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COMMUNICATION

A COUNTEREXAMPLE TO A CONJECTURE OF EDMONDS AND GILES*

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We give a counterexample to the following conjecture of Jack Edmonds and Rick Giles [1]: given a directed graph $D = (V, A)$ and a subset C of the arrow set A, such that each directed cut of D contains at least k arrows in C, then C can be partitioned into k *coverings*, i.e., into sets $C_1, \ldots, C_k$ such that each C_i intersects each directed cut.

Here a set A' of arrows is a *directed cut* if A' is the set of arrows entering some non-empty set V' of vertices with $V' \neq V$, provided that no arrow leaves V'. Hence a set C' of arrows is a covering if and only if the contraction of the arrows in C' makes D strongly connected.

The counterexample, for the case $k = 2$, is given in Fig. 1, where the arrows in C are represented by heavy lines.

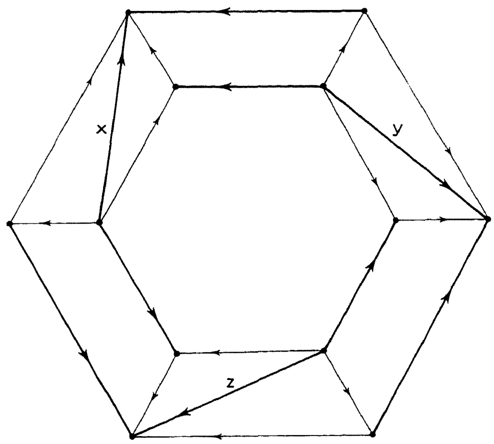

Fig. 1.

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To see that C intersects every directed cut at least twice, it suffices to show that for each arrow c in C the set $C\setminus\{c\}$ is a covering, which is easy since there are essentially only two types of arrows in C.

To show that C cannot be split into coverings C_1 and C_2, observe that each of these C_i must contain exactly one of the two arrows in C meeting any source or sink (indicated by black dots). Moreover, each C_i must contain at least one of the arrows labeled x, y, z, since the set of arrows from the inner hexagon to the outer hexagon forms a directed cut. Hence we may assume without loss of generality that C_1 contains the arrows x and y, but not z. But then C_1 does not intersect the directed cut of those arrows going from the right half of the figure to the left half.

Note that the counterexample is planar, and that therefore the, in the planar sense dual assertion (on directed cycles and their coverings) also cannot be true. An, in another sense dual assertion, where the roles of directed cut and covering are interchanged, was proved by C. L. Lucchesi and D. H. Younger [2].

References

[1] Jack Edmonds and Rick Giles, A min-max relation for submodular functions on graphs, in: P.L. Hammer, E.L. Johnson, and B.H. Korte, eds. "Studies in integer programming", Annals Discrete Math. 1 (1977) 185–204.

[2] C.L. Lucchesi and D.H. Younger, A minimax relation for directed graphs, J. London Math. Soc. (2) 17 (1978) 369–374.