each freely homotopic to C if and only if*

$$\mathrm{cr}(G, D) \geqslant k \cdot \mathrm{mincr}(C, D) \tag{5}$$

for each closed curve D on T.

Proof. Necessity of the condition is trivial. To prove sufficiency, suppose that the condition is satisfied. Let $k \geqslant 1$.

Consider the torus T as the quotient space of $\mathbb{C}\setminus\{0\}$ by identifying any $y, z \in \mathbb{C}\setminus\{0\}$ if $z = 2^u y$ for some integer u. Let $\pi : \mathbb{C}\setminus\{0\} \to T$ be the quotient map. We make this construction in such a way that $\pi^{-1}(C)$ is the union of a collection of pairwise disjoint simple closed curves Γ_t ($t \in \mathbb{Z}$) in $\mathbb{C}\setminus\{0\}$, each enclosing 0. We choose indices so that $\Gamma_{t+1} = 2\Gamma_t$ for each integer t.

The inverse image $H := \pi^{-1}[G]$ of G is an infinite graph embedded in $\mathbb{C}\setminus\{0\}$. Then each face of H is a bounded subset of $\mathbb{C}$, since otherwise there would exist a curve D not intersecting G such that $\mathrm{mincr}(C, D) \geqslant 1$, contradicting the condition.

For any curve P in $\mathbb{C}\setminus\{0\}$, let $\mathrm{cr}(H, P)$ denote the number of times P intersects H (counting multiplicities). Now for each integer i, let $\mathcal{R}_i$ be the set of faces F of H so that there exists an integer t and a curve P such that:

(i) P starts in a face enclosed by Γ_t and ends in F;

(ii) $\pi \circ P$ does not traverse any face of G more than once; (6)

(iii) $\mathrm{cr}(H, P) \leqslant i - kt$.

Since by (6)(ii) each P traverses at most f faces of H, where f is the number of faces of G, we know that $\bigcup \mathcal{R}_i$ is bounded, for each integer i.

Clearly, each face F enclosed by Γ_t belongs to $\mathcal{R}_{kt}$ (since we can take for P any curve remaining in F). Moreover, $\mathcal{R}_{i+k} = 2\mathcal{R}_i$.

The faces in $\mathcal{R}_i$ induce a connected subgraph of the dual graph of H, as one easily checks. Hence the arcs on the boundary of the unbounded connected component of $\mathbb{C}\setminus\overline{\bigcup \mathcal{R}_i}$ form a simple closed curve in H, call it Δ_i.

Then for each integer i, Δ_i is enclosed by Δ_{i+1}, without intersections. This follows from the fact that if F belongs to $\mathcal{R}_i$, then each face F' having a vertex in common with F belongs to $\mathcal{R}_{i+1}$. Indeed, by definition of $\mathcal{R}_i$, there exists a t and a curve P satisfying (6). We can extend P to a curve P' ending in F', by traversing a vertex incident with both F and F'.

If face $\pi[F']$ of G is not traversed by $\pi \circ P$, then t, P' satisfy (6) with respect to $i+1$ and F'. If face $\pi[F']$ of G is traversed by $\pi \circ P$, we may assume that $P' = P_1 \cdot P_2$, so that $\pi \circ P_2$ is a closed curve and so that $\pi \circ P_1$ does not traverse any face of G more than once.

Let P_2 go from z to $2^u z$ for some $u \in \mathbb{Z}$. So $\mathrm{mincr}(C, \pi \circ P_2) = |u|$. Hence $\mathrm{cr}(H, P_2) = \mathrm{cr}(G, \pi \circ P_2) \geqslant ku$. Therefore, the curve $2^u P_1$ starts in a face enclosed by Γ_{t+u} and ends in F', and

$$\begin{aligned} i+1 &\geqslant kt + \mathrm{cr}(H, P') = kt + \mathrm{cr}(H, P_1) + \mathrm{cr}(H, P_2) \\ &\geqslant kt + \mathrm{cr}(H, 2^u P_1) + ku. \end{aligned} \tag{7}$$

So the pair $t+u$, $2^u P_1$ satisfies (6) with respect to $i+1$ and F'.

Since also $\Delta_{i+k} = 2\Delta_i$ for each i, it follows that $\pi \circ \Delta_1, \ldots, \pi \circ \Delta_k$ are disjoint closed curves in G, each freely homotopic to C. □

This theorem was extended to directed graphs by Seymour [19] (cf. Ding et al. [4]).

4. Graphs on the torus and norms and polygons in $\mathbb{R}^2$

Theorem 1 can be formulated in terms of a certain convex set (in fact, polygon) associated with a graph embedded on the torus. Let G be a graph embedded on the torus T. Define for each $(m, n) \in \mathbb{Z}^2$

$$\phi_G(m, n) := \mathrm{mincr}(G, C_{m,n}). \tag{8}$$

It is not difficult to see that:

$$\begin{aligned} &\text{(i)} \quad \phi_G(m+m', n+n') \leqslant \phi_G(m, n) + \phi_G(m', n') \text{ and} \\ &\text{(ii)} \quad \phi_G(km, kn) = |k| \cdot \phi_G(m, n) \end{aligned} \tag{9}$$

hold for all (m, n), $(m', n') \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$. (The inequality in (i) follows from the fact that if C is freely homotopic to $C_{m,n}$ and C' is freely homotopic to $C_{m',n'}$ and (m, n) and (m', n') are linearly independent, then C has a crossing with C'. We can concatenate C and C' at this crossing so as to obtain a closed curve C'' freely homotopic to $C_{m+m',n+n'}$ with $\mathrm{cr}(G, C'') = \mathrm{cr}(G, C) + \mathrm{cr}(G, C')$. The equality in (ii) is easy.)

Hence, if each face of G is an open disk, there exists a unique norm $\|\cdot\|_G$ in $\mathbb{R}^2$ with the property that $\|(m, n)\|_G = \phi_G(m, n)$ for each $(m, n) \in \mathbb{Z}^2$.

Now we can associate with G a convex set P_G as follows:

$$P_G := \{x \in \mathbb{R}^2 \mid c^T x \leqslant \phi_G(c) \text{ for each } c \in \mathbb{Z}^2\}. \tag{10}$$

So P_G is closed, convex and 0-symmetric. (A set P is *0-symmetric* if $-P = P$.) It is standard convexity theory to show that (9) implies that

$$\max\{c^T x \mid x \in P_G\} = \phi_G(c) \tag{11}$$

for each $c \in \mathbb{Z}^2$.

Now Theorem 1 expressed in terms of P_G is as follows.

Theorem 2. *Let G be an undirected graph embedded on the torus, and let $m, n \in \mathbb{Z}$. Then G contains k pairwise disjoint circuits, each freely homotopic to $C_{m,n}$ if and only if the vector $k \cdot (n, -m)$ belongs to P_G.*

This follows directly from Theorem 1 combined with (3) and (10).

5. Nontrivial closed curves

In Theorem 1 we specified the homotopy type of the disjoint curves to be found. We can relax this question by just asking for a large number of pairwise disjoint *noncontractible* closed curves in a graph embedded on the torus. (The curves necessarily are all of the same free homotopy type, but we do not prescribe it in advance.)

A closed curve C is called *contractible* if $C \sim C_{0,0}$. Any other closed curve on the torus is called *noncontractible* or *nontrivial.* For any graph G embedded on the torus T, the *face width* (or *representativity*) $r(G)$ of G is the minimum of $\mathrm{cr}(G, D)$, where D ranges over all nontrivial closed curves on T.

Theorem 3. (i) *Any graph G embedded on the torus contains at least $\lfloor \frac{3}{4} r(G) \rfloor$ pairwise disjoint nontrivial circuits.*

(ii) *The factor $\frac{3}{4}$ is best possible.*

(Here $\lfloor\ \rfloor$ denotes the lower integer part of x.)

Theorem 3 is equivalent to the following theorem in the geometry of numbers (cf. Cassels [2], Lekkerkerker [8]). For any compact convex 0-symmetric set P in $\mathbb{R}^n$ let

$$\lambda(P) := \min\{\lambda \mid \lambda \cdot P \text{ contains a nonzero integer vector}\}. \tag{12}$$

Moreover, for any convex set $P \in \mathbb{R}^n$ the *polar* P^* is defined by

$$P^* := \{x \in \mathbb{R}^n \mid y^T x \leqslant 1 \text{ for each } y \in P\}. \tag{13}$$

As is well known, if P is closed, convex, and 0-symmetric then $P^{**} = P$.

Now Theorem 3 is equivalent to the following.

Theorem 4. (i) *For any 0-symmetric compact convex set P in $\mathbb{R}^2$ one has $\lambda(P) \cdot \lambda(P^*) \leqslant \frac{4}{3}$.*

(ii) *The number $\frac{4}{3}$ is best possible.*

For a proof of Theorem 4 we refer to [17].

We now prove the implications Theorem 4(i)$\Rightarrow$Theorem 3(i) and Theorem 3(ii)$\Rightarrow$Theorem 4(ii).

To see the first implication, let G be a graph embedded on the torus. Apply Theorem 4(i) to $P = P_G$. The face width $r(G)$ of G is equal to the minimum value of

$$\mathrm{mincr}(G, C_{m,n}) = \phi_G(m, n) \tag{14}$$

taken over all nonzero integer vectors (m, n). Now using (11),

$$(m, n) \in \lambda \cdot P_G^* \Leftrightarrow \max\{c^T x \mid x \in P_G\} \leqslant \lambda \Leftrightarrow \phi_G(m, n) \leqslant \lambda$$

where $c = (m, n)$. Hence $r(G) = \lambda(P_G^*)$.

Therefore, by Theorem 4(i), $\lambda(P_G) \leqslant 4/(3r(G))$. So $(4/(3r(G)) \cdot P_G$ contains a nonzero integer vector, say (m, n). Then $(3r(G)/4) \cdot (m, n)$ belongs to P_G, and hence also $\lfloor 3r(G)/4 \rfloor \cdot (m, n)$ belongs to P_G. Then by Theorem 2, G contains $\lfloor 3r(G)/4 \rfloor$ pairwise disjoint circuits each freely homotopic to $C_{n,-m}$. So G contains $\lfloor 3r(G)/4 \rfloor$ pairwise disjoint nontrivial circuits.

This construction also shows that Theorem 3(ii) implies Theorem 4(ii), since any better factor in Theorem 4(i) would imply a better factor in Theorem 3(i).

6. Integer norms and integer polygons

In the proof above we associated with any graph G on the torus a 0-symmetric convex set P_G. We show that P_G in fact is a polygon, with integer vertices. This follows from a slight extension of a result of Hoffman [7] in integer linear programming, which can be derived from the 'cutting plane theorem' of Chvátal [3].

A *polytope* is the convex hull of a finite set of vectors. A polytope P is called *integer* if each vertex of P is an integer vector.

Theorem 5. *Let C be a non-empty compact convex set in $\mathbb{R}^n$. Then C is an integer polytope if and only if* $\max\{c^T x \mid x \in C\}$ *is an integer for each integer vector* $c \in \mathbb{R}^n$.

This theorem implies the following.

Theorem 6. *For each graph G embedded on the torus, P_G is an integer polygon.*

Proof. By (11), $\max\{c^T x \mid x \in P_G\} = \phi_G(c)$ for each vector $c \in \mathbb{Z}^2$. Since $\phi_G(c)$ is an integer, the maximum is an integer for each integer vector c. Hence Theorem 5 gives Theorem 6. □

In Section 11 below we show Theorem 25 which implies the following.

Theorem 7. *For each* 0*-symmetric integer polygon P in $\mathbb{R}^2$ there exists a graph G embedded on the torus such that $P_G = P$.*

This theorem enables us to give a proof of the implications Theorem 3(i) $\Rightarrow$ Theorem 4(i) and Theorem 4(ii) $\Rightarrow$ Theorem 3(ii).

We first show the first implication. Let P be a 0-symmetric convex body in $\mathbb{R}^2$ not containing any nonzero integer vector. We show $\lambda(P) \cdot \lambda(P^*) \leqslant \frac{4}{3}$. By continuity we may assume that P is an integer polygon and that $\lambda(P^*)$ is a multiple of 4.

By Theorem 7, there exists a graph G embedded on the torus such that $P_G = P$. As above this implies $\lambda(P^*) = \lambda(P_G^*) = r(G)$. Hence by Theorem 3(i), G contains $k := \frac{3}{4}\lambda(P^*)$ pairwise disjoint nontrivial circuits. They are all mutually freely homotopic, say they are all freely homotopic to $C_{m,n}$. Then by Theorem 2, $k \cdot (n, -m)$ belongs to P_G, and hence $(n, -m)$ belongs to $(1/k) \cdot P_G$. So $\lambda(P) = \lambda(P_G) \leqslant 1/k = 4/(3\lambda(P^*))$.

Again, any better factor in Theorem 3(i) would imply a better factor in Theorem 4(i). This gives the implication Theorem 4(ii)$\Rightarrow$Theorem 3(ii).

7. Tight graphs and minimally crossing systems of curves

We now turn to considering pairwise *edge*-disjoint circuits. Here the complication arises that edge-disjoint circuits can cross each other, and hence they need not be all of the same free homotopy type. We are interested in the question: given a graph G embedded on the torus T and closed curves $C_1, \ldots, C_k$ on T, under which conditions does G contain pairwise edge-disjoint closed walks $\tilde{C}_1, \ldots, \tilde{C}_k$, where $\tilde{C}_i$ is freely homotopic to C_i $(i = 1, \ldots, k)$?

We would like to have an analogue of Theorem 1; that is, we would like to have 'cut type' conditions. It turns out that such conditions are sufficient if we restrict ourselves to Eulerian graphs. (In this paper, a graph is *Eulerian* if each vertex has even degree. (We do not require connectedness.))

Let $G = (V, E)$ be an undirected graph embedded on the torus T, and let D be a closed curve on T. In studying the edge-disjoint case we use the following number:

$$\mathrm{mincr}'(G, D) := \min\{\mathrm{cr}(G, \tilde{D}) \mid \tilde{D} \sim D, \tilde{D} \text{ does not traverse } V\}. \tag{15}$$

So $\mathrm{mincr}'(G, D)$ represents the minimum 'edge cut' capacity, over all cuts freely homotopic to D. The analogue of ϕ_G is the function $\phi'_G : \mathbb{Z}^2 \to \mathbb{Z}$ defined by

$$\phi'_G(m, n) := \mathrm{mincr}'(G, C_{m,n}). \tag{16}$$

Again one can show that (if each face of G is an open disk) there exists a unique norm $\|\cdot\|'_G$ in $\mathbb{R}^2$ such that $\|(m, n)\|'_G = \phi'_G(m, n)$ for each $(m, n) \in \mathbb{Z}^2$. Moreover, one can define the closed, 0-symmetric, convex set P'_G by

$$P'_G := \{x \in \mathbb{R}^2 \mid c^T x \leqslant \phi'_G(c) \text{ for each } c \in \mathbb{Z}^2\}. \tag{17}$$

The following operation is essential. Let G be an Eulerian graph embedded on the torus T. Consider a vertex v, of degree d say. 'Open' the graph at v as in

Fig. 1,

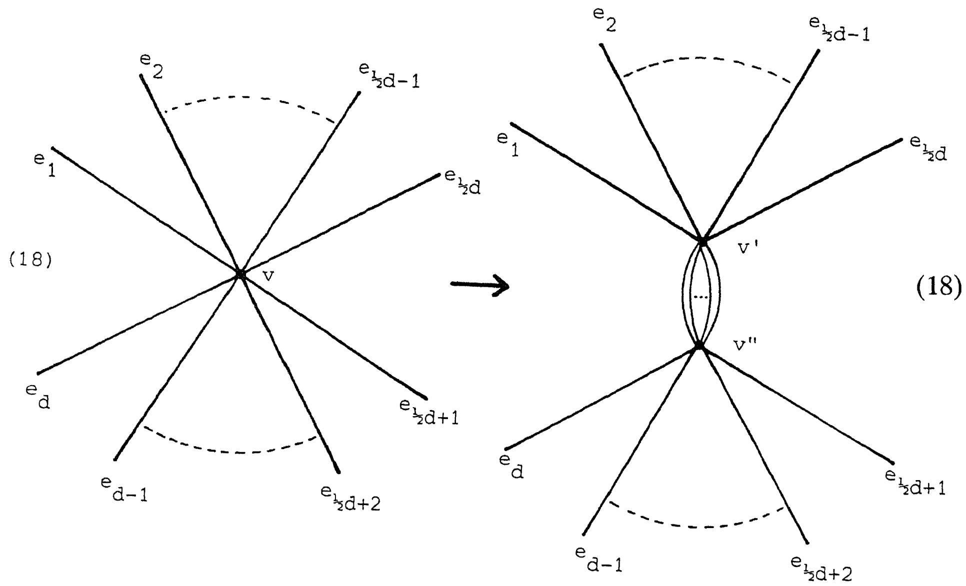


Fig. 1.

where there are $\frac{1}{2}d-2$ edges connecting the new vertices v' and v''. (Note that if $d=4$, we get a real opening in the graph.) We call any graph H obtained by a finite sequence of these operations an *opening* of G.

Clearly, if H is an opening of G then $P'_H \subseteq P'_G$. We call an Eulerian graph G embedded on T *tight* if for each proper opening H of G one has $P'_H \neq P'_G$. (By *proper* we mean: we apply the above operation at least once.)

In [15], tight graphs are characterized in the following way. Let $G=(V, E)$ be an Eulerian graph embedded on the torus T. In any vertex v, of degree $2d$ say, order the edges in cyclic order as $e_1, \ldots, e_{2d}$. Call edge e_{d+i} *opposite to* e_i *at* v, for $i=1, \ldots, 2d$, taking indices mod $2d$. Call a closed walk $(v_0, e_1, v_2, e_3, v_3, \ldots, e_s, v_s)$ (with $v_s=v_0$) *straight* if for each $j=1, \ldots, s$ edge e_{j-1} is opposite at v_j to edge e_j (taking indices mod s). A *straight decomposition* is a decomposition of the edge set E into straight closed walks $C_1, \ldots, C_k$ such that each edge of G is traversed exactly once by $C_1 \cup \cdots \cup C_k$. Clearly, G has a unique straight decomposition, up to reversing closed walks, up to changing the starting (= end) vertices, and up to permuting subscripts.

Call a system $C_1, \ldots, C_k$ of simple closed curves on the torus T *minimally crossing* if

$$\operatorname{cr}(C_i, C_j) = \operatorname{mincr}(C_i, C_j) \tag{19}$$

for all $i, j \in \{1, \ldots, k\}$, $i \neq j$.

Now the following holds.

Theorem 8. *Let G be an Eulerian graph embedded on the torus. Then G is tight if and only if any straight decomposition of G forms a minimally crossing system of simple closed curves.*

For a proof we refer to [15] (where an appropriate extension to general compact orientable surfaces is given). Theorem 8 turns out to be useful, since minimally crossing systems of simple closed curves satisfy the following property, which is not difficult to prove.

Theorem 9. *Let $C_1, \ldots, C_k$ be a minimally crossing system of simple closed curves on the torus. Then for any closed curve D on the torus there exists a closed curve $\tilde{D} \sim D$ such that*

$$\mathrm{cr}(C_i, \tilde{D}) = \mathrm{mincr}(C_i, D) \tag{20}$$

for all $i = 1, \ldots, k$.

So there exists a closed curve $\tilde{D}$ that attains the minimum number of crossings with all C_i *simultaneously*.

8. Edge-disjoint closed walks

The results on tight graphs described in the previous section are helpful in obtaining results on collections of edge-disjoint closed walks in graphs on the torus. Theorem 9 gives that Theorem 8 implies the following.

Theorem 10. *Let G be an Eulerian graph embedded on the torus T. Then there exist pairwise edge-disjoint closed walks $C_1, \ldots, C_k$ in G such that for each closed curve D on T one has*

$$\mathrm{mincr}'(G, D) = \sum_{i=1}^{k} \mathrm{mincr}(C_i, D). \tag{21}$$

(Note that the inequality $\geqslant$ in (21) holds trivially for *any* set of pairwise edge-disjoint closed walks in G—the content of the theorem is that we can choose $C_1, \ldots, C_k$ such that equality holds.)

Proof. We may assume that G is tight, since if we could open G at some vertex in some direction without changing $\mathrm{mincr}'(G, D)$ for any closed curve D, we make this opening and apply induction.

Now let $C_1, \ldots, C_k$ be a stright decomposition of G. We show that (21) holds for each closed curve D. By Theorem 8, $C_1, \ldots, C_k$ form a minimally crossing system of simple closed curves. By Theorem 9, there exists a closed curve $\tilde{D} \sim D$ such that (20) holds. We may assume that $\tilde{D}$ does not traverse any vertex of G.

Then we have

$$\text{mincr}'(G, D) \leq \text{cr}(G, \tilde{D}) = \sum_{i=1}^{k} \text{cr}(C_i, \tilde{D}) = \sum_{i=1}^{k} \text{mincr}(C_i, D). \tag{22}$$

The reverse inequality in (21) being easy, this shows Theorem 10. □

By surface duality one derives from Theorem 10 the following theorem. Given any graph G embedded on the torus T and any closed curve C on T define

$$\text{minlength}(G, C) := \min\{\text{length}(\tilde{C}) \mid \tilde{C} \sim C, \tilde{C} \text{ contained in } G\}. \tag{23}$$

Here length $(\tilde{C})$ is the number of edges of G traversed by $\tilde{G}$, counting multiplicities.

Theorem 11. *Let $G = (V, E)$ be a bipartite graph embedded on the torus T. Then there exist closed curves $D_1, \ldots, D_t$ on T not intersecting V, such that each edge of G is crossed exactly once by $D_1 \cup \cdots \cup D_t$ and such that for each closed curve C on T one has*

$$\text{minlength}(G, C) = \sum_{j=1}^{t} \text{mincr}(C, D_j). \tag{24}$$

Proof. If each face of G is an open disk, apply Theorem 10 to the surface dual of G, and the result follows directly. If G has a face that is not an open disk (but an annulus), Theorem 11 can be easily shown directly. □

By Farkas' lemma (cf. [12]), Theorem 11 implies a 'homotopic circulation theorem', stating that a cut condition is sufficient for the existence of a 'fractional' solution to the closed walk packing problem (note that if each $\lambda_{i,j}$ below equals 1, we obtain a collection of edge-disjoint closed walks).

Theorem 12. *Let $G = (V, E)$ be a graph embedded on the torus T and let $C_1, \ldots, C_k$ be simple closed curves on T. Then there exist closed walks $C_{i,j}$ in G and rational numbers $\lambda_{i,j} > 0$ ($i = 1, \ldots, k$; $j = 1, \ldots, n_i$) such that:*

$$\begin{aligned} &\text{(i)} \quad C_{i,j} \sim C_j \textit{ for each } i = 1, \ldots, k \textit{ and } j = 1, \ldots, n_i, \\ &\text{(ii)} \quad \sum_{j=1}^{n_i} \lambda_{i,j} = 1 \textit{ for each } i = 1, \ldots, k, \\ &\text{(iii)} \quad \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{i,j} \chi(C_{i,j}, e) \leq 1 \textit{ for each edge } e \textit{ of } G, \end{aligned} \tag{25}$$

if and only if for each closed curve D on T not traversing V one has

$$\text{cr}(G, D) \geq \sum_{i=1}^{k} \text{mincr}(C_i, D). \tag{26}$$

(Here for any closed walk C in G and any edge e of G, $\chi(C, e)$ denotes the number of times C traverses e.)

Proof. Necessity being easy, we show sufficiency. With Farkas' lemma one can show that the $C_{i,j}$, $\lambda_{i,j}$ as required exist if and only if for each 'length' function $l: E \to \mathbb{R}_+$ one has

$$\sum_{i=1}^{k} \operatorname{minlength}_l(G, C_i) \leqslant \sum_{e \in E} l(e). \tag{27}$$

Here for any closed curve C, $\operatorname{minlength}_l(G, C)$ is equal to the minimum value of $l(\tilde{C})$ where $\tilde{C}$ ranges over all $\tilde{C} \sim C$ and where $l(\tilde{C}) := \sum_{e \in E} l(e)\chi(\tilde{C}, e)$. In fact, it is not difficult to see that we may restrict ourselves to functions l that take positive even integer values.

To show (27), let l be a positive even integer valued function on E. Replace each edge e by a path of length $l(e)$. (That is, $l(e) - 1$ new vertices are 'inserted' on e.) This way we obtain a bipartite graph H, to which we apply Theorem 11. Let $D_1, \ldots, D_t$ be the closed curves found. So $D_1 \cup \cdots \cup D_t$ intersects any edge e of G exactly $l(e)$ times. Hence we have

$$\begin{aligned} \sum_{e \in E} l(e) &= \sum_{j=1}^{t} \operatorname{cr}(G, D_j) \geqslant \sum_{j=1}^{t} \sum_{i=1}^{k} \operatorname{mincr}(C_i, D_j) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{t} \operatorname{mincr}(C_i, D_j) = \sum_{i=1}^{k} \operatorname{minlength}_l(G, C_i), \end{aligned} \tag{28}$$

thus proving (27). $\square$

From this fractional theorem it is derived in [5] that in fact an *integer* solution exists if the following *parity condition* holds:

> for each closed curve D on T, not intersecting vertices of G, the number of crossings of D with edges of G, plus the number of crossings with $C_1, \ldots, C_k$ is an even number. (29)

One easily checks that the parity condition implies that G is Eulerian.

Theorem 13. *Let $G = (V, E)$ be a graph embedded on the torus T, and let $C_1, \ldots, C_k$ be simple closed curves on T, such that the parity condition holds. Then there exist pairwise edge-disjoint closed walks $\tilde{C}_1, \ldots, \tilde{C}_k$ in G (not traversing any edge more than once) so that $\tilde{C}_i \sim C_i$ $(i = 1, \ldots, k)$ if and only if the 'cut condition'*

$$\operatorname{cr}(G, D) \geqslant \sum_{i=1}^{k} \operatorname{mincr}(C_i, D) \tag{30}$$

holds for each closed curve D on T not traversing V.

(We do not require the $\tilde{C}_i$ to be simple—they may have self-intersections at vertices of G. It can be shown that the parity or the simple-ness condition cannot be deleted.)

For a proof we refer to [5].

9. Closed curves and polygons

We will now show that the results above can also be used in deriving the existence of certain minors in graphs embedded on the torus. To this end it is convenient to associate polygons also with systems of curves. For any closed curve C on the torus T define $\phi_C:\mathbb{Z}^2\to\mathbb{Z}$ by

$$\phi_C(m, n) := \text{mincr}(C, C_{m,n}) \tag{31}$$

for $(m, n)\in\mathbb{Z}^2$. Moreover, define

$$P_C := \{x\in\mathbb{R}^2 \mid c^Tx \leq \phi_C(c) \text{ for each } c\in\mathbb{Z}^2\}. \tag{32}$$

It is not difficult to see that P_C is equal to the line segment connecting $(-n, m)$ and $(n, -m)$ if $C\sim C_{m,n}$.

There is the following relation between sums of functions ϕ_C and sums of polygons P_C (as usual, $P+Q := \{x+y \mid x\in P, y\in Q\}$ for $P, Q\subseteq\mathbb{R}^2$).

Theorem 14. *Let $C_1, \ldots, C_k$ be closed curves on the torus T. Then*

$$P_{C_1}+\cdots+P_{C_k} = \left\{x\in\mathbb{R}^2 \mid c^Tx \leq \sum_{i=1}^{k}\phi_{C_i}(c) \text{ for each } c\in\mathbb{Z}^2\right\}. \tag{33}$$

This can be shown easily using standard convexity theory.

Let P be any 0-symmetric integer polygon in $\mathbb{R}^2$, with vertices $z_1, \ldots, z_{2m}$, in cyclic order (clockwise). So $z_{m+i} = -z_i$ for $i = 1, \ldots, m$. Let $y_i := z_{i+1} - z_i$ for $i = 1, \ldots, 2m$, taking indices mod $2m$. Let P_i be the line segment connecting $\frac{1}{2}y_i$ and $-\frac{1}{2}y_i$. Then it is elementary plane geometry to show that

$$P = P_1+\cdots+P_m. \tag{34}$$

Now if $\frac{1}{2}y_i$ is an integer vector, there exists a closed curve C_i such that $P_i = P_{C_i}$ (viz. the closed curve $C_{m,n}$ if $\frac{1}{2}y_i = (-n, m)$). Hence we have the following.

Theorem 15. *Let P be a 0-symmetric integer polygon in $\mathbb{R}^2$ such that for each two vertices z, z' of P the vector $z-z'$ has even components only. Then there exist simple closed curves $C_1, \ldots, C_k$ on the torus T such that $P = P_{C_1}+\cdots+P_{C_k}$. These closed curves are unique up to permuting indices and up to reversing closed curves.*

Proof. To obtain the *simple* closed curves, we write each $\frac{1}{2}y_i$ above as $s \cdot y_i'$, where s is an integer and where y_i' has relatively prime integer components. Then we take s copies of $C_{m,n}$ where m, n are such that $y_i' = (-n, m)$.

Uniqueness follows from the fact that the decomposition (34) is unique, up to trivial operations. □

This theorem implies that a system of simple closed curves $C_1, \ldots, C_k$ is determined by $P_{C_1} + \cdots + P_{C_k}$. This gives the following theorem.

Theorem 16. *Let $C_1, \ldots, C_k$ and $C_1', \ldots, C_{k'}'$ be simple closed curves on the torus T. Then the following are equivalent:*

(i) *$k = k'$ and there exists a permutation π of $\{1, \ldots, k\}$ such that $C_{\pi(i)}' \sim C_i$ or $C_{\pi(i)}' \sim C_i^{-1}$ for each $i = 1, \ldots, k$;*

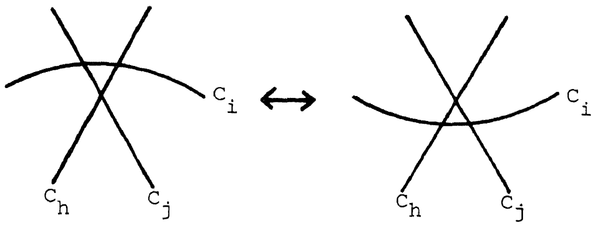
(ii) *for each closed curve D on T one has*

$$\sum_{i=1}^{k} \operatorname{mincr}(C_i, D) = \sum_{i=1}^{k'} \operatorname{mincr}(C_i', D). \tag{35}$$

Proof. The implication (i)$\Rightarrow$(ii) being trivial, we show (ii)$\Rightarrow$(i). If (35) holds for all D, then $\phi_{C_1} + \cdots + \phi_{C_k} = \phi_{C_1'} + \cdots + \phi_{C_{k'}'}$, and hence $P_{C_1} + \cdots + P_{C_k} = P_{C_1'} + \cdots + P_{C_{k'}'}$. By the uniqueness of the decomposition (35) the theorem follows. □

This theorem was shown for general compact orientable surfaces in [13]. Moreover, it was shown in [16] that the following is true.

Theorem 17. *Let $C_1, \ldots, C_k$ and $C_1', \ldots, C_k'$ be two minimally crossing systems of simple closed curves on the torus T, such that $C_i \sim C_i'$ for each $i = 1, \ldots, k$ and such that $C_1 \cup \cdots \cup C_k$ and $C_1' \cup \cdots \cup C_k'$ form 4-regular graphs. Then $C_1, \ldots, C_k$ can be moved to $C_1', \ldots, C_{k'}'$ by repeated application of the operation shown in Fig. 2.*



(36)

Fig. 2.

This jump is similar to the type III move given by Reidemeister [10].

10. Tight graphs and polygons

Theorem 14 implies that Theorem 10 is equivalent to the following.

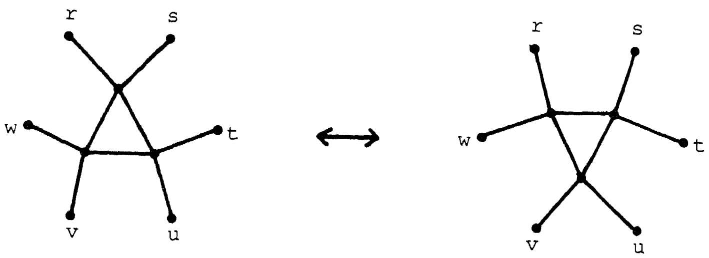
Theorem 18. *Let G be an Eulerian graph embedded on the torus T. Then the edges of G can be decomposed into closed walks $C_1, \ldots, C_k$ such that $P'_G = P_{C_1} + \cdots + P_{C_k}$.*

Similarly, Theorem 13 is equivalent to the following.

Theorem 19. *Let G be a graph embedded on the torus T and let $C_1, \ldots, C_k$ be simple closed curves on T such that the parity condition holds. Then G contains pairwise edge-disjoint closed walks $\tilde{C}_1, \ldots, \tilde{C}_k$ (not traversing any edge more than once) such that $\tilde{C}_i \sim C_i$ for $i = 1, \ldots, k$, if and only if $P_{C_1} + \cdots + P_{C_k} \subseteq P'_G$.*

Another consequence is the following.

Theorem 20. *Let G and H be 4-regular tight graphs. Then $P'_G = P'_H$ if and only if H can be obtained from G by repeated application of the operation (up to shifting the graph over the torus) shown in Fig. 3.*



(37)

Fig. 3.

Proof. Sufficiency is easy, since the operation given does not change P'_G. To see necessity, let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ be straight decompositions of G and H, respectively. Then by Theorems 8 and 10,

$$P_{C_1} + \cdots + P_{C_k} = P'_G = P'_H = P_{C'_1} + \cdots + P'_{C'_{k'}}. \tag{38}$$

So $\phi_{C_1} + \cdots + \phi_{C_k} = \phi_{C'_1} + \cdots + \phi_{C'_{k'}}$, and hence by Theorems 16 and 17 we can shift $C_1, \ldots, C_k$ to $C'_1, \ldots, C'_{k'}$ by steps as in (36). Hence G can be moved to H as in (37). $\square$

We also have the following directly from Theorem 15.

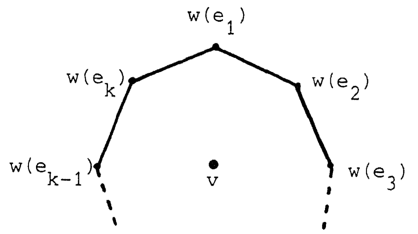
Theorem 21. *Let P be a 0-symmetric integer polygon in $\mathbb{R}^2$ such that for each two vertices z, z' one has that $z - z'$ has integer components only. Then there exists a tight graph G such that $P = P'_G$. This graph is unique up to the operation (37).*

11. Kernels

Let G be a graph embedded on the torus T. A *minor* of G is any graph obtained from G by a series of deletions and contractions of edges (contracting loops only if they enclose a face). Any minor of G has a natural embedding on T derived from the embedding of G. If we fix the embedding of the minor this way, we speak of an *embedded minor*. A minor is *proper* if at least one edge is deleted or contracted.

Clearly, if H is an embedded minor of G then $P_H \subseteq P_G$. Call a graph G embedded on the torus T a *kernel* if for each proper embedded minor H of G one has $P_H \neq P_G$.

There is a close connection between kernels and tight graphs, through the concept of medial graph, introduced by Steinitz [20] and in reverse form by Tait [22, 23]. Let G be a graph embedded on the torus T. Put a new vertex $w(e)$ in the 'middle' of each edge e of G, and connect, for each vertex v of G the points $w(e)$ for those edges e incident with v, by a new circuit as in Fig. 4.



(39)

Fig. 4.

Doing this for each edge and vertex, the new vertices and new edges form a graph, the *medial graph* $M(G)$ of G. (It is unique up to shifting over T.) The medial graph $M(G)$ is 4-regular. If $H = M(G)$, we say that G is a *radial graph* of H. Note that the radial graph is unique up to surface duality (and shifting over the surface). Each 4-regular graph H embedded on the torus T so that each face is an open disk and so that the faces can be bicoloured, is a medial graph, and hence has a radial graph. Note also that each two radial graphs of a cellularly embedded graph can be obtained from each other by homotopic shifts and taking surface duals.

Since trivially $2\text{mincr}(G, D) = \text{mincr}'(M(G), D)$ we have

$$\phi'_{M(G)} = 2\phi_G \quad \text{and hence} \quad P'_{M(G)} = 2P_G. \tag{40}$$

Now one has the following.

Theorem 22. *Let G be a graph embedded on the torus. Then G is a kernel if and only if the medial graph $M(G)$ is tight.*

Proof. Observe that the two types of opening a vertex of $M(G)$ correspond to deletion and contraction. Then by (40) the theorem follows. □

Theorem 22 combined with Theorem 20 gives the following.

Theorem 23. *Let G and H be kernels with $P_G = P_H$. Then H can be obtained from G by a series of the following operations:*

(i) *shifting the graph over the torus;*
(i) *taking the surface dual of the graph;*
(iii) *ΔY-exchange.* (41)

(Here *ΔY-exchange* means replacing a triangular face F by a vertex in the face connected to the three vertices incident with F, or conversely. (This operation was introduced by Steintz [20], who called it the θ-process.))

Proof. Note that the radial graph is unique up to duality, and that the operation (37) in the medial graph $M(G)$ corresponds to ΔY-exchange in G. $\square$

The extension of Theorem 23 to general compact orientable surfaces is given in [16]. For the torus, a stronger statement can be proved (cf. [18]). Let G and H be graphs embedded on the torus. We call a graph H a *ΔY-minor* of G if H arises from some embedded minor of G by the operations (41) (maintaining the embedding throughout).

Theorem 24. *Let G and H be graphs embedded on the torus, where G is a kernel. Then G is a ΔY-minor of H if and only if $P_G \subseteq P_H$.*

Proof. Necessity of the condition is easy, since P_G is maintained under the operations (41), while $P_G \subseteq P_H$ if G is a minor of H.

To see sufficiency, assume $P_G \subseteq P_H$. We assume that each face of H is an open disk (in particular, H is connected)—if this is not the case we can derive the theorem quite directly from Theorem 2. Let $M(G)$ and $M(H)$ be the medial graphs of G and H respectively. Since G is a kernel, $M(G)$ is tight, and hence the straight decomposition of $M(G)$ is a minimally crossing system of simple closed curves $D_1, \ldots, D_k$.

Then $P_{D_1} + \cdots + P_{D_k} \subseteq P'_{M(G)} = 2 \cdot P_G \subseteq 2 \cdot P_H = P'_{M(H)}$. Hence by Theorem 19, $M(H)$ contains closed curves $D'_1, \ldots, D'_k$, so that no edge of G is traversed more than once, and so that $D'_i \sim D_i$ for $i = 1, \ldots, k$. (The parity condition can be easily checked.)

It is not difficult to see that we may assume that $D'_1, \ldots, D'_k$ form a minimally crossing system of simple nontrivial closed curves, traversing each edge of $M(H)$ exactly once.

Now at any 'touching' of two D'_i and D'_j (possibly $i = j$), we can 'open' the graph as in (18). Doing this at each touching we have transformed $M(H)$ to a graph M' that is the union of a minimally crossing system of simple closed curves $D''_1, \ldots, D''_k$, with $D''_i \sim D'_i$ for $i = 1, \ldots, k$. Since openings of $M(H)$ correspond to deleting and contracting edges of H, M' is the medial graph of some minor H' of H.

By Theorem 8, M' is tight, and hence, by Theorem 22, H' is a kernel. By (40), $P_{H'} = \frac{1}{2}P'_{M(H')} = \frac{1}{2}P'_{M(G)} = P_G$. Concluding, by Theorem 23, H' arises by operations (41) from G. So G is ΔY-minor of H. $\square$

This theorem states that for each 0-symmetric integer polygon P there exists a unique minor-minimal graph among all graphs G with $P_G \supseteq P$—unique up to the operations (41). This is more general than Theorem 23, which states that there exists a unique minor-minimal graph among all graphs G with $P_G = P$.

Finally we note that the above also implies the following.

Theorem 25. *Let P be a 0-symmetric integer polygon in $\mathbb{R}^2$. Then there exists a kernel G such that $P = P_G$. This graph is unique up to the operations* (41).

Proof. By Theorem 21, there exists a tight graph M such that $P'_M = 2P$. We can move the curves making M slightly so as to make it 4-regular. Since each vertex of P has even integers as components, we know that M is the medial graph of some kernel G. Then $2P = P'_{M(G)} = 2P_G$. Uniqueness follows directly from Theorem 23. $\square$

In particular we have Theorem 7.

12. Grid minors

We describe a consequence of the results described in Sections 9–11 to 'toroidal grids' given in [6]. Let $k \geqslant 3$. The product $C_k \times C_k$ of two copies of the k-circuit C_k is called the *toroidal k-grid*. Clearly, the toroidal k-grid can be embedded on the torus, in fact in a unique way, up to homeomorphisms of the torus and of the grid.

Let H be the embedding of $C_k \times C_k$ on the torus, consisting of k disjoint circuits freely homotopic to $C_{1,0}$ crossed by k disjoint circuits freely homotopic to $C_{0,1}$.

Then H is a kernel (as its medial graph $M(H)$ is tight, as is easy to check with Theorem 8). Since it is self-dual and does not allow ΔY-exchange (as all vertices have degree 4 and each face is bounded by 4 edges), Theorem 24 implies the following.

Theorem 26. *Let G be a graph embedded on the torus. Then G contains H as an embedded minor if and only if P_G contains $(k, 0)$ and $(0, k)$.*

Proof. This follows directly from Theorem 24, since P_H is the convex hull of $\pm(k, 0)$ and $\pm(0, k)$. $\square$

By Theorem 2, Theorem 26 is equivalent to the plausible statement: a graph G embedded on the torus contains H as an embedded minor, if and only if G contains k pairwise disjoint closed curves each freely homotopic to $C_{1,0}$ and G contains k pairwise disjoint closed curves each freely homotopic to $C_{1,0}$. (We do not have a direct proof of this statement.)

Theorem 26 directly gives the following.

Theorem 27. *Let G be a graph embedded on the torus, and let $k \geqslant 3$. Then G contains a toroidal k-grid as a minor if and only if $(1/k)P_G$ contains two linearly independent integer vectors.*

Proof. Directly from Theorem 26. □

We apply this theorem as follows. Robertson and Seymour [11] showed

> for each graph H embedded on a compact surface S there exists an integer ρ_H so that each graph G embedded on S with $rG) \geqslant \rho_H$ contains H as a minor. (42)

The following is shown in [6].

Theorem 28. *Each graph G embedded on the torus T contains a $k \times k$ grid as a minor, with $k := \lfloor \frac{2}{3} r(G) \rfloor$. The factor $\frac{2}{3}$ is best possible.*

In fact, in [6] it is shown that for each fixed $r(G) \geqslant 5$ the number $\lfloor \frac{2}{3} r(G) \rfloor$ is best possible. Since each graph embedded on the torus is obviously a minor of some toroidal grid, Theorem 28 implies (42) for the torus.

Theorem 28 again is equivalent to a result in the geometry of numbers. For any 0-symmetric compact convex set P in $\mathbb{R}^n$ and $i = 1, \ldots, n$ one defines:

$$\lambda_i(P) := \min\{\lambda \mid \lambda \cdot P \text{ contains } i \text{ linearly independent integer vectors}\}. \quad (43)$$

So $\lambda_1(P) = \lambda(P)$. Then (see [6]) we have the following.

Theorem 29. *For each 0-symmetric compact convex set P in $\mathbb{R}^2$ one has $\lambda_2(P) \cdot \lambda_1(P^*) \leqslant \frac{3}{2}$. The number $\frac{3}{2}$ is best possible.*

Similarly to the equivalence of Theorems 3 and 4 one sees (using Theorem 27) the equivalence of Theorems 28 and 29.

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