A DEFECT CORRECTION METHOD FOR PARABOLIC SINGULAR PERTURBATION PROBLEMS ON A RECTANGLE

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On a rectangle we consider the Dirichlet problem for a singularly perturbed parabolic PDE of reaction-diffusion type with the perturbation parameter $\varepsilon \in (0,1]$. The solution of the problem has parabolic boundary layers which are one-dimensional in the neighbourhood of the smooth boundaries and two-dimensional (corner) in the neighbourhood of the vertices of the rectangle. For such problems we know a special difference scheme, which converges ε-uniformly. This base scheme is classical finite difference approximations of the differential equation on piecewise uniform meshes refined in the neighbourhood of the boundary layers. Its solution converges with the order of accuracy $\mathcal{O}\left(N^{-2}\ln^2 N + N_0^{-1}\right)$, where $N = \min[N_1, N_2]$, and N_0+1, N_i+1 is the number of nodes, respectively, in the time mesh and in the space mesh along the axis x_i, $i=1,2$. It is of great interest to develop difference schemes with a higher order of convergence. In this work we construct ε-uniformly convergent schemes for which the order of accuracy is more than one with respect to the time variable. For this we adapt the system of discrete problems (i.e., difference schemes) constructed with using the base scheme. The problems in the system are solved sequentially. These problems are constructed by such a way that the grid solutions (and their derivatives) already obtained are used then in the correction procedure for increasing the consistency order of the next discrete problem. This correction method allows us to find the approximate solution with high-order time-accuracy, uniform in ε.

1. Introduction

In this paper we continue to study ε-uniform schemes for time-dependent singular perturbation problems. Earlier, in [1], [2] we investigated ε-uniformly convergent difference schemes for parabolic singularly perturbed Dirichlet's problems on an interval for the cases when the problem data are sufficiently smooth and the parabolic equation does not contain convection terms. There we have constructed a new discrete method based on defect correction, which can achieve an arbitrary high order of accuracy with respect to the time variable $\mathcal{O}\left(N^{-2}\ln^2 N + K^{-n}\right)$, $n > 3$, where N and K denote, respectively, the number of intervals in the space and time discretizations.

Here we consider the Dirichlet problem on a rectangle for a singularly perturbed parabolic PDE of reaction-diffusion type. The perturbation parameter ε multiplying the highest derivative takes any value from the half-interval (0,1]. The solution of the problem has singularities such as parabolic boundary layers which are one-dimensional in the neighbourhood of the smooth boundaries and two-dimensional (corner) in the neighbourhood of the vertices of the rectangle. For such a boundary value problem we know a special difference scheme, the solution of which converges ε-uniformly. This base scheme is constructed by classical finite difference approximations of the differential equation on piecewise uniform meshes (see, e.g., [3]) refined in the neighbourhood of the boundary layers. Its solution converges with the order of accuracy $\mathcal{O}\left(N^{-2}\ln^2 N + N_0^{-1}\right)$, where $N = \min[N_1, N_2]$, and N_0+1, N_i+1 is the number of nodes, respectively, in the time mesh and in the space mesh along the axis x_i, $i=1,2$. For the problem stated above it is of great interest to develop difference schemes with a higher order of convergence.

In this work we develop special ε-uniformly convergent schemes for which the order of accuracy is more than one with respect to the time variable. For solving the problem we adapt the system of discrete problems (i.e., difference schemes) constructed with using the base scheme. The discrete problems in the system are solved sequentially. These problems are constructed in such a way that the grid solutions (and their derivatives) already obtained are used then in the correction procedure, similarly to [2], for increasing of the consistency order of the next discrete problem. This correction method allows us to find the approximate solution with high-order time-accuracy, uniform in ε. To achieve ε-uniform convergence, we use a grid with nodes that are condensed in the neighbourhood of the boundary layer.

2. The class of boundary value problems studied

On the domain $G = D \times (0,T]$ with the boundary $S = \overline{G} \setminus G$, where $D = \{x : 0 < x_s < d_s,\ s = 1,2\}$, we consider the following singularly perturbed parabolic equation with Dirichlet boundary conditions[1]:

$$L_{(2.1)}u(x,t) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x,t)\frac{\partial^2}{\partial x_s^2}u(x,t) - c(x,t)u(x,t) - p(x,t)\frac{\partial u}{\partial t}(x,t) = f(x,t), \quad (x,t) \in G, \qquad (2.1a)$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \qquad (2.1b)$$

Here $S = S_0 \cup S^L$, S^L is the lateral boundary, $S^L = \{(x,t) : x \in \Gamma,\ 0 < t \leq T\}$, $S_0 = \{(x,t) : x \in \overline{D},\ t = 0\}$, $\Gamma = \overline{D} \setminus D$. In (2.1) $a_s(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions which satisfy

$$0 < a_0 \leq a_s(x,t), \quad 0 < p_0 \leq p(x,t), \quad c(x,t) \geq 0, \quad (x,t) \in \overline{G}.$$

The real parameter ε may take any small positive value, say $\varepsilon \in (0,1]$.

When the parameter ε tends to zero in (2.1a), the solution of the problem has parabolic boundary layers in the neighbourhood of the lateral boundary. In the neighbourhood of the smooth parts of the boundary S^L these layers are described by a one-dimensional parabolic equation, and in the neighbourhood of the vertices from the S^L the corner layer is described by a two-dimensional equation.

3. An arbitrary nonuniform mesh

To solve problem (2.1), we give a classical finite difference scheme. On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0, \quad \overline{D}_h = \overline{\omega}_1 \times \overline{\omega}_2, \qquad (3.1)$$

where $\overline{\omega}_s$ is, generally speaking, a nonuniform mesh on the interval $[0, d_s]$ on the axis x_s, $s = 1,2$; let $h_s^i = x_s^{i+1} - x_s^i$, $x_s^i, x_s^{i+1} \in \overline{\omega}_s$; $h_s = \max_i h_s^i$, $h = \max_s h_s$; $\overline{\omega}_0$ is a uniform mesh on the interval $[0,T]$ with the step τ. By $N_s + 1$ and $K + 1$ we denote the number of nodes in the meshes $\overline{\omega}_s$ and $\overline{\omega}_0$, respectively; $\tau = T K^{-1}$; let $h \leq M N^{-1}$, where $N = \min[N_1, N_2]$.

Here and below we denote by M (or m) sufficiently large (or small) positive constants which do not depend on the value of the parameter ε or on the difference operators.

For problem (2.1) we use the difference scheme (cf [4])

$$\Lambda_{(3.2)}z(x,t) = f(x,t), \qquad (x,t) \in G_h, \qquad (3.2a)$$

$$z(x,t) = \varphi(x,t), \qquad (x,t) \in S_h. \qquad (3.2b)$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$,

$$\Lambda_{(3.2)}z(x,t) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x,t)\delta_{\overline{xs}\,\widehat{xs}}z(x,t) - c(x,t)z(x,t) - p(x,t)\delta_{\overline{t}}z(x,t),$$

$$\delta_{\overline{xs}\,\widehat{xs}}\,z(x,t) = 2(h_s^{i-1} + h_s^i)^{-1}\left(\delta_{xs}z(x,t) - \delta_{\overline{xs}}z(x,t)\right), \quad \delta_{\overline{t}}z(x,t) = \tau^{-1}\left(z(x,t) - z(x,t-\tau)\right),$$

the difference operator $\delta_{\overline{xs}\,\widehat{xs}}\,z(x,t))$ is an approximation of the operator $(\partial^2/\partial x_s^2)u(x,t))$ on the nonuniform mesh, $\delta_{xs}z(x,t)$ and $\delta_{\overline{xs}}z(x,t)$, $\delta_{\overline{t}}z(x,t)$ are the forward and backward differences, for example,

$$\delta_{\overline{x1}}z(x,t) = (h_1^{i-1})^{-1}\left(z(x,t) - z(x_1^{i-1}, x_2, t)\right), \quad \delta_{x1}z(x,t) = (h_1^i)^{-1}\left(z(x_1^{i+1}, x_2, t) - z(x,t)\right), \quad x = (x_1^i, x_2).$$

[1] The notation is such that the operator $L_{(a.b)}$ is first introduced in Eq. $(a.b)$.

From [4] we know that the difference scheme (3.2), (3.1) is monotone. By means of the maximum principle and taking into account estimates of the derivatives we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter ε:

$$| u(x,t) - z(x,t) | \leq M(\varepsilon^{-1}N^{-1} + \tau), \quad (x,t) \in \overline{G}_h. \tag{3.3}$$

This error bound for the classical difference scheme is clearly not ε-uniform. The proof of (3.3) follows the lines of the classical convergence proof for monotone difference schemes (cf [4]; see [5] for more details).

We denote by $H^{(\alpha)}(\overline{G}) = H^{\alpha,\alpha/2}(\overline{G})$ the Hölder space, where α is an arbitrary positive number [5]. We suppose that the functions $f(x,t)$ and $\varphi(x,t)$ satisfy compatibility conditions at the corner points, so that the solution of the boundary value problem is smooth for every fixed value of the parameter ε.

For simplicity, we assume that at the corner points $S^* = \{S_0 \cap \overline{S}^L\} \cup S^c$, where $S^c = \Gamma^c \times (0,T]$, and Γ^c is the set of the corner points of $\overline{D}$, the following conditions hold

$$\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\,\partial x_2^{k_2}\,\partial t^{k_0}}\varphi(x,t) = 0, \quad k = k_1 + k_2, \quad k + 2k_0 \leq [\,\alpha\,] + 2n, \tag{3.4}$$
$$\frac{\partial^{k+k_0}}{\partial x_1^{k_1}\,\partial x_2^{k_2}\,\partial t^{k_0}}f(x,t) = 0, \quad k = k_1 + k_2, \quad k + 2k_0 \leq [\,\alpha\,] + 2n - 2,$$

where $[\,\alpha\,]$ is the integer part of the number α, $\alpha > 0$, $n \geq 0$ is an integer. We also suppose that $[\,\alpha\,]+2n \geq 2$.

We rely on the a-priori estimates for the solution of problem (2.1) on the domain $G = D \times [0,T]$, and its derivatives as derived for elliptic and parabolic equations in [6]–[8]. Taking into account these a-priori estimates, we are thus led to such a general result.

Theorem 1. *Assume in Eq.* (2.1) *that* $a \in H^{(\alpha+2n-1)}(\overline{G})$, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geq 0$ *and let the condition* (3.4) *be fulfilled. Then, for a fixed value of the parameter* ε, *the solution of* (3.2), (3.1) *converges to the solution of* (2.1) *with an error bound given by* (3.3).

4. The ε-uniformly convergent method

In this section we discuss an ε-uniformly convergent method for (2.1) by taking a special mesh, condensed in the neighbourhood of the boundary layers. The location of the nodes is derived from a-priori estimates of the solution and its derivatives. The way to construct the mesh for problem (2.1) is the same as in [1] and [8]. More specificly, we take

$$\overline{G}_h^{\,*} = \overline{D}_h^{\,*} \times \overline{\omega}_0, \quad \overline{D}_h^{\,*} = \overline{\omega}_1^{\,*} \times \overline{\omega}_2^{\,*}\ , \tag{4.1}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(3.1)}$, $\overline{\omega}_s^* = \overline{\omega}_s^*(\sigma_s)$ is a special *piecewise* uniform mesh on $[0, d_s]$, σ_s is a parameter determining the redistribution of nodes in the mesh $\overline{\omega}_s^*$; the value of σ_s depends on ε and N_s. The steps of the mesh $\overline{\omega}_s^*$ on the intervals $[0,\sigma_s]$, $[\sigma_s,\, d_s - \sigma_s]$, $[\,d_s - \sigma_s,\, d_s]$ are constant and equal to $h_s^{(1)} = 4\,\sigma_s\,N_s{-}1$ and $h_s^{(2)} = 2\,(d_s - 2\,\sigma_s)\,N_s{-}1$ on the intervals $[0,\sigma_s]$, $[d_s - \sigma_s,\, d_s]$ and $[\sigma_s,\, d_s - \sigma_s]$, respectively. We take $\sigma_s = \sigma_s(\varepsilon, N_s) = \min[4^{-1}\,d_s,\ M\,\varepsilon\,\ln N_s]$, where M is an arbitrary positive number. The mesh $\overline{G}_h^*$ is constructed.

Theorem 2. *Let the conditions of Theorem* (1) *hold for* $n = 1$. *Then the solution of* (3.2), (4.1) *converges* ε*-uniformly to the solution of* (2.1) *and the following estimate holds:*

$$|u(x,t) - z(x,t)| \leq M(N^{-2}\ln^2 N + \tau), \quad (x,t) \in \overline{G}_h^{\,*}. \tag{4.2}$$

The proof of this theorem can be found in [6], [9].

5. Improved time-accuracy

5.1. *A scheme based on defect correction*

In this section we construct a new discrete method based on defect correction, which also converges ε-uniformly to the solution of the boundary value problem, but with an order of accuracy (with respect to τ) higher than in (4.2).

The idea is similar to that published in [1]. For the difference scheme (3.2), (4.1) the error in the approximation of the partial derivative $(\partial/\partial t)\,u(x,t)$ is caused by the divided difference $\delta_{\overline{t}}\,z(x,t)$ and is associated with the truncation error given by the relation

$$\frac{\partial u}{\partial t}(x,t) - \delta_{\overline{t}}\,u(x,t) = 2^{-1}\,\tau\frac{\partial^2 u}{\partial t^2}(x,t) - 6^{-1}\,\tau^2\frac{\partial^3 u}{\partial t^3}(x,t-\vartheta), \quad \vartheta \in [0,\tau].$$

Therefore, we now use for the approximation of $(\partial/\partial t)\,u(x,t)$ the expression $\delta_{\bar t}\,u(x,t)+\tau\delta_{\bar t\bar t}\,u(x,t)/2$, where $\delta_{\bar t\bar t}\,u(x,t)\equiv\delta_{t\bar t}\,u(x,t-\tau)$. Notice that $\delta_{t\bar t}\,u(x,t)$ is the second central divided difference. We can evaluate a better approximation than (3.2a) by defect correction

$$\Lambda_{(3.2)}z^c(x,t)=f(x,t)-2^{-1}p(x,t)\tau\,\frac{\partial^2 u}{\partial t^2}(x,t), \tag{5.1}$$

with $x\in\overline{\omega}$ and $t\in\overline{\omega}_0$, where $\overline{\omega}$ and $\overline{\omega}_0$ are as in (3.1); τ is the step-size of the mesh $\overline{\omega}_0$; $z^c(x,t)$ is the "corrected" solution. Instead of $(\partial^2/\partial t^2)\,u(x,t)$ we shall use $\delta_{\bar t\bar t}\,z(x,t)$, where $z(x,t)$, $(x,t)\in G_{h(4.1)}$ is the solution of the difference scheme (3.2), (4.1). We may expect that the new solution $z^c(x,t)$ has an accuracy of $\mathcal{O}\left(\tau^2\right)$ with respect to the time variable. This is true, as will be shown in Section 5.2.

Moreover, in a similar way we can construct a difference approximation with a convergence order higher than two (with respect to the time variable) and $\mathcal{O}\left(N^{-2}\ln^2 N\right)$ with respect to the space variable, uniformly in ε (see Section 5.3).

5.2. *Modified difference schemes of the second order accuracy in* τ

We denote by $\delta_{k\bar t}z(x,t)$ the backward difference of order k:

$$\delta_{0\bar t}\;z(x,t)=z(x,t),\quad \delta_{k\bar t}\;z(x,t)=\left(\delta_{k-1\,\bar t}\;z(x,t)-\delta_{k-1\,\bar t}\;z(x,t-\tau)\right)/\tau,\quad (x,t)\in\overline{G}_h,\quad t\geq k\tau,\quad k\geq 1.$$

When constructing difference schemes of second order accuracy in τ in (5.1), instead of $(\partial^2/\partial t^2)u(x,t)$ we use $\delta_{2\,\bar t}\,z(x,t)$, which is the second divided difference of the solution to the discrete problem (3.2), (4.1). On the mesh $\overline{G}_h$ we consider the finite difference scheme (3.2), writing

$$\Lambda_{(3.2)}z^{(1)}(x,t)=f(x,t),\quad (x,t)\in G_h,\quad z^{(1)}(x,t)=\varphi(x,t),\quad (x,t)\in S_h. \tag{5.2}$$

Then for the boundary value problem (2.1) we now get for the difference equations for $t=\tau$ and $t\geq 2\tau$ respectively:

$$\Lambda_{(3.2)}z^{(2)}(x,t)=f(x,t)+\frac{p(x,t)}{2}\,\tau\,\frac{\partial^2 u}{\partial t^2}(x,0),\quad t=\tau, \tag{5.3}$$

$$\Lambda_{(3.2)}z^{(2)}(x,t)=f(x,t)+\frac{p(x,t)}{2}\,\tau\,\delta_{2\bar t}z^{(1)}(x,t),\quad t\geq 2\tau,\quad (x,t)\in G_h,$$

$$z^{(2)}(x,t)=\varphi(x,t),\quad (x,t)\in S_h.$$

Here $z^{(1)}(x,t)$ is the solution of the discrete problem (5.2), (4.1), and the derivative $\dfrac{\partial^2 u}{\partial t^2}(x,0)$ is obtained from Eq. (2.1a). We shall call $z^{(2)}(x,t)$ the solution of difference scheme (5.3), (5.2), (4.1) (or shortly, (5.3), (4.1)).

In what follows, for simplicity, we suppose that the coefficients $a_s(x,t)$ do not depend on t

$$a_s(x,t)=a_s(x),\quad (x,t)\in\overline{G} \tag{5.4}$$

and we take a homogeneous initial condition:

$$\varphi(x,0)=0,\quad x\in[0,1]. \tag{5.5}$$

Under the conditions (5.4), (5.5), the following estimate holds for the solution of problem (5.3), (4.1)

$$\left|u(x,t)-z^{(2)}(x,t)\right|\leq M\;\left[\;N^{-2}\ln^2 N+\tau^2\;\right],\quad (x,t)\in\overline{G}_h. \tag{5.6}$$

Theorem 3. *Let conditions* (5.4), (5.5) *hold and assume in Eq.* (2.1) *that* $a\in H^{(\alpha+2n-1)}(\overline{G})$, c, p, $f\in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi\in H^{(\alpha+2n)}(\overline{G})$, $\alpha>4$, $n=2$ *and let condition* (3.4) *be satisfied for* $n=2$. *Then for the solution of difference scheme* (5.3), (4.1) *the estimate* (5.6) *holds.*

The proof of this Theorem and Theorem 4 is similar to the proof in [2].

5.3. *The difference scheme of the third order accuracy in time*

Analogously we construct a difference scheme with the third order accuracy in τ. On the mesh $\overline{G}_h$ we consider the difference scheme

$$\Lambda_{(3.2)}z^{(3)}(x,t) = f(x,t) + p(x,t)\left(C_{11}\tau\frac{\partial^2}{\partial t^2}u(x,0) + C_{12}\tau^2\frac{\partial^3 u}{\partial t^3}(x,0)\right), \quad t=\tau, \tag{5.7a}$$

$$\Lambda_{(3.2)}z^{(3)}(x,t) = f(x,t) + p(x,t)\left(C_{21}\tau\frac{\partial^2 u}{\partial t^2}(x,0) + C_{22}\tau^2\frac{\partial^3 u}{\partial t^3}(x,0)\right), \quad t=2\tau,$$

$$\Lambda_{(3.2)}z^{(3)}(x,t) = f(x,t) + p(x,t)\left(C_{31}\tau\delta_{2\bar{t}}z^{(2)}(x,t) + C_{32}\tau^2\delta_{3\bar{t}}z^{(1)}(x,t)\right), \quad t\geq 3\tau, \quad (x,t)\in G_h,$$

$$z^{(3)}(x,t) = \varphi(x,t), \quad (x,t)\in S_h.$$

Here $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ are the solutions of problems (5.2), (4.1) and (5.3), (4.1), respectively, the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are obtained from Eq. (2.1a), the coefficients C_{ij} are determined below. They are chosen such that the following conditions are satisfied

$$\frac{\partial u}{\partial t}(x,t) = \delta_{\bar{t}}u(x,t) + C_{11}\tau\frac{\partial^2 u}{\partial t^2}(x,t-\tau) + C_{12}\tau^2\frac{\partial^3 u}{\partial t^3}(x,t-\tau) + O\left(\tau^3\right),$$

$$\frac{\partial u}{\partial t}(x,t) = \delta_{\bar{t}}u(x,t) + C_{21}\tau\frac{\partial^2 u}{\partial t^2}(x,t-2\tau) + C_{22}\tau^2\frac{\partial^3 u}{\partial t^3}(x,t-2\tau) + O\left(\tau^3\right),$$

$$\frac{\partial u}{\partial t}(x,t) = \delta_{\bar{t}}u(x,t) + C_{31}\tau\delta_{2\bar{t}}u(x,t) + C_{32}\tau^2\delta_{3\bar{t}}u(x,t) + O\left(\tau^3\right).$$

It follows that

$$C_{11} = C_{21} = C_{31} = 1/2, \quad C_{12} = C_{32} = 1/3, \quad C_{22} = 5/6. \tag{5.7b}$$

We shall call $z^{(3)}(x,t)$ the solution of the difference scheme (5.7), (5.3), (5.2), (4.1) (or shortly, (5.7), (4.1)). Again we assume the homogeneous initial condition

$$\varphi(x,0) = 0, \quad f(x,0) = 0, \quad x\in[0,1]. \tag{5.8}$$

Under conditions (5.4), (5.8) the following estimate holds for the solution of difference scheme (5.7), (4.1)

$$\left|u(x,t) - z^{(3)}(x,t)\right| \leq M\,\left[\,N^{-2}\ln^2 N + \tau^3\,\right], \quad (x,t)\in\overline{G}_h. \tag{5.9}$$

Theorem 4. *Let conditions* (5.4), (5.8) *hold and assume in Eq.* (2.1) *that* $a\in H^{(\alpha+2n-1)}(\overline{G})$, c, p, $f\in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi\in H^{(\alpha+2n)}(\overline{G})$, $\alpha>4$, $n=3$ *and let condition (3.4) be satisfied with* $n=3$. *Then for the solution of scheme* (5.7), (4.1) *the estimate* (5.9) *is valid.*

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References

[1] P. W. Hemker, G. I. Shishkin and L. P. Shishkina, *The use of defect correction for the solution of parabolic singular perturbation problems*, ZAMM - Z. Angew. Math. Mech. **76** (1997), pp. 59–74.

[2] P. W. Hemker, G. I. Shishkin and L. P. Shishkina, *ε-Uniform Schemes with High- Order Time-Accuracy for Parabolic Singular Perturbation Problems*, IMA J. Numer. Anal. **14** (1998), pp. 1–24.

[3] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, **Fitted Numerical Methods for Singular Perturbation Problems** (World Scientific, 1996).

[4] A. A. Samarskii, **Theory of difference schemes** (Nauka, 1989) (in Russian).

[5] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, **Linear and Quasi-linear Equations of Parabolic Type** (AMS, 1968).

[6] G. I. Shishkin. **Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations**, (Ur. O. RAN, Ekaterinburg, 1992) (in Russian).

[7] G. I. Shishkin, *Approximation of solutions to singularly perturbed boundary value problems with corner boundary layers*. Zh. Vychisl. Mat. Mat. Fiz.. **27** (1987), pp. 1360–1374 (in Russian).

[8] G. I. Shishkin, *Approximation of solutions to singularly perturbed boundary value problems with parabolic boundary layers*. Zh. Vychisl. Mat. Mat. Fiz.. **29** (1989), pp. 963–977. (In Russian).

[9] G. I. Shishkin, *Grid approximation of singularly perturbed elliptic equation in domain with characterictic bounds*. Sov. J. Numer. Anal. Math. Modelling. **5** (1990), pp. 327–343.