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MEDIAN GRAPHS AND HELLY HYPERGRAPHS

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One-to-one correspondences are established between the following combinatorial structures: (i) median interval structures (or median segments, introduced by Sholander); (ii) maximal Helly hypergraphs such that with each edge also its complement is in the hypergraph; and (iii) median graphs (connected graphs such that for any three vertices u, v, w there is exactly one vertex x such that $d(u,v)=d(u,x)+d(x,v)$, $d(v,w)=d(v,x)+d(x,w)$ and $d(w,u)=d(w,x)+d(x,u)$, where d is the distance function of the graph).

Introduction

In this paper one-to-one correspondences will be established between three at first sight fairly distinct concepts. These concepts are:

(i) *median interval structures* introduced by M. Scholander [7, 8] under the name of median segments (see Section 1.1.);

(ii) *maximal Helly copair hypergraphs* (i.e. simple Helly hypergraphs, the edge-set of which contains with edge its complement, and which are maximal with respect to this property; see Section 1.2); and

(iii) *median graphs*, introduced in Section 1.3.

The one-to-one correspondences are established in Section 2.

In Section 3 we elaborate how to construct a maximal Helly copair hypergraph from a median graph, using results of Sholander [9].

With minor adaptations we adopt the terminology of Berge [1] on hypergraphs, of Wilson [10] on graphs and of Birkhoff [2] on lattice theory.

1. Definitions and preliminaries

Throughout this paper V denotes a fixed finite set.

1.1. *Interval structures*

A function $I: V\times V \to \mathcal{P}(V)$ is called an *interval structure* on V if

(I1) $\quad x, y \in I(u,v)$ iff $I(x,y)\subset I(u,v)$, $\qquad (x,y,u,v\in V)$;

(I2) $\quad I(u,v)\cap I(v,w)\cap I(w,u)\neq\emptyset$, $\qquad (u,v,w\in V)$.

Each set $I(u, v)$ is called an *interval*. A subset U of V is *I-convex* if, for all $u, v \in U$ the interval $I(u, v)$ is contained in U. The notion of interval structure was introduced in [3]. Examples of interval structures on V can be obtained from trees with vertex-set V (then take $I(u, v) = \{w \in V \mid w$ lies on the shortest u, v-path$\}$), and from lattices $(V, \leqslant)$ (in this case $I(u, v) = \{w \in V \mid u \wedge v \leqslant w \leqslant u \vee v\}$).

If I satisfies condition (I1) and the following condition

$$(\mathrm{I}2') \quad |I(u, v) \cap I(v, w) \cap I(w, u)| = 1, \qquad (u, v, w \in V),$$

then I is called a *median interval structure* on V. Interval structures obtained from trees as indicated above are median interval structures. An interval structure obtained from a lattice is a median interval structure iff the lattice is distributive (see [2]). Sholander [8] has given the following characterization of median interval structures (he used the term median segments).

Theorem 1 (Sholander [8]). *A function $I: V \times V \to \mathcal{P}(V)$ is a median interval structure on V iff*

$$\text{if} \quad w \in I(u, v), \quad \text{then} \quad I(u, w) \subset I(u, v) \cap I(v, u) \quad (u, v \in V),$$

$$|I(u, v) \cap I(v, w) \cap I(w, u)| = 1 \quad (u, v, w \in V),$$

$$I(v, v) = \{v\} \quad (v \in V).$$

1.2. *Hypergraphs*

In this paper a *hypergraph* $H = (V, \mathscr{E})$ consists of a *vertex-set* V and a family $\mathscr{E} \subset \mathcal{P}(V)$ of nonvoid subsets of V, the members of which are called *edges*. Occasionally we shall write $\mathscr{E}$ instead of $(V, \mathscr{E})$.

A hypergraph is a *Helly hypergraph* if it satisfies the *Helly property*, i.e. every subfamily of $\mathscr{E}$, any two members of which meet, has a non-empty intersection. For vertices u and v of the hypergraph $(V, \mathscr{E})$, we define

$$I_{\mathscr{E}}(u, v) = \cap\{B \in \mathscr{E} \mid u, v \in B\}.$$

A theorem of Gilmore (see [5], or [1, p. 396]) can be formulated as follows.

Theorem 2 (Gilmore). *A hypergraph $(V, \mathscr{E})$ satisfies the Helly property iff $I_{\mathscr{E}}$ is an interval structure on V.*

As a consequence of Gilmore's theorem we have: *let I be an interval structure on V; any family $\mathscr{E}$ of nonvoid I-convex subsets of V satisfies the Helly property.*

A hypergraph $(V, \mathscr{E})$ with the property that $V \setminus B \in \mathscr{E}$ for all $B \in \mathscr{E}$ will be called a *copair hypergraph*. We call the set $\{B, V \setminus B\}$ a copair of V and $\{\emptyset, V\}$ the trivial copair. A Helly copair hypergraph of course is a copair hypergraph that satisfies the Helly property. Finally a *maximal Helly copair hypergraph* $(V, \mathscr{E})$ is a

Helly copair hypergraph such that: if $\{A, V\setminus A\}$ is a non-trivial copair and $\mathscr{E}\cup\{A, V\setminus A\}$ satisfies the Helly property, then $A\in\mathscr{E}$.

A hypergraph $(V,\mathscr{E})$ is said to *separate vertices* if for any two distinct vertices $u, v\in V$ there exists an edge $A\in\mathscr{E}$ such that $u\in A$ and $v\notin A$.

Lemma 3. *Let $(V,\mathscr{E})$ be a Helly copair hypergraph. Then $(V,\mathscr{E})$ is maximal iff $(V,\mathscr{E})$ separates vertices.*

Proof. Note that $(V,\mathscr{E})$ separates vertices iff $I_{\mathscr{E}}(v, v)=\{v\}$ for all $v\in V$.

Assume that $\mathscr{E}$ does not separate vertices. That is there exists a vertex $v\in V$ such that $I_{\mathscr{E}}(v, v)$ contains, besides v, another vertex. Using Gilmore's theorem it can be verified that in this case $\mathscr{E}\cup\{\{v\}, V\setminus\{v\}\}$ satisfies the Helly property. Therefore $\mathscr{E}$ is not maximal.

To prove sufficiency of vertex separation let $\{A, V\setminus A\}$ be a non-trivial copair of V not in $\mathscr{E}$. Take a vertex $u\in A$ and a vertex $v\in V\setminus A$ such that $|I_{\mathscr{E}}(u, v)|$ is as small as possible. We assert that $I_{\mathscr{E}}(u, v)\cap A=\{u\}$ and $I_{\mathscr{E}}(u, v)\setminus A=\{v\}$.

For suppose $I_{\mathscr{E}}(u, v)\cap A\neq\{u\}$ and let $w\in I_{\mathscr{E}}(u, v)\cap A$ with $w\neq u$. Since $\mathscr{E}$ separates vertices, there exists an edge $C\in\mathscr{E}$ such that $w\in C$ and $u\notin C$. Then we have that $v\in C$. So $u\notin I_{\mathscr{E}}(w, v)\subset I_{\mathscr{E}}(u, v)$, contradicting the minimality of $I_{\mathscr{E}}(u, v)$. In the same way we prove $I_{\mathscr{E}}(u, v)\setminus A=\{v\}$. Hence $I_{\mathscr{E}}(u, v)=\{u, v\}$.

Let $B\in\mathscr{E}$ be an edge such that $v\in B$ and $u\notin B$. Then $A\cap B\neq\emptyset$ or $(V\setminus A)\cap(V\setminus B)\neq\emptyset$, since $A\notin\{B, V\setminus B\}\subset\mathscr{E}$. Say $A\cap B\neq\emptyset$. Now the set of edges that contain both u and v, together with A and B, forms a family of subsets of V, any two members of which meet. The intersection of this family equals

$$I_{\mathscr{E}}(u, v)\cap A\cap B=\{u, v\}\cap A\cap B,$$

which clearly is empty. Thus $\mathscr{E}\cup\{A, V\setminus A\}$ does not satisfy the Helly property. $\square$

From this lemma a lower bound can be obtained for the number of edges in a maximal Helly copair hypergraph. That this lower bound is best possible follows from the results of Section 3.3.

Corollary 4. *Let $(V,\mathscr{E})$ be a maximal Helly copair hypergraph. Then*

$$|\mathscr{E}|\geqslant 2\lceil {}^2\log|V|\rceil$$

Proof. Fix $x\in V$. For each $v\in V$ let $\mathscr{E}_v=\{E\in\mathscr{E}\mid v, x\in E\}$. Since $(V,\mathscr{E})$ separates vertices, $\mathscr{E}_v\neq\mathscr{E}_w$ if $v\neq w$. Therefore, $|V|\leqslant 2^{|\mathscr{E}_x|}=2^{|\mathscr{E}|/2}$. $\square$

1.3. *Median graphs*

Let G be a simple loopless graph with vertex-set V and distance function d. G will be called a ***median graph*** if it is connected and satisfies the ***graph** median property*, i.e. for any $u, v, w\in V$ there exists precisely one vertex $x\in V$, called the

Proof. We shall prove that G_I is connected and that $I_{G_I} = I$. Then clearly G_I is a median graph.

First observe that for $u, v, w \in V$ we have

$$w \in I(u, v) \quad \text{iff} \quad I(u, w) \cap I(w, v) = \{w\}.$$

Thus for $w \in I(u, v) \setminus \{u, v\}$ holds $u \notin I(w, v) \subset I(u, v)$ and $v \notin I(u, w) \subset I(u, v)$. Using this it is easily verified by induction on $|I(u, v)|$ that $I(u, v)$ induces a connected subgraph of G_I for all $u, v \in V$. Hence G_I is connected.

To prove that $I(u, v) = I_{G_I}(u, v)$ for all $u, v \in V$ we use induction on $d(u, v)$. Clearly $I(u, v) = I_{G_I}(u, v)$ for all $u, v \in V$ with $d(u, v) \leq 1$. So take vertices $u, v \in V$ with $d(u, v) > 1$.

Let $w \in I_{G_I}(u, v) \setminus \{u, v\}$. Then $d(u, w) < d(u, v)$ and $d(w, v) < d(u, v)$, so $I_{G_I}(u, w) = I(u, w)$ and $I_{G_I}(w, v) = I(w, v)$. Since clearly $I_{G_I}(u, w) \cap I_{G_I}(w, v) = \{w\}$, we have $w \in I(u, v)$ and thus $I_{G_I}(u, v) \subset I(u, v)$.

Assume $I(u, v) \setminus I_{G_I}(u, v) \neq \emptyset$. For any vertex $w \in I(u, v) \setminus I_{G_I}(u, v)$ we must have $I(u, w) \cap I_{G_I}(u, v) = \{u\}$, and similarly $I(w, v) \cap I_{G_I}(u, v) = \{v\}$. For if $w' \in I(u, w) \cap I_{G_I}(u, v)$, with $w' \neq u$, then $w \in I(w', v)$ and by the induction hypothesis $I(w', v) = I_{G_I}(w', v) \subset I_{G_I}(u, v)$. Hence $w \in I_{G_I}(u, v)$, contradicting the choice of w.

Since $I(u, v)$ induces a connected subgraph of G_I, there exists a path P from u to v, all of whose internal vertices lie in $I(u, v) \setminus I_{G_I}(u, v)$. Clearly the length of P exceeds $d(u, v)$ so P has at least two distinct internal vertices, say x and y.

Since $d(u, v) \geq 2$, there exists a vertex $z \in I_{G_I}(u, v) \setminus \{u, v\}$. By the induction hypothesis we have $I(u, z) = I_{G_I}(u, z)$ and $I(z, v) = I_{G_I}(z, v)$. Now

$$u \in I(u, z) \cap I(u, x) = I_{G_I}(u, z) \cap I(u, x) \subset I_{G_I}(u, v) \cap I(u, x) = \{u\}.$$

So $u \in I(z, x)$. Similarly $v \in I(z, x)$ and thus $I(u, v) \subset I(z, x) \subset I(u, v)$. In the same way it follows that $I(u, v) = I(z, y)$. But then

$$x, y \in I(x, y) = I(z, x) \cap I(x, y) \cap I(y, z),$$

contradicting the fact that I is a median interval structure. Conclusion: $I(u, v) = I_{G_I}(u, v)$. □

In the proof of the preceding proposition we have seen that $I_{G_I} = I$ holds for a median interval structure I. Furthermore, from Propositions 6 and 7 it follows immediately that, when G is a median graph, we have $G_{I_G} = G$.

Proposition 8. *Let $(V, \mathscr{E})$ be a maximal Helly copair hypergraph. Then $I_{\mathscr{E}}$ is a median interval structure on V.*

Proof. Assume that there exist vertices $u, v, w \in V$ such that $x, y \in I_{\mathscr{E}}(u, v) \cap I_{\mathscr{E}}(v, w) \cap I_{\mathscr{E}}(w, u)$ for vertices $x, y \in V$, with $x \neq y$. According to Lemma 3, there is an edge $B \in \mathscr{E}$ such that $x \in B$ and $y \notin B$. Then one of the edges B and $V \setminus B$, say B, must contain at least two of the three vertices u, v and w, say u and v. But then $y \notin I_{\mathscr{E}}(u, v)$. Contradiction. □

graph median of u, v and w, such that

$$\begin{cases} d(u, x)+d(x, v)=d(u, v), \\ d(v, x)+d(x, w)=d(v, w), \\ d(w, x)+d(x, u)=d(w, u). \end{cases}$$

Note that trees and n-cubes are median graphs. It can be seen that each median graph is bipartite.

2. The theorem

Theorem 5. *There exists a one-to-one correspondence between the median interval structures on V, the maximal Helly copair hypergraphs with vertex-set V, and the median graphs with vertex-set V, as follows:*

(i) *Let I be a median interval structure on V. Then*
  $-(V, \mathscr{E})$ *is a maximal Helly copair hypergraph, where* $\mathscr{E}=\{B \subset V \mid \emptyset \neq B \neq V,$ *B and $V \setminus B$ are I-convex*$\}$;
  $-(V, E)$ *is a median graph, where $uv \in E$ iff $u \neq v$ and $I(u, v)=\{u, v\}$.*

(ii) *Let $(V, \mathscr{E})$ be a maximal Helly copair hypergraph. Then*
  $-I$ *is a median interval structure on V, where* $I(u, v)=\bigcap\{B \in \mathscr{E} \mid u, v \in B\}$;
  $-(V, E)$ *is a median graph, where $uv \in E$ iff $u \neq v$ and* $\bigcap\{B \in \mathscr{E} \mid u, v \in B\}=\{u, v\}$.

(iii) *Let (V, E) be a median graph. Then*
  $-I$ *is a median interval structure on V, where $I(u, v)=\{w \in V \mid w$ lies on a shortest u, v-path in $(V, E)\}$;*
  $-(V, \mathscr{E})$ *is a maximal Helly copair hypergraph, where $\mathscr{E}$ consists of the canonical copairs of (V, E) (see Section 3.3).*

The proof of the theorem amounts to the following propositions. (The direct correspondence between median graphs and maximal Helly copair hypergraphs will be explained in Section 3.)

For vertices u and v of the graph $G=(V, E)$ define

$$I_G(u, v)=\{w \in V \mid w \text{ lies on a shortest } u, v\text{-path in } G\}.$$

Proposition 6. *Let $G=(V, E)$ be a median graph. Then I_G is a median interval structure on V.*

Proof. I_G satisfies the conditions mentioned in Theorem 1. □

Proposition 7. *Let I be a median interval structure on V. Define the graph G_I with vertex-set V by*

$$uv \in E(G_I) \text{ iff } u \neq v \text{ and } I(u, v)=\{u, v\} \quad (u, v \in V).$$

Then G_I is a median graph.

Proposition 9. *Let I be a median interval structure on V and let*

$$\mathscr{E}_I = \{B \subset V \mid \emptyset \neq B \neq V,\ B \text{ and } V \setminus B \text{ are } I\text{-convex}\}.$$

Then $(V, \mathscr{E}_I)$ is a maximal Helly copair hypergraph.

Proof. Clearly $(V, \mathscr{E}_I)$ is a Helly copair hypergraph. By Lemma 3 it suffices to show that $\mathscr{E}_I$ separates vertices. So suppose that for vertices $u, v \in V$, with $u \neq v$, there is no edge B such that $u \in B$ and $v \notin B$. Assume furthermore that u and v are such that $|I(u, v)|$ is as small as possible.

We first prove that $I(u, v) = \{u, v\}$. Suppose $w \in I(u, v) \setminus \{u, v\}$. Since $|I(u, w)| < |I(u, v)|$, there exists an edge A such that $u \in A$ and $w \notin A$. It follows that $v \in A$ (u and v cannot be separated). So $w \in I(u, v) \subset A$, for A is I-convex, contradicting $w \notin A$. Therefore $I(u, v) = \{u, v\}$.

Now let $B = \{z \in V \mid v \notin I(u, z)\}$. Then $V \setminus B = \{z \in V \mid u \notin I(z, v)\}$, since $I(u, z) \cap I(z, v) \cap \{u, v\}$ is a singleton. We assert that B and $V \setminus B$ are I-convex, that is $B \in \mathscr{E}_I$. Since $u \in B$ and $v \notin B$ this contradicts our assumption that $\mathscr{E}_I$ does not separate vertices.

We only prove that B is I-convex (the I-convexity of $V \setminus B$ can be treated similarly). Note that for each $z \in B$ we have $I(u, z) \subset B$, since $v \notin I(u, z)$. Let $x, y \in B$ and suppose $I(x, y) \not\subset B$. Take $w \in I(x, y) \setminus B$. Since $I(u, x) \subset I(v, x)$ and $I(u, y) \subset I(v, y)$ we have that

$$\{z\} = I(u, x) \cap I(x, y) \cap I(y, u) = I(v, x) \cap I(x, y) \cap I(y, v)$$

for some $z \in B$. Now also

$$\{z\} \subset I(z, w) \cap I(z, v) \subset I(x, y) \cap I(x, v) \cap I(y, v) = \{z\},$$

since $z, w \in I(x, y)$ and $z \in I(u, x) \cap I(u, y) \subset I(v, x) \cap I(v, y)$. This implies $z \in I(w, v)$ according to the observation made at the beginning of the proof of Proposition 7. So $I(z, v) \subset I(w, v)$. But, since $w \notin B$, $u \notin I(w, v)$ and thus $u \notin I(z, v)$, that is $z \in V \setminus B$, contradicting the fact that $z \in B$. $\square$

From Propositions 8 and 9 we deduce: let I be a median interval structure on V, then $I_{\mathscr{E}_I} = I$; and let $(V, \mathscr{E})$ be a maximal Helly copair hypergraph, then $\mathscr{E}_{I_{\mathscr{E}}} = \mathscr{E}$.

3. Median graphs and helly hypergraphs

In this section the direct correspondence between median graphs and maximal Helly copair hypergraphs with vertex-set V, mentioned in the theorem, is further elaborated.

3.1. *Median semilattices*

Let $(V, \leqslant)$ be a partially ordered set (poset). v is said to *cover* $u (u, v \in V)$, if $u < v$ and there is no $w \in V$ such that $u < w < v$. A *semilattice* $(V, \leqslant)$ is a poset, in which any two elements u, v have a greatest lower bound $u \wedge v$. For $u, v \in V$ set $[u, v] = \{w \in V \mid u \leqslant w \leqslant v\}$. The semilattice $(V, \leqslant)$ is called *distributive* if $([u, v], \leqslant)$ is a distributive lattice for all $u, v \in V$. The semilattice is said to satisfy the *coronation property* if for any three elements $u, v, w \in V$, such that the three least upper bounds $u \vee v$, $v \vee w$, $w \vee u$ exist, there exists a least upper bound $u \vee v \vee w$.

A *median semilattice* is a distributive semilattice, which satisfies the coronation property. This concept was introduced by Sholander [9].

On a median semilattice $(V, \leqslant)$ the ternary operation $(u, v, w) = (u \wedge v) \vee (v \wedge w) \vee (w \wedge u) \in V$ can be defined, called the *median* of u, v and w (Sholander [9] also characterized medians).

We review some results of Sholander [9] reformulating them in our terminology:

(A) *Each median semilattice* $(V, \leqslant)$ *yields a median interval structure* $I_{\leqslant}$ *on* V, *where*

$$I_{\leqslant}(u, v) = \{w \mid w \text{ is the median of } u, v, w\} \quad (u, v \in V).$$

(B) *Let I be a median interval structure on V and* $u \in V$. *Define an ordering* $\leqslant_{I,u}$ *on V by*

$$v \leqslant_{I,u} w \quad \text{iff} \quad v \in I(u, w) \quad (v, w \in V).$$

Then $(V, \leqslant_{I,u})$ *is a median semilattice. Furthermore the correspondences given in* (A) *and* (B) *commute.*

(C) *Let* $(V, \leqslant)$ *be a median semilattice. Then* $(V, \leqslant)$ *can be embedded in a Boolean algebra by an order preserving mapping, which also preserves the covering relation in* $(V, \leqslant)$.

3.2. *Cutset colourings*

A *cutset colouring* of a connected graph is a colouring of the edges in such a way that the edges of any colour form a matching as well as a cutset (i.e. a minimal disconnecting edge-set). If we want to establish a cutset colouring of a graph we are forced to colour non-adjacent edges in each circuit of length four with the same colour. So the n-cube admits a cutset colouring with n colours, which is uniquely determined up to the labelling of the colours. Deleting the edges with a given colour from the n-cube breaks the graph up into two components, which both are $(n-1)$-cubes.

Note that not all connected graphs admit a cutset colouring. Necessary conditions for the existence of a cutset colouring of the edges of a connected graph are for instance that the graph is simple, loopless and bipartite and that it does not contain $K_{2,3}$ as a subgraph.

3.3. *Median graphs and maximal Helly copair hypergraphs*

The *diagraph* of a poset $(V, \leqslant)$ is the graph with vertex-set V, in which two vertices are joined by an edge iff one of the two covers the other in the poset.

Clearly, the diagraph of the Boolean algebra on 2^n elements is the n-cube. As a consequence of (A) and (B) and Propositions 6 and 7 we have

Proposition 10. *Let G be a graph. Then G is a median graph iff G is the diagraph of a median semilattice.*

Proposition 11. *Let G be a graph. Then G is a median graph iff G is a connected induced subgraph of an n-cube such that with any three vertices of G their graph median in the n-cube also is a vertex of G.*

Proof. The only if part follows from Proposition 10 and (C).

The if part follows as soon as we have proved that the distance in G between two vertices equals their distance in the n-cube. Let d be the distance function of G and e that of the n-cube. Assume that there are vertices u, v of G with $d(u, v) \neq e(u, v)$ and let $k := d(u, v)$ be as small as possible. Note that $k > 2$.

Let w be a vertex of G with $d(u, w) = 2$ and $d(w, v) = k-2$. Then $e(u, w) = 2$ and $e(w, v) = k-2$. Let z be the graph median of u, v and w in the n-cube. Thus z is a vertex of G.

If $z = w$, then $e(u, v) = e(u, w) + e(w, v) = 2 + k - 2$. So $z \neq w$. But then, since $e(u, w) = 2 = d(u, w)$, z is a common neighbour of u and w. Now $e(z, v) = e(w, v) - e(w, z) = k - 2 - 1 = k - 3$. Thus

$$d(u, v) \leq d(u, z) + d(z, v) = 1 + e(z, v) = k - 2 < k,$$

which is a contradiction. $\square$

Let G be a median graph with vertex-set V. Embed G in an n-cube K with n as small as possible. Since G is connected, G has at least one edge of each colour from the cutset colouring of K.

The cutset colouring of K induces an edge colouring of G. According to Proposition 11 with any two vertices u and v of G a shortest u, v-path of K lies entirely in G. So the induced edge colouring of G in fact is a cutset colouring. Any cutset from this colouring induces a copair of V: after deleting the cutset from G the graph breaks up into two components, the vertex-sets of which form the complementary subsets of the copair. In this way the cutset colouring of G induces a copair hypergraph $(V, \mathscr{C}_G)$. Since G is an induced subgraph of K it follows that $\mathscr{C}_G$ consists of I_G-convex subsets of V. Besides it follows that $\mathscr{C}_G$ separates vertices. And thus, according to Lemma 3, $(V, \mathscr{C}_G)$ is a maximal Helly copair hypergraph. Furthermore uv is an edge in G iff $u \neq v$ and $\bigcap \{B \in \mathscr{C}_G \mid u, v \in B\} = \{u, v\}$. That is $G_{I_{\mathscr{C}_G}} = G$.

Starting with a maximal Helly copair hypergraph $(V, \mathscr{C})$ then $G_{\mathscr{C}} = G_{I_{\mathscr{C}}}$ is a median graph with vertex-set V. Moreover $\mathscr{C}$ consists of $I_{G_{\mathscr{C}}}$-convex subsets of V. But also $\mathscr{C}_{G_{\mathscr{C}}}$ is a Helly copair hypergraph consisting of $I_{G_{\mathscr{C}}}$-convex subsets of V. Since both $\mathscr{C}$ and $\mathscr{C}_{G_{\mathscr{C}}}$ are maximal, we have that $\mathscr{C} = \mathscr{C}_{G_{\mathscr{C}}}$.

The preceding observations imply that a median graph G, with vertex-set V, admits only one cutset colouring which induces a maximal Helly copair hypergraph. Let us call the copairs of V induced by this cutset colouring of G the *canonical copairs* of G. (In fact it can be proved that up to the labelling of the colours a median graph admits exactly one cutset colouring of its edges, see [6].)

Recapitulating we have proved:

Proposition 12. *The hypergraph* $(V, \mathscr{E})$ *is a maximal Helly copair hypergraph iff* $\mathscr{E}$ *consists of the canonical copairs of a median graph with vertex-set* V.

3.4. *Concluding remarks*

Let G be a connected graph with n vertices, which admits a cutset colouring. Since each cutset contains edges of a spanning tree, the number of colours in the cutset colouring is at most $n-1$.

Lemma 13. *Let* G *be a connected graph with* n *vertices admitting a cutset colouring. Then the number of colours in the cutset colouring is* $n-1$ *iff* G *is a tree.*

Proof. The if part of this lemma is trivial. To prove the only if part let T be a spanning tree of G. Then T has $n-1$ edges, so the edges of T all have different colours. Thereby every edge of T determines exactly one cutset of the colouring. Assume that there is an edge joining u and v in G, which is not in T. The u, v-path in T must contain at least two edges, say $f_1, f_2, \ldots$. But then the edge uv is in the cutset determined by f_1 and in the cutset determined by f_2, which is a contradiction. □

The term *maximum* will be used in the sense of: with a maximal number of edges.

Proposition 14. *The hypergraph* $(V, \mathscr{E})$ *is a maximum Helly copair hypergraph iff* $\mathscr{E}$ *consists of the canonical copairs of a tree with vertex-set* V.

Corollary 15. *Let* $(V, \mathscr{E})$ *be a Helly copair hypergraph. then*

$$|\mathscr{E}| \leq 2(|V|-1).$$

Corollary 16 (E.C. Milner, see [4]). *Let* $(V, \mathscr{E})$ *be a Helly hypergraph. Then*

$$|\mathscr{E}| \leq 2^{|V|-1} + |V| - 1.$$

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