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Induced Circuits in Planar Graphs

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In McDiarmid, B. Reed, A. Schrijver, and B. Shepherd (Univ. of Waterloo Tech. Rep., 1990) a polynomial-time algorithm is given for the problem of finding a minimum cost circuit without chords (induced circuit) traversing two given vertices of a planar graph. The algorithm is based on the ellipsoid method. Here we give an $O(n^2)$ combinatorial algorithm to determine whether two nodes in a planar graph lie on an induced circuit. We also give a min-max relation for the problem of finding a maximum number of paths connecting two given vertices in a planar graph so that each pair of these paths forms an induced circuit.

Let $G=(V, E)$ be an undirected graph without loops, and let s, t be distinct nonadjacent vertices. We call two $s-t$ paths P', P'' *separate* if there is no edge joining an internal vertex of P' and an internal vertex of P''. We consider the problem of finding a maximum number of pairwise separate $s-t$ paths. For general graphs this is an NP-hard problem; this follows from Fellows [1] in which it is shown that it is NP-complete to decide if there exists an induced circuit containing s and t.

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We show that the problem can be solved in polynomial time for planar graphs. Moreover, we give a good characterization, based on the following concepts. Assume that G is embedded in the two-sphere S_2. Let C be a closed curve in S_2, not traversing s or t. The *winding number* $w(C)$ of C is, roughly speaking, the number of times that C separates s and t. More precisely, consider any curve P from s to t, crossing C only a finite number of times. Let λ be the number of times C crosses P from left to right, and let ρ be the number of times C crosses P from right to left (fixing some orientation of C, and orienting P from s to t). Then $w(C) = |\lambda - \rho|$. (This number can be seen to be independent of the choice of P.)

We call a closed curve C *alternate* if C does not traverse s or t and there exists a sequence

$$(F_0, w_1, F_1, w_2, F_2, ..., w_l, F_l) \tag{1}$$

(where $l \geqslant 0$) such that

(i) $F_0, ..., F_l$ are faces of G, with $F_0 = F_l$;

(ii) w_i is a vertex or edge of G $(i = 1, ..., m)$;

(iii) C traverses vertices, edges, and faces of G in the order (1).

Here, by definition, C *traverses* an edge e if C follows e from one end vertex to the other.

Let $l(C)$ denote the number l in (1). Now

THEOREM A. *Let $G = (V, E)$ be a graph embedded in the two-sphere S_2 and let s, t be distinct nonadjacent vertices.*

(i) *There exist k pairwise separate $s - t$ paths if and only if $l(C) \geqslant k \cdot w(C)$ for each alternate closed curve C.*

(ii) *The curves C in* (i) *can be restricted to those with $l(C) < |V|$ and*

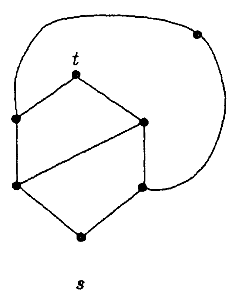


FIGURE 1

whose intersection with G is contained in a subgraph with maximum degree two (i.e., no three w_i's of (1) *are mutually incident).*

(iii) *There is an $O(|V|^2)$ algorithm which finds a maximum set of pairwise separate $s-t$ paths or an appropriate alternate curve.*

Before proving the theorem, let us give a small example of a graph where an alternate curve with winding number at least two must be used. Note that a proof of nonexistence of an induced circuit containing s, t, by means of an alternate curve with winding number one, is equivalent to there being a vertex cut set which is a clique of size at most two. It is easily verified that the graph of Fig. 1 does not contain any induced s, t circuit and yet neither does it have such a clique cut set. Other examples can be constructed for $k>2$ (see [4]).

Proof of Theorem A. I. *Necessity in* (i). Let $P_1, ..., P_k$ be pairwise separate $s-t$ paths, and let C be an alternate closed curve. Then C intersects each P_i at least $w(C)$ times. It is not hard to see that for each i, at least $w(C)$ of the w_j in (1) are incident to a vertex in P_i (defining two vertices v', v'' to be *incident* if $v'=v''$). Since distinct P_i and $P_{i'}$ are separate, there should be at least $k \cdot w(C)$ w_j's, i.e., $l(C) \geqslant k \cdot w(C)$.

II. *Algorithm.* We next describe an algorithm finding for any k, either k pairwise separate $s-t$ paths or an alternate closed curve C with $l(C)<k \cdot w(C)$.

First we introduce some notation and terminology. Any $s-t$ path will be oriented from s to t. Let O be an open disk whose boundary contains s and t. An edge e (of G) contained in the closure $\bar{O}$ of O, connecting two points on the boundary of O, is called a *belt* relative to O, if any curve from s to t contained in O, must cross e. Let P', P'' be two edge-disjoint $s-t$ paths, without crossings. Then $R(P', P'')$ denotes the region encircled by the closed curve $P' \cdot (P'')^{-1}$ in clockwise orientation. We call the pair (P', P'') *internally separate* if $R(P', P'')$ is an open disk not containing a belt. Note that even if (P', P'') is internally separate, P' and P'' can have a vertex $v \neq s, t$ in common. Note, moreover, that P' and P'' are separate if and only if both (P', P'') and (P'', P') are internally separate.

For $k=1$ the algorithm is trivial: either there exists an $s-t$ path, or there exists a closed curve C not intersecting G with $w(C)=1$ (implying $l(C)=0<1 \cdot w(C)$).

Suppose now that $k>1$, and that we have found $k-1$ pairwise separate $s-t$ paths $P_1, ..., P_{k-1}$. In the case that $k=2$ we assume that there exist two internally disjoint $s-t$ paths P, Q. If no such pair exists, then it is easy to find an appropriate alternate curve with the help of Menger's theorem. For $k=2$ we may furthermore choose P_1 to be P.

Without loss of generality the first edges of $P_1, \ldots, P_{k-1}$ occur in this order clockwise at s. Let P_k be a path "parallel" to the left of P_1. That is, we add to each edge traversed by P_1 a parallel edge at the left-hand side (with respect to the orientation of P_1), and P_k follows these new edges. (Note that adding parallel edges does not change our problem and in the case $k=2$ we have chosen P_1 so that (P_1, P_2) is internally separate.) Then the first edges of $P_1, \ldots, P_k$ occur in this order clockwise at s, and each pair (P_{i-1}, P_i) is internally separate $(i=2, \ldots, k)$.

Now for $n=k, k+1, k+2, \ldots$ we do the following. We have pairwise edge-disjoint $s-t$ paths $P_{n-k+1}, \ldots, P_n$, without crossings, so that the first edges of $P_{n-k+1}, \ldots, P_n$ occur in this order clockwise at s, and each pair (P_{i-1}, P_i) is internally separate $(i=n-k+2, \ldots, n)$.

If also the pair (P_n, P_{n-k+1}) is internally separate, then $P_{n-k+1}, \ldots, P_n$ are pairwise separate, and hence we have k pairwise separate $s-t$ paths as required. If (P_n, P_{n-k+1}) is not internally separate, let P_{n+1} be the path in $\bar{R}(P_{n-k+1}, P_{n-k+2})$ such that (P_n, P_{n+1}) is internally separate and such that $R(P_{n+1}, P_{n-k+2})$ is as large as possible. If P_{n+1} uses an edge in P_{n-k+2}, then as with P_k, we let P_{n+1} use a new parallel edge to the left. Then reset $n := n+1$, and repeat.

III. *Correctness and running time.* Suppose we do $|V|$ iterations and let $m := k + |V|$. Consider the surface U obtained in the following way. First, we cut out holes in S_2 at s and t. This transforms the sphere into a cylinder where the boundaries or holes at the ends are identified with s and t, respectively. Now make a cut from one end of the cyclinder to the other to obtain a rectangle. We then obtain U by taking an infinite number of copies of this rectangle and glueing them together to form an infinitely long strip whose two boundaries are again identified with the nodes s and t. What we have described is a special instance of what is called the *universal covering surface* of some fixed surface (see [5]). In our situation U is the universal covering surface of $S_2 \setminus \{s, t\}$.

Note that there are now many copies on U of each point of S_2. Denote by π the projection mapping $\pi: U \to S_2 \setminus \{s, t\}$ which maps a point of U back to its associated point on S_2. Thus π^{-1} maps each point of S_2 to an infinite set and so $\pi^{-1}[G \setminus \{s, t\}]$ is an infinite graph on U.

For any simple $s-t$ path P in G, a *lifting* of P is any copy of P in $\pi^{-1}(P)$. If Q is a lifting of P, we denote by Q^1 the lifting next to the right of Q. That is, Q^1 is to the right of Q (with respect to the lifted orientation of P from s to t), and there is no other lifting of P between Q and Q^1.

By our construction, there exist liftings $Q_1, \ldots, Q_m$ of $P_1, \ldots, P_m$, respectively, so that Q_n is to the right of Q_{n-1} (possibly touching) for $n=2, \ldots, m$, and such that $Q_{n-k+2}, \ldots, Q_n$ are contained in the region enclosed by Q_{n-k+1} and Q^1_{n-k+1} for $n=k, k+1, \ldots, m$.

For each $n=k+1, ..., m$, let V_n denote the set of internal vertices of Q_n which are not vertices of Q^1_{n-k}. Let V_k be the internal vertices of Q_k. Since we did keep shifting, each $V_n \neq \emptyset$. Note that for any $v \in V_n$, there is an internal vertex v' of Q_{n-1} and a curve C_v from v' to v such that either (a) C_v traverses a face which contains v' and v or (b) C_v traverses an edge $v'v''$ and then traverses a face containing v'' and v. For the curve C_v, we call v' its *starting* vertex and v its *end* vertex. In the case (b), the vertex v'' is a *middle* vertex. A vertex v is *active* in some iteration if $\pi(v)$ has a lifting which is an internal vertex of one of the current paths on this iteration. Otherwise it is called *inactive*. Otherwise it is called *inactive*. Note that if a vertex z is a middle vertex on iteration i, then it becomes inactive in iteration $i+1$ and will only become active again on an iteration when some lifting of $\pi(z)$ occurs as an end vertex.

We claim next that $v' \in V_{n-1}$. If this is not the case, then the internal vertex v' of Q^1_{n-k-1} is either a vertex of Q^1_{n-k} or is adjacent to an internal vertex of Q^1_{n-k}. This contradicts the fact that (P_{n-k-1}, P_{n-k}) is internally separate when $n-1>k$.

We now show how to construct an alternate curve. Choose $v_m \in V_m$ and for each $n=m-1, m-2, ..., k$, let v_n be the starting vertex of $C_{v_{n+1}}$. Since $m=k+|V|$, there exist n', n'' with $m \geqslant n'' > n' \geqslant k$ such that $\pi(v_{n''}) = \pi(v_{n'})$. Let D be the curve

$$C_{v_{n'+1}} \cdot C_{v_{n'+2}} \cdot \cdots \cdot C_{v_{n''}}, \tag{2}$$

and let C be the projection $\pi \circ D$ of D to S_2. So C is an alternate closed curve with $l(C)=n''-n'$. Next we show that $k \cdot w(C) > n''-n'$, proving sufficiency in (i).

For any lifting Q of any simple $s-t$ path P and any $i \geqslant 0$, let $Q^{(i)}$ be the ith lifting to the right of Q. That is, $Q^{(0)}=Q$ and $Q^{(i+1)}=(Q^{(i)})^1$.

Let $u := \lfloor (n''-n')/k \rfloor$. We must show $w(C)>u$. If $u=0$, then $w(C) > u=0$ since $v_{n''} \neq v_{n'}$. If $u>0$, then $v_{n''}$ is strictly to the right of $Q^1_{n''-k}$ and $Q^1_{n''-k}$ is to the right of $Q^{(u)}_{n'}$ (since $Q_{n''-k}$ is to the right of $Q^{(u-1)}_{n'}$, as $n''-k \geqslant n'+(u-1)k$). So $v_{n''}$ is strictly to the right of $Q^{(u)}_{n'}$. Therefore, $w(C)>u$.

We now analyze the running time of the algorithm and show that (ii) holds. For each $i=1, 2, ...$ the algorithm applies a shifting procedure. This procedure either returns i separate paths or an alternative curve. Each step of the shifting procedure is characterized by a path Q_{n-1} which is "too" close to a previous path. More precisely, there are internal vertices I of Q_{n-1} which are adjacent to or contained in the set of internal vertices of Q_{n-i}. Hence we must shift from the vertices of I. It is easily checked (and standard) that the time needed to shift from a single vertex is proportional to the size of the faces incident to the vertex. Thus since for a planar graph,

$$\sum_{F \text{ a face of } G} |V_F|$$

is $O(|V_G|)$, the shifting procedure sweeps across a single copy of G (i.e., rectangle of the universal covering surface) in linear time. The point is then how many rectangles must we shift across before we find an alternate curve?

Let K be the maximum number of separated paths. Then any alternate curve with length less than $|V|$ has winding number less than $|V|/K$ by necessity in (i), and so this is an upper bound on the number of rectangles. The time bound of $O(|V|^2)$ now follows. Note that after n iterations we may construct a curve C as in (2) which has length n say and suppose that C crosses r liftings of P_{i-1}. Each time it crosses such a lifting it touches a vertex v contained in a partial curve in which it is either a start, middle, or end vertex. But as we have seen, if two vertices v', v'' occur respectively as start and end vertices on C such that $\pi(v') = \pi(v'')$ and v' is to the left of v'', then the portion of C between v' and v'' determines an alternate curve. This implies that if we have not found an alternate curve after r crossings, then each internal vertex of P_{i-1} has at most one lifting which occurs as a start vertex of C and similarly at most one occurs as an end vertex. In fact it is straightforward to see that there is at most one lifting which occurs as a middle vertex; this can be seen by considering the curve C being built in the other direction. This now easily implies (ii).

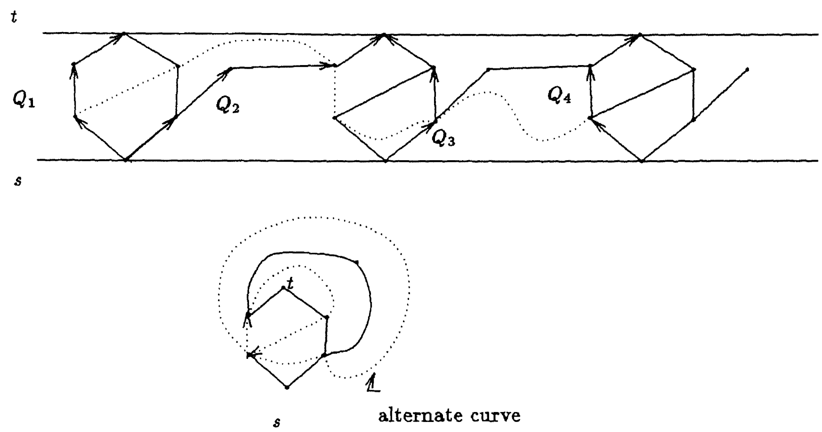
The algorithm given in the proof of the theorem can be extended for any fixed surface S and any fixed k, to find k pairwise separate $s-t$ paths in any graph embedded on S. It can also be shown [4] that the problem of finding a minimum-weight induced circuit traversing two given vertices s and t in a planar graph, is solvable in polynomial time. Moreover, finding a set of k pairwise separate $s-t$ paths of minimum total weight, is solvable in polynomial time for planar graphs.

The proof of the theorem can also be extended to solve a directed version of the problem for planar diagraphs $D = (V, A)$. A collection of $s-t$ dipaths is pairwise separate if there is no arc connecting internal vertices of distinct dipaths in the collection. We call a closed curve C (with clockwise orientation relative to s) *di-alternate* if C does not traverse s or t, and there exist a sequence

$$(C_0, a_1, C_1, a_2, C_2, ..., a_l, C_l) \tag{3}$$

such that

(i) a_i is an arc of $D\backslash\{s, t\}$ with endpoints s_i, t_i $(i = 1, ..., l)$;



alternate curve
projected onto S_2
FIGURE 2

(ii) C_i is a (noncrossing) curve of positive length from t_{i-1} to s_i and these are the only vertices of D that C_i intersects ($i = 1, ..., l$ and $C_0 = C_l$);

(iii) C traverses the paths and curves given in (3) in the described order;

(iv) each C_i may cross arcs only from right to left (relative to the orientation derived from C) and may not cross any a_i.

Informally, condition (iv) requires that any arcs crossed by C_i must be directed towards s.

THEOREM B. *For a plane digraph* $D = (V, A)$:

(i) *There exist* k *pairwise separate* $s - t$ *dipaths if and only if* $l(C) \geq k \cdot w(C)$ *for each di-alternate closed curve* C.

(ii) *A maximum number of pairwise separate* $s - t$ *dipaths can be found in polynomial time.*

(iii) *The curves* C *in* (i) *can be restricted to those with* $l(C) < |V|$.

We note that in the directed case we do not require the paths in the collection to be induced, i.e., they may have backwards arcs. In fact, Fellows *et al.* [2] have shown that the problem of determining whether there is a single induced $s - t$ dipath in a planar digraph is NP-complete.

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References

1. M. R. Fellows, The Robertson–Seymour theorems: A survey of applications, *Contemporary Math.* **89** (1989).
2. M. Fellows, J. Kratochvil, M. Mittendorf, and F. Pfeifer, manuscript, 1991 (unpublished).
3. C. McDiarmid, B. Reed, A. Schrijver, and B. Shepherd, Centrum voor Wiskunde en Informatica Technical Report, 1991 (unpublished).
4. C. McDiarmid, B. Reed, A. Schrijver, and B. Shepherd, University of Waterloo Technical Report, 1990 (unpublished).
5. W. S. Massey, *Algebraic Topology: An Introduction*, Gradute Texts in Math., Vol. 56 (Springer-Verlag, New York/Berlin, 1967).