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On a Minor-Monotone Graph Invariant

HEIN VAN DER HOLST

CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands

MONIQUE LAURENT*

LIENS, Ecole Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France

AND

ALEXANDER SCHRIJVER

CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands and Department of Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

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For any undirected graph $G=(V, E)$ let $\lambda(G)$ be the largest d for which there exists a d-dimensional subspace X of $\mathbb{R}^V$ with the property that for each nonzero $x \in X$, the positive support of x induces a nonempty connected subgraph of G. (Here the *positive support* of x is the set of vertices v with $x(v)>0$.) We show that $\lambda(G)$ is monotone under taking minors and clique sums. Moreover, we show that $\lambda(G) \leqslant 3$ if and only if G has no K_5- or V_8-minor; that is, if and only if G arises from planar graphs by taking clique sums and subgraphs.

1. INTRODUCTION

In this paper we study a graph invariant $\lambda(G) \in \mathbb{N}$, defined for any undirected graph $G=(V, E)$ as follows: $\lambda(G)$ is the largest d for which there exists a d-dimensional subspace X of $\mathbb{R}^V$ such that:

$$\text{for each nonzero } x \in X,\ \langle \operatorname{supp}_+(x) \rangle \text{ is a nonempty connected graph.} \tag{1}$$

Here $\operatorname{supp}_+(x)$ denotes the *positive support* of x; that is, the set $\{v \in V \mid x(v)>0\}$. Moreover, for any $U \subseteq V$, $\langle U \rangle$ denotes the subgraph of G *induced by* U; that is, the subgraph with vertex set U and edges all edges of G contained in U. In this paper, all graphs are assumed to be simple.

Clearly, (1) implies that also the *negative support* $\operatorname{supp}_-(x)$ of any nonzero $x \in X$ induces a nonempty connected subgraph of G (where $\operatorname{supp}_-(x) := \{v \in V \mid x(v)<0\}$).

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The function $\lambda(G)$ was motivated by the graph invariant $\mu(G)$ introduced by Colin de Verdière [3] (cf. [4]), although we do not know a relation between the two numbers. (It might be that $\lambda(G) \leqslant \mu(G)$ holds for each graph G.)

There is a direct equivalent characterization of $\lambda(G)$. Let $G = (V, E)$ be a graph and let $d \in \mathbb{N}$. Call a function ϕ: $V \to \mathbb{R}^d$ a *valid representation* if

$$\text{for each halfspace } H \text{ of } \mathbb{R}^d\text{, the set } \phi^{-1}(H) \text{ is nonempty and induces a connected subgraph of } G. \tag{2}$$

In this paper, a subset H of $\mathbb{R}^d$ is called a *halfspace* if $H = \{x \in \mathbb{R}^d \mid c^T x > 0\}$ for some nonzero $c \in R^d$. Note that if ϕ: $V \to \mathbb{R}^d$ is a valid representation, then the vectors $\phi(v)$ ($v \in V$) span $\mathbb{R}^d$ (since otherwise there would exist a halfspace H with $\phi^{-1}(H) = \emptyset$).

Now $\lambda(G)$ is equal to the largest d for which there is a valid representation ϕ: $V \to \mathbb{R}^d$. This is easy to see. Suppose X is a d-dimensional subspace of $\mathbb{R}^V$ satisfying (1). Let $x_1, ..., x_d$ form a basis of X. Define $\phi(v) := (x_1(v), ..., x_d(v))$ for each $v \in V$. This gives a valid representation.

Conversely, let ϕ: $V \to \mathbb{R}^d$ be a valid representation. Define for any $c \in \mathbb{R}^d$ the function $x_c \in \mathbb{R}^V$ by: $x_c(v) := c^T \phi(v)$ for $v \in V$. Then $X := \{x_c \mid c \in \mathbb{R}^d\}$ satisfies (1).

It is easy to show that the function $\lambda(G)$ is monotone under taking minors. (A *minor* of a graph arises by a series of deletions and contractions of edges and deletions of isolated vertices, suppressing multiple edges and loops.) That is:

THEOREM 1. *If G' is a minor of G then $\lambda(G') \leqslant \lambda(G)$.*

Proof. If G' arises from G by deleting an isolated vertex v_0, the inequality $\lambda(G') \leqslant \lambda(G)$ is easy: if ϕ: $V(G') \to \mathbb{R}^d$ is a valid representation for G' with $d = \lambda(G')$, then defining $\phi(v_0) := 0$ gives a valid representation for G.

So we may assume that $G' = (V', E')$ arises from $G = (V, E)$ by deleting or contracting one edge $e = uw$. Let ϕ': $V' \to \mathbb{R}^d$ be a valid representation for G' with $d = \lambda(G')$. If G' arises from G by deleting e, then $V = V'$, and ϕ' is also a valid representation for G. Hence $\lambda(G) \geqslant d = \lambda(G')$.

If G' arises from G by contracting e, let v_0 be the vertex of G' which arised by contracting e. Define $\phi(u) := \phi(w) := \phi'(v_0)$, and define $\phi(v) := \phi'(v)$ for all other vertices v of G. Then ϕ is a valid representation of G. ∎

Having Theorem 1, one can derive from the work of Robertson and Seymour [8] that for each fixed n there is a finite class $\mathscr{G}_n$ of graphs such that for any graph G: $\lambda(G) \geqslant n$ if and only if G contains a graph in $\mathscr{G}_n$ as a minor.

We observe that trivially $\lambda(G)=0$ if and only if G has exactly one vertex. So $\mathscr{G}_1$ consists only of the graph $\overline{K_2}$.

For the complete graph one has:

THEOREM 2. $\lambda(K_n)=n-1$.

Proof. Let V be the vertex set of K_n. To see $\lambda(K_n)<n$, suppose X is a subspace of $\mathbb{R}^V$ satisfying (1) of dimension n. Then $X=\mathbb{R}^V$, and hence the function $x(v)=-1$ $(v\in V)$ belongs to X, contradicting (1).

On the other hand, $\lambda(K_n)\geqslant n-1$, since the set X of functions $x\in\mathbb{R}^V$ with $\sum_{v\in V}x(v)=0$ satisfies (1). ▮

It is easy to see that if $n\geqslant 3$, each proper minor G' of K_n satisfies $\lambda(G)\leqslant n-2$. So if $n\geqslant 3$, K_n belongs to $\mathscr{G}_{n-1}$. (This is not true for $n=2$, since the graph G with two isolated vertices also satisfies $\lambda(G)=1$).

Theorem 2 gives that Hadwiger's conjecture implies that $\gamma(G)\leqslant\lambda(G)+1$, where $\gamma(G)$ denotes the (vertex-)chromatic number of G. So by the results of Appel and Haken [1], Appel, Haken, and Koch [2] (the four-colour theorem), and Robertson, Seymour, and Thomas [11], the inequality $\gamma(G)\leqslant\lambda(G)+1$ holds if $\lambda(G)\leqslant 4$.

It is easy to see that if $G'=(V',E')$ arises from $G=(V,E)$ by deleting a vertex u of G, then $\lambda(G)\leqslant\lambda(G')+1$. Indeed, let X be a d-dimensional subspace of $\mathbb{R}^V$ satisfying (1), where $d:=\lambda(G)$. Then $X':=\{x\in\mathbb{R}^V \mid x(u)=0\}$ has dimension at least $d-1$. Deleting coordinate u gives a subspace of $\mathbb{R}^{V'}$ (satisfying (1) with respect to G') of dimension at least $d-1=\lambda(G)-1$.

This implies that contracting or deleting any edge uw of G decreases $\lambda(G)$ by at most 1, as the new graph contains as a subgraph the graph G' obtained from G by deleting u.

Similarly to the chromatic number, also the function $\lambda(G)$ cannot be increased by "clique sums", as we shall see in Section 2. This directly gives that $\lambda(G)\leqslant 1$ if and only if G has no K_3-minor, that is, if and only if G is a forest; and that $\lambda(G)\leqslant 2$ if and only if G has no K_4-minor, that is, if and only if G is a series-parallel graph.

Let V_8 be the graph with vertices $v_1, ..., v_8$, where v_i and v_j are adjacent if and only if $|i-j|\in\{1,4,7\}$. In Section 3 we show that $\lambda(G)\leqslant 3$ if and only if G has no K_5- or V_8-minor; that is, if and only if G can be obtained from planar graphs by taking clique sums and subgraphs. The kernel of the proof here is to show that $\lambda(G)\leqslant 3$ for any planar graph G. Having this, a fundamental decomposition theorem of Wagner [12] then implies the full characterization.

Note that the inequality $\lambda(G)\geqslant 3$ is easy for 3-connected planar graphs: in that case G can be represented as the vertices and edges of a full-dimensional convex polytope in $\mathbb{R}^3$. We may assume that this polytope contains the origin in its interior. Then this embedding of V in $\mathbb{R}^3$ is a valid representation.

More generally, if G is the 1-skeleton of a d-dimensional convex polytope, then $\lambda(G) \geqslant d$. (The *1-skeleton* of a convex polytope P is the graph made by the vertices and edges of P.) However, in general one can have $\lambda(G) > d$, since Gale [5] showed that for each $n \geqslant 5$, K_n is the 1-skeleton of a 4-dimensional polytope.

In Section 4 we give a few observations concerning the class of graphs G with $\lambda(G) \leqslant 4$.

Finally in Section 5 we study a related graph invariant $\kappa(G)$ for connected graphs $G = (V, E)$. This is the largest d for which there exists a function ϕ: $V \to \mathbb{R}^d$ such that $\phi(V)$ affinely spans a full-dimensional affine space and such that for each affine halfspace H the set $\phi^{-1}(H)$ induces a connected subgraph of G (possibly empty). (Here an *affine halfspace* is a subset of $\mathbb{R}^d$ of the form $\{x \in \mathbb{R}^d \mid c^T x > \delta\}$ for some nonzero $c \in \mathbb{R}^d$ and some $\delta \in \mathbb{R}$.)

Again it is easy to show that $\kappa(G)$ is monotone under taking minors. Moreover, one has $\kappa(G) \leqslant \lambda(G)$. In Section 5 we show that $\kappa(G) \leqslant d$ if and only if G does not have a K_{d+2}-minor. So for this invariant the class of forbidden minors is exactly known for each d.

2. Clique Sums

In this section we show that the function $\lambda(G)$ does not increase by taking clique sums, and from this we derive characterizations of the classes of graphs G satisfying $\lambda(G) \leqslant 1$ and $\lambda(G) \leqslant 2$.

We first give an auxiliary result. For any finite subset Z of $\mathbb{R}^d$ let cone(Z) denote the smallest nonempty convex cone containing Z; that is, it is the intersection of all closed halfspaces $\{x \in \mathbb{R}^d \mid c^T x \geqslant 0\}$ containing Z. (Thus cone($\emptyset$) $= \{0\}$, while cone(Z) $= \mathbb{R}^d$ if there are no halfspaces containing Z.)

THEOREM 3. *Let ϕ: $V \to \mathbb{R}^d$ be a valid representation of a graph $G = (V, E)$ and let $U \subseteq V$. Assume that* cone($\phi(U)$) *is not a hyperplane in $\mathbb{R}^d$. Then there is at most one component K of $G - U$ for which the inclusion $\phi(K) \subseteq$* cone($\phi(U)$) *does not hold.*

Proof. We may assume that cone($\phi(U)$) $\neq \mathbb{R}^d$. Since cone($\phi(U)$) is not a hyperplane in $\mathbb{R}^d$, the set

$$C := \{c \in \mathbb{R}^d \mid c \neq 0,\ c^T \phi(v) \leqslant 0 \text{ for each } v \in U\}, \qquad (3)$$

is nonempty and topologically connected (as the polar cone $C \cup \{0\}$ of cone($\phi(U)$) is not a line). For $c \in \mathbb{R}^d$, let $H_c := \{x \in \mathbb{R}^d \mid c^T x > 0\}$. Let $K_1, ..., K_t$ be the components of $G - U$. Let C_i be the set of vectors $c \in C$ for which H_c intersects $\phi(K_i)$. So if $i \neq j$ then $C_i \cap C_j = \emptyset$, since if $c \in C$ then $\phi^{-1}(H_c)$ is connected and is disjoint from U. As $C_1 \cup \cdots \cup C_t = C$ and since each C_i is an open subset of C, it follows that $C_i = \emptyset$ for all but one i. Hence $\phi(K_i) \subseteq \text{cone}(\phi(U))$ for all but one i. ∎

Let $G = (V, E)$ be a graph and let V_1 and V_2 be subsets of V such that $K := V_1 \cap V_2$ is a clique in G and such that there is no edge connecting $V_1 \backslash K$ and $V_2 \backslash K$. Then G is called a *clique sum* of the graphs $G_1 := \langle V_1 \rangle$ and $G_2 := \langle V_2 \rangle$.

THEOREM 4. *If G is a clique sum of G_1 and G_2 then $\lambda(G) = \max\{\lambda(G_1), \lambda(G_2)\}$ (except if G_1 and G_2 each consist of one vertex and G of two nonadjacent vertices).*

Proof. Since G_1 and G_2 are subgraphs of G, we have $\lambda(G) \geqslant \max\{\lambda(G_1), \lambda(G_2)\}$. So it suffices to show that $\lambda(G) = \lambda(G_i)$ for some $i = 1, 2$. Assume that $\lambda(G) > \max\{\lambda(G_1), \lambda(G_2)\}$. Let $d := \lambda(G)$, $G = (V, E)$, and $G_i = (V_i, E_i)$ for $i = 1, 2$.

Let $\phi \colon V \to \mathbb{R}^d$ be a valid representation of G. As $d > \lambda(G_i)$, $\phi \mid V_i$ is not a valid representation of G_i for $i = 1$ and $i = 2$. Let $K := V_1 \cap V_2$ and $t := |K|$. We may assume that we have chosen the counterexample so that $|K|$ is as small as possible.

Then $\langle V_1 \backslash K \rangle$ has a component L such that each vertex in K is adjacent to at least one vertex in L. Otherwise G would be a repeated clique sum of subgraphs of G_1 and G_2 with common clique being smaller than K. In that case $\lambda(G) = \max\{\lambda(G_1), \lambda(G_2)\}$ would follow by the minimality of K.

So G_1 has a K_{t+1}-minor. So $\lambda(G_1) \geqslant t$, and hence $\lambda(G) > t = |K|$. Therefore, cone($\phi(K)$) is not a hyperplane in $\mathbb{R}^d$. (Here we use that it is not the case that $K = \emptyset$ and $d = 1$.) So by Theorem 3, we may assume that $\phi(V_1) \subseteq \text{cone}(\phi(K))$.

As $d > \lambda(G_2)$, there exists a halfspace H of $\mathbb{R}^d$ such that $\langle \phi^{-1}(H) \cap V_2 \rangle$ is empty or disconnected. If it is empty, then $\phi(v) \in H$ for some $v \in V_1 \backslash K$, contradicting the facts that $\phi(v) \in \text{cone}(\phi(K))$ and that $\phi(K) \cap H = \emptyset$. So it is disconnected. But then also $\phi^{-1}(H)$ would induce a disconnected subgraph of G, as K is a clique. This is a contradiction. ∎

This theorem directly implies characterizations of those graphs G satisfying $\lambda(G) \leqslant 1$ and $\lambda(G) \leqslant 2$.

COROLLARY 4a. *For any graph G, $\lambda(G) \leqslant 1$ if and only if G does not have a K_3-minor; that is, if and only if G is a forest.*

Proof. If $\lambda(G) \leqslant 1$ then G has no K_3-minor, as $\lambda(K_3) = 2$.

Conversely, if G is a forest, then G arises by taking clique sums and subgraphs from the graph K_2. As $\lambda(K_2) = 1$, Theorem 4 gives the corollary. ∎

COROLLARY 4b. *For any graph G, $\lambda(G) \leqslant 2$ if and only if G does not have any K_4-minor; that is, if and only if G is a series-parallel graph.*

Proof. If $\lambda(G) \leqslant 2$ then G has no K_4-minor, as $\lambda(K_4) = 3$.

Conversely, if G is a series-parallel graph, then G arises by taking clique sums and subgraphs from the graph K_3. As $\lambda(K_3) = 2$, Theorem 4 gives the corollary. ∎

3. GRAPHS SATISFYING $\lambda(G) \leqslant 3$

We next give a characterization of those graphs G satisfying $\lambda(G) \leqslant 3$. To this end we first show:

THEOREM 5. *If G is planar then $\lambda(G) \leqslant 3$.*

Proof. Suppose $G = (V, E)$ is a planar graph with $\lambda(G) \geqslant 4$. Choose G such that $|V|$ is minimal. Then G is 4-connected, since otherwise it would be a subgraph of a clique sum of two smaller planar graphs, contradicting by Theorem 4 the minimality of $|V|$. (In this paper, graph H is *smaller than* graph G if H has fewer vertices than G.)

Let $\phi \colon V \to \mathbb{R}^4$ be a valid representation. Let $X \subseteq \mathbb{R}^V$ be the 4-dimensional space corresponding to ϕ; that is, $X = \{x_c \mid c \in \mathbb{R}^4\}$, where $x_c(v) := c^T\phi(v)$ for $v \in V$.

By the minimality of $|V|$ we know that $\phi(v) \neq 0$ for each $v \in V$ (otherwise we can delete v). So we may assume that $\|\phi(v)\| = 1$ for each $v \in V$.

Assume that E has been chosen such that, fixing V and ϕ,

$$\sum_{e = uw \in E} (\angle(\phi(u), \phi(w)))^2 \tag{4}$$

is as small as possible. (Here $\angle(x, y)$ denotes the angle between vectors x and y.)

We assume that G is embedded on the 2-sphere S^2. For any face f of G, let V_f be the set of vertices incident with f.

We observe:

$$\text{for any face } f, \text{ if } u, w \in V_f \text{ then } \phi(u) \neq \phi(w). \tag{5}$$

Otherwise, we could identify u and w, contradicting the minimality of $|V|$.

Moreover:

$$\text{if } u \text{ and } w \text{ are adjacent, then } \phi(u) \neq \pm\phi(w). \tag{6}$$

Indeed, if $\phi(u) = \phi(w)$ we contradict (5). If $\phi(u) = -\phi(w)$, we can delete the edge uw without violating (2), contradicting the minimality of the sum (4).

Let L_f be the linear space generated by $\phi(V_f)$. For $i = 1, ..., 4$, let F_i denote the set of faces f with $\dim L_f = i$. Note that (6) implies that $F_1 = \emptyset$. We next have:

for any face f, if u, v, $w \in V_f$ and if u and v are adjacent, then $\phi(w) \notin \text{cone}(\{\phi(u), \phi(v)\})$. (7)

Otherwise we could remove edge uv and add edges uw and vw (if they do not already exist), thereby decreasing sum (4).

Next we show:

$$F_4 = \emptyset. \tag{8}$$

Suppose $f \in F_4$. Let $X_f := \{x | V_f \mid x \in X\}$. (Here $x | V_f$ denotes the restriction of x to V_f. As $\dim L_f = 4$ we have $\dim X_f = 4$. Let $X'_f := \{y \in X_f \mid \sum_{v \in V_f} y_v = 0\}$. Then X'_f has dimension at least 3 and for each nonzero $y \in X'_f$ one has $\text{supp}_+(y) \neq \emptyset$. So, as $\langle V_f \rangle$ is a series-parallel graph (indeed, a circuit), by Corollary 4b, X'_f contains a vector y with $\text{supp}_+(y)$ having at least two components on V_f. Let $x \in X$ satisfy $y = x | V_f$, and let $c \in \mathbb{R}^V$ be such that $x_c = x$ (that is, $x_v = c^T\phi(v)$ for each $v \in V$).

Let $U := \text{supp}_+(x)$. As $c^T\phi(v) > 0$ for each $v \in U$, $\text{cone}(\phi(U))$ is a pointed cone. Now for each $v \in V \backslash \text{supp}_+(x)$ we have $c^T\phi(v) \leqslant 0$. As $\phi(v) \neq 0$, we have that $\phi(v) \notin \text{cone}(\phi(U))$ for each $v \in V \backslash \text{supp}_+(x)$. Therefore, by Theorem 3, $G - \text{supp}_+(x)$ has only one component. As G is planar, this contradicts the facts that $\text{supp}_+(y)$ has at least two components on V_f and that $\langle \text{supp}_+(x) \rangle$ is connected. So we have proved (8).

Next we show:

Let f' and f'' be two faces having an edge in common, with $\dim L_{f'} = \dim L_{f''}$. Then $L_{f'} = L_{f''}$. (9)

If $\dim L_{f'} = 2$ the statement is trivial, so assume $\dim L_{f'} = 3$. Let $e = uw$ be the common edge of f' and f''. Suppose $L_{f'} \neq L_{f''}$. Then we can select $v' \in V_{f'}$ and $v'' \in V_{f''}$ such that $\phi(u)$, $\phi(w)$, $\phi(v')$, and $\phi(v'')$ form a basis of $\mathbb{R}^4$. Hence there exists a $c \in \mathbb{R}^4$ such that $c^T\phi(u) = 0$, $c^T\phi(w) = 0$, $c^T\phi(v') > 0$, and $c^T\phi(v'') > 0$. Hence for $x := x_c \in X$ one has that $x(u) = 0$, $x(w) = 0$, $x(v') > 0$, and $x(v'') > 0$. Let G' be the subgraph of G induced by $V \backslash \text{supp}_+(x)$. Since $\langle \text{supp}_+(x) \rangle$ and $\langle \text{supp}_-(x) \rangle$ are connected, we may assume that $\text{supp}_-(x)$ is not contained in the same component of $G' - e$ as u.

Now there exists a $y \in X$ such that $y(u) < 0$ and $y(w) = 0$. This follows from the fact that $\phi(u) \neq \pm\phi(w)$. Then for small enough $\varepsilon > 0$, the function $z := x + \varepsilon y$ has $\text{supp}_+(z) \supseteq \text{supp}_+(x)$ and $\text{supp}_-(z) \supseteq \text{supp}_-(x)$, while $u \in \text{supp}_-(z)$ and $w \notin \text{supp}_-(z)$. This contradicts the connectedness of $\langle \text{supp}_-(z) \rangle$. This proves (9).

This implies more strongly:

> Let f' and f'' be two faces having a vertex in common, with $\dim L_{f'} = \dim L_{f''} = 3$. Then $L_{f'} = L_{f''}$. (10)

Let v be a common vertex of f' and f''. If all faces f incident with v have $\dim L_f = 3$, the statement directly follows from (9). So we may assume that there is a face f incident with v with $\dim L_f = 2$. Let u and w be the two vertices in V_f incident with v, chosen in such a way that u, w, f', f'' occur in this order cyclically around v. Assume $L_{f'} \neq L_{f''}$. Then there exist vertices $v' \in V_{f'}$ and $v'' \in V_{f''}$ such that the vectors $\phi(u)$, $\phi(v)$, $\phi(v')$, and $\phi(v'')$ are linearly independent. Hence there is a $c \in \mathbb{R}^4$ such that $c^T\phi(u) > 0$, $c^T\phi(v) = 0$, $c^T\phi(v') > 0$, and $c^T\phi(v'') < 0$. Hence for $x := x_c \in X$ we have $x(u) > 0$, $x(v) = 0$, $x(v') > 0$, and $x(v'') < 0$.

We show that $x(w) < 0$, that is, $c^T\phi(w) < 0$. Assume $c^T\phi(w) \geqslant 0$. Since $\dim L_f = 2$, there exist λ and μ such that $\phi(w) = \lambda\phi(u) + \mu\phi(v)$. Hence $c^T\phi(w) = \lambda c^T\phi(u) + \mu c^T\phi(v) = \lambda c^T\phi(u)$. As $c^T\phi(u) > 0$ and $c^T\phi(w) \geqslant 0$ one has $\lambda \geqslant 0$. Now $\lambda \neq 0$ since otherwise v and w are linearly dependent, contradicting (6). So $\lambda > 0$. However, if $\mu \geqslant 0$ then $\phi(w) \in \text{cone}(\{\phi(u), \phi(v)\})$, contradicting (7); and if $\mu < 0$ then $\phi(u) \in \text{cone}(\{\phi(v), \phi(w)\})$, contradicting (7) again.

It follows that $x(w) < 0$. This however contradicts the connectedness of the graphs induced by $\text{supp}_+(x)$ and $\text{supp}_-(x)$. Thus we have (10).

Now $F_3 \neq \emptyset$, since otherwise $L_f = L_{f'}$ for any two faces f, f', implying that $\dim \phi(V) = 2$. Consider a component K of the space $S := \bigcup_{f \in F_3} \bar{f}$. ($\bar{f}$ denotes the topological closure of f.)

By (10), there is a 3-dimensional subspace L of $\mathbb{R}^4$ such that for each vertex v contained in K one has $\phi(v) \in L$. As $\phi(V)$ has dimension 4, there exists a vertex v_0 such that $\phi(v_0) \notin L$. As $v_0 \notin K$, there is a simple closed curve C not intersecting vertices of G, such that each face traversed by C belongs to F_2 and such that C separates K and v_0. So by (9) there exists a 2-dimensional subspace M of $\mathbb{R}^4$ such that $\phi(V_f) \subseteq M$ for each face f traversed by C.

We may assume that C traverses at least one face that has an edge in common with K. Hence $M \subset L$. Let U be the set of all vertices incident with faces traversed by C. As $\phi(v_0) \notin L$, $\phi(v_0) \notin M$. Moreover, since $\dim(\phi(U)) = 2$ and $\dim(\phi(K)) = 3$, there is a vertex $v_1 \in K$ with $\phi(v_1) \notin M$.

So $\phi(v_0) \notin \text{cone}(\phi(U))$ and $\phi(v_1) \notin \text{cone}(\phi(U))$. As v_0 and v_1 belong to different components of $G - U$, this contradicts Theorem 3. ∎

Having Theorem 5, Theorem 4 gives that $\lambda(G) \leqslant 3$ also holds for graphs G obtained from planar graphs by taking clique sums and subgraphs. This characterizes the graphs G with $\lambda(G) \leqslant 3$, as follows from the following two results.

THEOREM 6. *If G has no K_5- or V_8-minor, then G can be obtained by taking clique sums and subgraphs from planar graphs.*

Proof. Suppose G is not planar. If G is not 3-connected, then it is easy to see that G is a subgraph of a clique sum of two smaller graphs not having any K_5- or V_8-minor. So we may assume that G is 3-connected.

Then by Wagner's theorem [12], G can be obtained as a subgraph of a 3-clique sum of two smaller graphs G_1 and G_2 both with no K_5-minor. Let K be the clique.

It suffices to show that G_1 and G_2 have no V_8-minor. Suppose to the contrary that G_1, say, has a V_8-minor. As V_8 does not contain any triangle, the V_8-minor in G_1 does not need all three edges of K. So $G_1 - e$ has a V_8-minor for some edge e in K. However, $G_1 - e$ is a minor of G (by the 3-connectedness of G), contradicting the fact that G does not have a V_8-minor. ∎

THEOREM 7. $\lambda(V_8) = 4$.

Proof. The inequality $\lambda(V_8) \leqslant 4$ follows from the fact that for any vertex v of V_8, the graph $V_8 - v$ is planar. Hence $\lambda(V_8) \leqslant \lambda(V_8 - v) + 1 \leqslant 4$ by Theorem 5.

We next show $\lambda(V_8) \geqslant 4$. Again, represent V_8 as the graph G with vertex set $V = \{v_1, ..., v_8\}$, where v_i and v_j are adjacent if and only if $|i-j|$ is 1, 4 or 7. We define $\phi: V \to \mathbb{R}^4$ as follows:

$$\begin{aligned} &\phi(v_1) = (1, 1, 1, 3), \phi(v_2) = (1, 0, 0, 0), \phi(v_3) = -(1, 2, 3, 6), \\ &\phi(v_4) = (0, 1, 0, 0), \phi(v_5) = (1, 3, 3, 3), \phi(v_6) = (0, 0, 1, 0), \qquad (11) \\ &\phi(v_7) = -(1, 2, 1, 2), \phi(v_8) = (0, 0, 0, 1). \end{aligned}$$

We first show that for $i = 1, ..., 8$:

$$\phi(v_i) \text{ belongs to } \text{cone}(\{\phi(v_{i-1}), \phi(v_{i+1}), \phi(v_{i+4})\}) \qquad (12)$$

(taking indices mod 8). Indeed:

$$
\begin{aligned}
(1,1,1,3) &= 2(0,0,0,1)+\tfrac{2}{3}(1,0,0,0)+\tfrac{1}{3}(1,3,3,3),\\
(1,0,0,0) &= 2(1,1,1,3)-(1,2,3,6)+(0,0,1,0),\\
-(1,2,3,6) &= 2(1,0,0,0)+4(0,1,0,0)-3(1,2,1,2),\\
(0,1,0,0) &= -(1,2,3,6)+(1,3,3,3)+3(0,0,0,1),\\
(1,3,3,3) &= 2(0,1,0,0)+2(0,0,1,0)+(1,1,1,3),\\
(0,0,1,0) &= \tfrac{2}{3}(1,3,3,3)-(1,2,1,2)+\tfrac{1}{3}(1,0,0,0),\\
-(1,2,1,2) &= 2(0,0,1,0)+4(0,0,0,1)-(1,2,3,6),\\
(0,0,0,1) &= -(1,2,1,2)+(1,1,1,3)+(0,1,0,0).
\end{aligned}
\tag{13}
$$

To show that (2) holds, consider an open halfspace H of $\mathbb{R}^4$. Then $W := \phi^{-1}(H)$ is nonempty, since at least one of $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(-1,-2,-3,-6)$ belongs to H.

Assume that W induces a disconnected subgraph of V_8. Let $U := V \backslash W$, and let K_1 and K_2 be two of the components of $\langle W \rangle$. Then $|K_i| \geqslant 2$, since otherwise K_i would consist of one vertex, contradicting (12). So $|U| \leqslant 4$. Since V_8 is 3-connected, since each cut set of size 3 consists of the set of vertices adjacent with one vertex v_i, and since U separates K_1 and K_2, it follows that $|U| = 4$, and that the subgraph induced by W consists of two disjoint edges.

Now note that for each edge $e = v_i v_{i+1}$ of V_8, each other edge e' of V_8 disjoint from e contains at least one vertex that is adjacent to at least one vertex in e. It follows that $W = \{v_1, v_3, v_5, v_7\}$ or $W = \{v_2, v_4, v_6, v_8\}$.

First assume $W = \{v_1, v_3, v_5, v_7\}$. However, $\phi(v_1)$ belongs to cone$(\{\phi(v_2), \phi(v_4), \phi(v_6), \phi(v_8)\})$, contradicting the fact that $\phi(v_1) \in H$ while $\phi(v_i) \notin H$ for $i = 2, 4, 6, 8$.

Next assume $W = \{v_2, v_4, v_6, v_8\}$. Now $\phi(v_2)$ belongs to cone$(\{\phi(v_1), \phi(v_3), \phi(v_5), \phi(v_7)\})$ (as $(1,0,0,0) = 3(1,1,1,3) + \frac{3}{2}(-1,-2,-3,-6) + (1,3,3,3) + \frac{3}{2}(-1,-2,-1,-2)$), contradicting the fact that $\phi(v_2) \in H$ while $\phi(v_i) \notin H$ for $i = 1, 3, 5, 7$. ∎

Thus we have the following theorem:

THEOREM 8. *Let G be a graph. Then $\lambda(G) \leqslant 3$ if and only if G has no K_5- or V_8-minor; that is, if and only if G arises by taking clique sums and subgraphs from planar graphs.*

Proof. Directly from Theorems 2, 4, 5, 6, and 7. ∎

4. Graphs Satisfying $\lambda(G) \leqslant 4$

We do not know a characterization of the class of graphs G satisfying $\lambda(G) \leqslant 4$. By Theorem 2, $G = K_6$ is a forbidden minor for this class. Any other graph G in the "Petersen family" of graphs however satisfies $\lambda(G) \leqslant 4$. The *Petersen family* consists of all graphs that can be obtained from K_6 by a series of ΔY- and $Y\Delta$-transformations.

(A *ΔY-transformation* consists of choosing a triangle uvw in G, deleting the three edges of the triangle, adding a new vertex r to G, and adding the three new edges ru, rv, and rw. A *$Y\Delta$-transformation* is the converse operation, starting with a vertex of degree 3.)

The Petersen family consists of seven graphs, including the Petersen graph. Robertson, Seymour, and Thomas [9] showed that the Petersen family is exactly the family of forbidden minors for the class of graphs that are linklessly embeddable in $\mathbb{R}^3$.

We first observe:

Theorem 9. *Let G be in the Petersen family with $G \neq K_6$. Then G is obtainable by taking clique sums and subgraphs from K_5.*

Proof. Inspection of the Petersen family (cf. Robertson, Seymour, and Thomas [10]) shows that G is either a subgraph of the graph obtained from K_7 by deleting the edges of a triangle, and this graph is a clique sum of three K_5's, or G arises from such a subgraph by one or more ΔY-transformations, that is, it is a subgraph of a clique sum with K_4's. ∎

This immediately implies that $\lambda(G) \leqslant 4$ for each graph $G \neq K_6$ in the Petersen family. Moreover, it follows that each such graph is obtainable by taking clique sums and subgraphs from linklessly embeddable graphs.

Linklessly embeddable graphs are good candidates for graphs G satisfying $\lambda(G) \leqslant 4$—and hence, by Theorem 4, so are all graphs obtainable from linklessly embeddable graphs by clique sums and subgraphs. Note that the graph G obtained from V_8 by adding a new vertex adjacent to all vertices of V_8, cannot be obtained from linklessly embeddable graphs by taking clique sums and subgraphs; but G does not have a K_6-minor.

In fact, it follows from the next result that this graph satisfies $\lambda(G) = 5$. However it is not minor minimal for the property $\lambda(G) \geqslant 5$.

Let G_1 denote the graph obtained from V_8 by adding a new vertex v_0 adjacent to v_2, v_4, v_6, v_7, v_8. Similarly, let G_2 denote the graph where the new vertex v_0 is adjacent to v_2, v_3, v_5, v_7, v_8.

Theorem 10. $\lambda(G_1) = \lambda(G_2) = 5$.

Proof. It suffices to give a representation in $\mathbb{R}^5$ of the graphs G_1 and G_2. This representation can be constructed as an extension of the representation ϕ of V_8 given in the proof of Theorem 7. Namely, for $k = 1, 2$, set $\phi_k(v_0) = (0, 0, 0, 0, 1)$ and $\phi_k(v_i) = (\phi(v_i), x_i^k)$ for $i = 1, ..., 8$, where $x^1 = (0, 0, -3, 0, 0, 0, -1, 0)$ and $x^2 = (1, 0, -3, 0, 3, 0, -2, 0)$. Then, for all $1 \leqslant i \leqslant 8$, $\phi_k(v_i)$ belongs to the cone generated by $\phi_k(u)$ for the vertices u adjacent to v_i in G_k. Moreover, $\phi_k(v_1)$ belongs to cone($\{\phi_k(v_0), \phi_k(v_2), \phi_k(v_4), \phi_k(v_6), \phi_k(v_8)\}$) and $\phi_k(v_2)$ belongs to cone($\{\phi_k(v_0), \phi_k(v_1), \phi_k(v_3), \phi_k(v_5), \phi_k(v_7)\}$). This permits to show that ϕ_k is a representation of G_k in the same way as in the proof of Theorem 7. ∎

The graphs G_1 and G_2 are minor minimal for the class of graphs satisfying $\lambda(G) \leqslant 5$. Indeed, every minor G of G_1 or G_2 satisfies $\lambda(G) \leqslant 4$. (For this, note that every such G has a node whose deletion produces a graph which is planar or a subgraph of a clique-sum of planar graphs.)

5. A Related Graph Invariant

We finally study a graph invariant related to $\lambda(G)$, for which the set of forbidden minors can be precisely characterized. For any connected graph $G = (V, E)$, define $\kappa(G)$ to be the largest d for which there exists a function $\phi: V \to \mathbb{R}^d$ such that:

$$\begin{array}{ll} \text{(i)} & \phi(V) \text{ affinely spans a } d\text{-dimensional affine space;} \\ \text{(ii)} & \text{for each affine halfspace } H \text{ of } \mathbb{R}^d, \phi^{-1}(H) \text{ induces a} \\ & \text{connected subgraph of } G \text{ (possibly empty).} \end{array} \tag{14}$$

Note that such a function ϕ does not exist for disconnected graphs; so $\kappa(G)$ would be undefined if G is disconnected.

Observe that if G is the 1-skeleton of a full-dimensional polytope in $\mathbb{R}^d$, then $\kappa(G) \geqslant d$, as the polytope gives the embedding in $\mathbb{R}^d$.

By similar arguments as used in the proof of Theorem 1 one shows that if G' is a connected minor of G then $\kappa(G') \leqslant \kappa(G)$. So again for each d there is a finite set of forbidden minors for the class of graphs satisfying $\kappa(G) \leqslant d$. This class of graphs equals $\{K_{d+2}\}$, as is shown in the next theorem.

First observe that

$$\kappa(G) \leqslant \lambda(G) \tag{15}$$

holds for each connected graph G, since if $\phi: V \to \mathbb{R}^d$ satisfies (14), then we may assume that the origin belongs to the interior of the convex hull of $\phi(V)$. But then trivially ϕ is a valid representation for G.

Basic in the characterization is the following observation (Grünbaum and Motzkin [7], Grünbaum [6]):

THEOREM 11. *If G is the 1-skeleton of a d-dimensional polytope P, then G contains a K_{d+1}-minor.*

Proof. By induction on d, the case $d=0$ being trivial. If $d>0$, let F be a facet of P. By the induction hypothesis, the 1-skeleton of F can be contracted to K_d. Moreover, the vertices of P not on F induce a connected subgraph of G, and hence can be contracted to one vertex. This yields a contraction of G to K_{d+1}, as each vertex of F is adjacent to at least one vertex of P not on F. ∎

This gives:

THEOREM 12. *For each connected graph G and each d, $\kappa(G) \geqslant d$ if and only if G has a K_{d+1}-minor.*

Proof. Sufficiency. One has $\kappa(K_{d+1}) = d$ since the vertices of a simplex in $\mathbb{R}^d$ give a function ϕ satisfying (14). So if G has a K_{d+1}-minor, then $\kappa(G) \geqslant d$.

Necessity. Let $G=(V, E)$ be a connected graph and let $d := \kappa(G)$, such that for each proper connected minor G' one has $\kappa(G') < d$. By Theorem 11 it suffices to show that G is the 1-skeleton of a d-dimensional polytope.

Let $\phi: V \to \mathbb{R}^d$ satisfy (14). Let P be the convex hull of $\phi(V)$. So P is a d-dimensional polytope in $\mathbb{R}^d$. We show that G is the 1-skeleton of P.

First observe that for each vertex x of P, the set $\phi^{-1}(x)$ induces a connected subgraph of G, as it is equal to $\phi^{-1}(H)$ for some affine halfspace H of $\mathbb{R}^d$. Hence if $\phi^{-1}(x)$ consists of more than one vertex of G, then we can contract this subgraph to one vertex, contradicting the minimality of G.

Similarly, for each edge xy of P, the set $\phi^{-1}(xy)$ induces a connected subgraph of G. Hence it contains a path from $\phi^{-1}(x)$ to $\phi^{-1}(y)$.

As this is true for each edge, G contains a subdivision of the 1-skeleton of P as a subgraph. By the minimality of G this implies that G is equal to the 1-skeleton of P. ∎

So Hadwiger's conjecture is equivalent to $\gamma(G) \leqslant \kappa(G)+1$ for each connected graph G.

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