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Edge-Disjoint Homotopic Paths in a Planar Graph with One Hole

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We prove the following theorem, conjectured by K. Mehlhorn: Let $G=(V, E)$ be a planar graph, embedded in the plane $\mathbb{C}$. Let O denote the interior of the unbounded face, and let I be the interior of some fixed bounded face. Let $C_1, ..., C_k$ be curves in $\mathbb{C}\setminus(I\cup O)$, with end points in $V\cap \mathrm{bd}(I\cup O)$, so that for each vertex v of G the degree of v in G has the same parity as the number of curves C_i beginning or ending in v (counting a curve beginning and ending in v for two). Then there exist pairwise edge-disjoint paths $P_1, ..., P_k$ in G so that P_i is homotopic to C_i in the space $\mathbb{C}\setminus(I\cup O)$ for $i=1, ..., k$, if and only if for each dual walk Q from $\{I, O\}$ to $\{I, O\}$ the number of edges in Q is not smaller than the number of times Q necessarily intersects the curves C_i. The theorem generalizes a theorem of Okamura and Seymour. We demonstrate how a polynomial-time algorithm finding the paths can be derived. © 1990 Academic Press, Inc.

1. THE THEOREM

We prove the following theorem, conjectured by K. Mehlhorn in relation to the automatic design of integrated circuits (cf. [1]).

THEOREM. *Let $G=(V, E)$ be a planar graph, embedded in the plane $\mathbb{C}$. Let O denote the interior of the unbounded face. Let I be the interior of some fixed bounded face. Let $C_1, ..., C_k$ be curves in $\mathbb{C}\setminus(I\cup O)$, with end points in $V\setminus \mathrm{bd}(I\cup O)$, so that for each vertex v of G*

$$\deg_G(v)+\deg_{C_1, ..., C_k}(v) \text{ is even} \qquad (\text{“parity condition”}). \tag{1}$$

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Then there exist pairwise edge-disjoint paths $P_1, ..., P_k$ *in* G *so that* $P_i \sim C_i$ *in* $\mathbb{C} \setminus (I \cup O)$ $(i = 1, ..., k)$ *if and only if for each dual walk* Q *from* $\{I, O\}$ *to* $\{I, O\}$ *we have*

$$e(Q) \geqslant \sum_{i=1}^{k} \operatorname{cr}(Q, C_i) \qquad (\text{"}\textit{cut condition}\text{"}). \tag{2}$$

We here use the following terminology and conventions. A graph may have multiple edges. We denote

$$\begin{aligned} \operatorname{bd}(F) &:= \text{boundary of } F; \\ \deg_G(v) &:= \text{the degree of } v \text{ in } G; \\ \deg_{C_1, ..., C_k}(v) &:= \textstyle\sum_{i=1}^{k} \rho_i, \text{ where} \\ &\qquad \rho_i := \text{the number of end points of } C_i \\ &\qquad\qquad \text{equal to } v \text{ (so } \rho_i \in \{0, 1, 2\}). \end{aligned} \tag{3}$$

By a *path* we mean a path not containing the same edge twice (it may contain vertices more than once). Each of the curves C_i is allowed to have self-intersections. By $P \sim C$ *in* $\mathbb{C} \setminus (I \cup O)$, or just $P \sim C$, we mean that P and C are *homotopic* in the space $\mathbb{C} \setminus (I \cup O)$ (i.e., there exists a continuous function F: $[0, 1] \times [0, 1] \to \mathbb{C} \setminus (I \cup O)$ so that $F(0, \cdot)$ follows P, $F(1, \cdot)$ follows C, $F(\cdot, 0)$ is constant, and $F(\cdot, 1)$ is constant; it implies that P and C have the same beginning point and have the same end point).

A *dual walk* (*from* $\{I, O\}$ *to* $\{I, O\}$) means a walk from one of I, O to one of I, O in the dual graph

$$Q = (F_0, e_1, F_1, e_2, F_2, ..., F_{t-1}, e_t, F_t), \tag{4}$$

where $F_0, ..., F_t$ are faces, where e_j is an edge separating F_{j-1} and F_j $(j = 1, ..., t)$, and *where* F_0 *and* F_t *are the only faces among* $F_0, ..., F_t$ *which belong to* $\{I, O\}$. (The edges $e_1, ..., e_t$ and faces $F_0, ..., F_t$ need not be distinct.) We denote by $e(Q)$ the number of edges in Q, counting multiplicities (so $e(Q) = t$ in (4)). Moreover,

$$\operatorname{cr}(Q, C) := \min\{|\tilde{Q} \cap \tilde{C}| \mid \tilde{Q} \sim Q, \tilde{C} \sim C\}. \tag{5}$$

Here we identify a dual walk in the obvious way with a curve in $\mathbb{C} \setminus (I \cup O)$, which is unique up to homotopy and up to the choice of the beginning and end points on the first and last edges of Q. (In $|\tilde{Q} \cap \tilde{C}|$ we count multiplicities.)

Note that I and O play a symmetric role: if the configuration is turned inside out, I and O can be interchanged.

It is not difficult to see that our theorem implies the following theorem due to Okamura and Seymour [4]:

Okamura–Seymour Theorem. *Let $G=(V,E)$ be a planar graph, embedded in the plane $\mathbb{C}$. Let O denote the unbounded face. Let $r_1, s_1, ..., r_k, s_k$ be vertices on the boundary of O, so that for each vertex v of G*

$$\deg_G(v)+\rho(\{v\}) \text{ is even.} \tag{6}$$

Then there exist pairwise edge-disjoint paths $P_1, ..., P_k$ in G so that P_i connects r_i and s_i $(i=1, ..., k)$, if and only if for each $U \subseteq V$

$$d_G(U) \geqslant \rho(U). \tag{7}$$

Here we denote

$$\begin{aligned} d_G(U) &:= \text{number of edges of } G \text{ having exactly one end point in } U,\\ \rho(U) &:= \text{number of } i=1, ..., k \text{ with exactly one of } r_i, s_i \text{ in } U. \end{aligned} \tag{8}$$

The Okamura–Seymour theorem can be derived from our theorem by replacing each pair r_i, s_i by an arbitrary curve in $\mathbb{C}\backslash O$ connecting r_i and s_i, and by adding, somewhere in O, a new vertex with a loop, whose interior we call I. We should remark however that our proof below makes use of the Okamura–Seymour theorem.

2. Proof of the Theorem

Since necessity of the cut condition (2) is trivial, we only show sufficiency. Suppose the implication does not hold. Then there exists a counterexample $G=(V,E)$, I, $C_1, ..., C_k$ such that each of the curves C_i is homotopically nontrivial and such that $2|E|-k$ is as small as possible. (Since $2|E|-k=|E|+\frac{1}{2}\sum_{v\in V}(\deg_G(v)-\deg_{C_1,...,C_k}(v)) \geqslant |E| \geqslant 0$, such a smallest counterexample exists.)

We may assume that G is embedded in the complex plane $\mathbb{C}$ so that 0 belongs to I. We identify G with its topological image.

For convenience we first show:

Claim 1. No edge of G is incident at both of its sides to face O. Similarly for face I.

Proof of Claim 1. Suppose to the contrary that edge e is incident at both sides to O. Then for the dual walk $Q=(O,e,O)$ we have $e(Q)=1$. Hence, by the parity and cut conditions, there is exactly one C_i with $\operatorname{cr}(Q,C_i)\neq 0$, for which C_i we have $\operatorname{cr}(Q,C_i)=1$. Without loss of generality, $i=1$. So C_1 passes edge e, and hence it can be decomposed as C_1', e, C_1'' for certain curves C_1' and C_1''. Then after deleting edge e and replacing C_1 by C_1' and C_1'' we are again in a situation where the parity and cut conditions hold. As in the new situation the number $2|E|-k$ is smaller, there exist pairwise edge-disjoint paths $P_1' \sim C_1'$, $P_1'' \sim C_1''$, $P_2 \sim C_2, ..., P_k \sim C_k$. Defining $P_1 := P_1', e, P_1''$ we obtain a packing of paths as required.

Similarly for face I. ∎

Claim 1 implies that we may assume that both I and $\mathbb{C} \setminus O$ are convex subsets of $\mathbb{C}$. We next consider the "projection function" $\tau: \mathbb{C} \to \mathbb{C} \setminus \{0\}$ given by $\tau(z) := e^{2\pi z}$. So for each $j \in \mathbb{Z}$, the restriction $\tau | \{z \in \mathbb{C} \mid j \leqslant \operatorname{Im} z < j+1\}$ is a bijection onto $\mathbb{C} \setminus \{0\}$. Then, as is well-known, for any curve C: $[0, 1] \to \mathbb{C} \setminus \{0\}$ and any $p \in \tau^{-1}(C(0))$, there exists a unique curve C': $[0, 1] \to \mathbb{C}$ such that $C'(0) = p$ and $C = \tau \circ C'$ (Lemma 3.1 in Chapter 5 of Massey [2]). The curve C' is called a *lifting* of C to $\mathbb{C}$.

For any $i = 1, ..., k$ and $j \in \mathbb{Z}$, let C_i^j be the lifting of C_i to $\mathbb{C}$ with $j \leqslant \operatorname{Im}(C_i^j(0)) < j+1$. Let $r_i^j := C_i^j(0)$ and $s_i^j := C_i^j(1)$. Consider the (infinite) graph $G' := \tau^{-1}[G]$, with vertex set $V' := \tau^{-1}[V]$. Then $O' := \tau^{-1}[O]$ and $I' := \tau^{-1}[I]$ are the two unbounded faces of G'. Now the cut condition (2) for G is equivalent to the "cut condition" for G'

$$\text{for each dual walk } Q \text{ in } G' \text{ from } \{I', 0'\} \text{ to } \{I', 0'\} \text{ we have } e(Q) \geqslant \rho(Q), \tag{9}$$

where

$$\rho(Q) := \text{the number of pairs } (i, j) \text{ such that } Q \text{ separates } r_i^j \text{ and } s_i^j \ (i = 1, ..., k; j \in \mathbb{Z}). \tag{10}$$

Here Q *separates* vertices v' and v'' if v' and v'' belong to different components when we delete from G' all edges occurring in Q.

We now first derive from the Okamura–Seymour theorem that (9) implies

Claim 2. There exist pairwise edge-disjoint paths P_i^j in G' such that P_i^j connects r_i^j and s_i^j $(i = 1, ..., k; j \in \mathbb{Z})$.

Proof of Claim 2. Let Q be a dual walk in G' from I' to O' with $e(Q) - \rho(Q)$ as small as possible. Clearly, Q is a simple walk (i.e., no face or edge occurs more than once in Q). For $j \in \mathbb{Z}$, let $Q_j := Q + j\mathbf{i}$ be the "copy" of Q obtained by replacing any edge e and face F in Q by their translate $e + j\mathbf{i}$ and $F + j\mathbf{i}$ ($\mathbf{i}$ denotes the complex number). Let $N := e(Q)!$. Let

$$\begin{aligned} D &:= \sup\{\operatorname{Im} p \mid p \text{ belongs to some edge in } Q\} + N\mathbf{i}, \\ C &:= \inf\{\operatorname{Im} p \mid p \text{ belongs to some edge in } Q\}. \end{aligned} \tag{11}$$

Let $V'' := \{v \in V' \mid C \leqslant \operatorname{Im} v \leqslant D\}$. Contract all vertices v of G' with $\operatorname{Im} v > D$ to one new vertex w. Contract all vertices v of G' with $\operatorname{Im} v < C$ to one new vertex u. This gives the finite graph G'' embedded in $\mathbb{C}$. Let for any vertex v of G':

$$\begin{aligned} \bar{v} &:= v && \text{if } C \leqslant \operatorname{Im} v \leqslant D, \\ &:= w && \text{if } \operatorname{Im} v > D, \\ &:= u && \text{if } \operatorname{Im} v < C. \end{aligned} \tag{12}$$

Let $K := \{(i, j) \mid i = 1, ..., k;\ j \in \mathbb{Z};\ \bar{r}_i^j \neq \bar{s}_i^j\}$. Then K is finite. Let $r := e(Q) - \rho(Q)$. Then for each $U \subseteq V'' \cup \{u, w\}$

$$\begin{aligned} &\text{(i)} \quad d_{G''}(U) \geqslant \rho(U), && \text{if } U \text{ does not separate } u \text{ and } w, \\ &\text{(ii)} \quad d_{G''}(U) \geqslant \rho(U) + r, && \text{if } U \text{ separates } u \text{ and } w, \end{aligned} \tag{13}$$

where

$$\begin{aligned} d_{G''}(U) &:= \text{number of edges of } G'' \text{ with exactly one end point in } U, \\ \rho(U) &:= \text{number of pairs } (i, j) \text{ in } K \text{ so that } U \text{ separates } \bar{r}_i^j \text{ and } \bar{s}_i^j. \end{aligned} \tag{14}$$

Here U *separates* v' and v'' if U contains exactly one of v' and v''.

Since by the parity condition (1),

$$\begin{aligned} d_{G''}(v) &\equiv \rho(\{v\}) && \pmod 2 \quad \text{for each } v \in V'', \\ d_{G''}(u) &\equiv \rho(\{u\}) + r && \pmod 2, \\ d_{G''}(w) &\equiv \rho(\{w\}) + r && \pmod 2, \end{aligned} \tag{15}$$

the Okamura–Seymour theorem gives us that in G'' there exist pairwise edge-disjoint paths $\bar{P}_i^j$ (for $(i, j) \in K$) and R_i (for $i = 1, ..., r$) such that $\bar{P}_i^j$ connects $\bar{r}_i^j$ and $\bar{s}_i^j$ and each R_i connects u and w.

Since $e(Q) = \rho(Q) + r$, and similarly $e(Q_h) = \rho(Q_h) + r$ for each $h \in \mathbb{Z}$, each edge in Q_h is contained in a unique path $\bar{P}_i^{j+h}$, with $(i, j) \in L$, or R_i, for $h = 0, ..., N$. Here

$$L := \{(i, j) \in K \mid Q \text{ separates } \bar{r}_i^j \text{ and } \bar{s}_i^j\}. \tag{16}$$

Since $e(Q) = \rho(Q) + r = |L| + r$, for each $h = 0, ..., N$ there exists a bijection

$$F_h \colon E(Q) \to L \cup \{1, ..., r\} \tag{17}$$

(where $E(Q)$ denotes the set of edges in Q), given by

$$\begin{aligned} F_h(e) &:= (i, j) \in L \text{ if } e + h\mathbf{i} \text{ belongs to } \bar{P}_i^{j+h}, \\ &:= i \in \{1, ..., r\} \text{ if } e + h\mathbf{i} \text{ belongs to } R_i. \end{aligned} \tag{18}$$

Therefore there exist two different $h, h' \in \{0, ..., N\}$ such that $F_h = F_{h'}$.

Having this, we can "glue" together copies of the part of G'' in between Q_h and $Q_{h'}$ to obtain G'. The packing of the paths $\bar{P}_i^j$ as it is between Q_h and $Q_{h'}$ extends to a packing of paths P_i^j as required. More precisely, path P_i^j is the path in G' consisting of those edges e for which the unique edge $e + k(h - h')\mathbf{i}$ satisfying: $e + k(h - h')\mathbf{i}$ is between Q_h and $Q_{h'}$ and $k \in \mathbb{Z}$, belongs to path $\bar{P}_i^{j+k(h-h')}$. ∎

What in fact is equivalent to what must be proved is that there exists a *periodic* packing of paths P_i^j of period 1; that is, one for which P_i^{j+1} arises from P_i^j by the translation $z \to z + \mathbf{i}$.

Since I is a convex subset of $\mathbb{C}$, we know that the boundary of I' is linearly ordered by $\operatorname{Im} p < \operatorname{Im} q$ for $p, q \in \operatorname{bd}(I')$. Similarly for $\operatorname{bd}(O')$. We next claim:

Claim 3. We may assume that $r_1^0 \in \operatorname{bd}(O')$, that P_1^0 contains a vertex v on $\operatorname{bd}(O')$ with $\operatorname{Im}(r_1^0) < \operatorname{Im}(v)$, and that, if $s_1^0 \in \operatorname{bd}(O')$, then $\operatorname{Im}(v) < \operatorname{Im}(s_1^0)$.

Proof of Claim 3. First note that for no i, j are the vertices r_i^j and s_i^j adjacent. Otherwise the curve C_i would be homotopic to one of the edges of G, and then deleting this edge and deleting this curve C_i would yield a counterexample with smaller value of $2|E| - k$.

If no P_i^j in the packing found in Claim 2 contains any edge e on $\operatorname{bd}(O')$, we can delete in G all edges on $\operatorname{bd}(O)$ without violating the cut condition (2) (as deleting all edges on $\operatorname{bd}(O')$ from G' does not violate condition (9), since a packing of paths P_i^j exists also in the remaining graph). This would yield again a counterexample with smaller value of $2|E| - k$. So at least one of the edges on $\operatorname{bd}(O')$ is used by one of the P_i^j. Similarly for $\operatorname{bd}(I')$.

Suppose now there is a path P_i^j having exactly one of its end points on $\operatorname{bd}(O')$ and containing a vertex v on $\operatorname{bd}(O')$ with $v \neq r_i^j$ and $v \neq s_i^j$. Then by renaming $(i \to 1, j \to 0)$ and possibly reorienting, we can arrive at the situation described in the claim.

So we may assume that no path P_i^j with exactly one of its end points on $\operatorname{bd}(O')$ has another point on $\operatorname{bd}(O')$. Similarly for $\operatorname{bd}(I')$. Since we know that at least one edge on $\operatorname{bd}(O')$ is used by some P_i^j, there must exist P_i^j with end points both on $\operatorname{bd}(O')$ or both on $\operatorname{bd}(I')$. Without loss of generality, let there exist paths P_i^j with both end points on $\operatorname{bd}(O')$. By interchanging r_i^j and s_i^j if necessary, we may assume $\operatorname{Im}(r_i^j) < \operatorname{Im}(s_i^j)$ for each of these paths. Choose i', j', i'', j'' such that $r_{i'}^{j'}, s_{i'}^{j'}, r_{i''}^{j''}, s_{i''}^{j''}$ all belong to $\mathrm{bd}(O')$, such that $\mathrm{Im}(r_{i'}^{j'}) < \mathrm{Im}(s_{i''}^{j''})$, and such that $\mathrm{Im}(s_{i''}^{j''}) - \mathrm{Im}(r_{i'}^{j'})$ is as small as possible (possibly $i' = i''$, $j' = j''$). Consider the edge e on $\mathrm{bd}(O')$ adjacent to $r_{i'}^{j'}$ in between $r_{i'}^{j'}$ and $s_{i''}^{j''}$ (i.e., $\mathrm{Im}(r_{i'}^{j'}) < \mathrm{Im}(p) < \mathrm{Im}(s_{i''}^{j''})$ for all points p on e).

If e is not used by any path P_i^j, then (by the parity condition (1)) e is contained in a circuit or in an infinite path consisting of edges all not used by any P_i^j. Then we can insert this circuit into $P_{i'}^{j'}$ or we can replace part of $P_{i'}^{j'}$ by part of this infinite path, so as to obtain that $P_{i'}^{j'}$ contains e. Hence we satisfy the claim (after renaming $i' \to 1$, $j' \to 0$).

So we may assume that e is used by some path P_i^j. This path cannot have exactly one of its end points on $\mathrm{bd}(O')$ (by the above), and hence r_i^j, $s_i^j \in \mathrm{bd}(O')$ or $r_i^j, s_i^j \in \mathrm{bd}(I')$. Write $P_i^j = (\alpha, r_{i'}^{j'}, e, \beta)$ for strings α, β. If β intersects $P_{i'}^{j'}$, say in vertex w, we can exchange the parts $r_{i'}^{j'}, \ldots, w$ of P_i^j and $P_{i'}^{j'}$, thus satisfying the claim (after renaming $i' \to 1$, $j' \to 0$).

If β does not intersect $P_{i'}^{j'}$, then $r_i^j, s_i^j \in \mathrm{bd}(O')$ and for the end point p of β we have $\mathrm{Im}(p) > \mathrm{Im}(r_{i'}^{j'})$. Hence $\mathrm{Im}(s_i^j) \geqslant \mathrm{Im}(p) > \mathrm{Im}(r_{i'}^{j'})$, and therefore $\mathrm{Im}(s_i^j) \geqslant \mathrm{Im}(s_{i''}^{j''})$ (by the minimality of $\mathrm{Im}(s_{i''}^{j''}) - \mathrm{Im}(r_{i'}^{j'})$). If also $\mathrm{Im}(r_i^j) \geqslant \mathrm{Im}(s_{i''}^{j''})$ then P_i^j and $P_{i''}^{j''}$ intersect each other at least twice, and we can exchange parts of P_i^j and $P_{i''}^{j''}$ so as to obtain that $P_{i''}^{j''}$ contains edge e, thus satisfying the claim (after renaming $i'' \to 1$, $j'' \to 0$).

If $\mathrm{Im}(r_i^j) < \mathrm{Im}(s_{i''}^{j''})$ then $\mathrm{Im}(r_i^j) \leqslant \mathrm{Im}(r_{i'}^{j'})$ (by the minimality of $\mathrm{Im}(s_{i''}^{j''}) - \mathrm{Im}(r_{i'}^{j'})$), and hence the claim is satisfied (after renaming $i \to 1$, $j \to 0$). ∎

We now distinguish three cases.

Case 1. $s_1^0 \in \mathrm{bd}(O')$ and

$$\mathrm{Im}(v) \leqslant \mathrm{Im}(r_1^0) + 1 \qquad \text{or} \qquad \mathrm{Im}(\mathrm{v}) \geqslant \mathrm{Im}(s_1^0) - 1. \tag{19}$$

(See Fig. 1.)

Define

$$\begin{aligned} \bar{r}_0^j &:= r_1^j, & \bar{s}_0^j &:= v + j\mathbf{i} & &(j \in \mathbb{Z}),\\ \bar{r}_1^j &:= v + j\mathbf{i}, & \bar{s}_1^j &:= s_1^j & &(j \in \mathbb{Z}),\\ \bar{r}_i^j &:= r_i^j, & \bar{s}_i^j &:= s_i^j & &(i = 2, \ldots, k;\ j \in \mathbb{Z}). \end{aligned} \tag{20}$$

We claim that we have the analogue of (9) for the new situation. That is,

$$e(Q) \geqslant \bar{\rho}(Q) \qquad \text{for each dual walk in } G' \text{ from } \{I', O'\} \text{ to } \{I', O'\}, \tag{21}$$

where

$$\bar{\rho}(Q) := \text{numer of pairs } (i, j) \text{ such that } Q \text{ separates } \bar{r}_i^j \text{ and } \bar{s}_i^j \ (\mathrm{i} = 0, \ldots, k;\ j \in \mathbb{Z}). \tag{22}$$

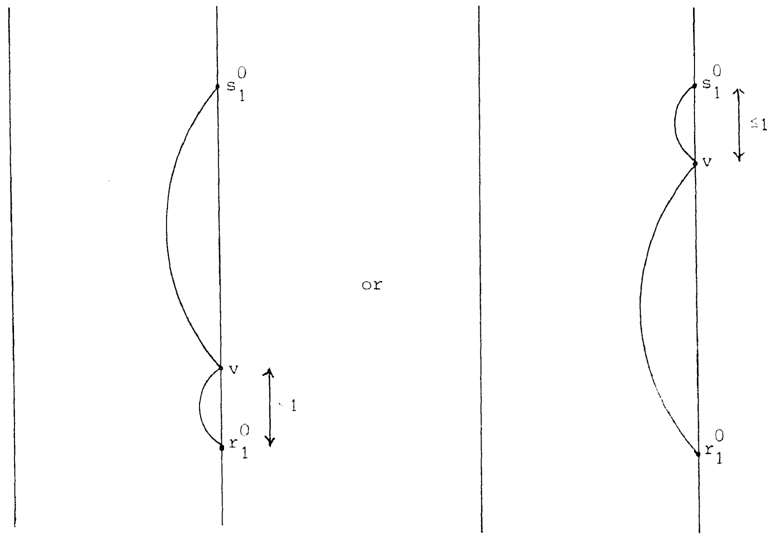

FIGURE 1

Note that

$$\bar{\rho}(Q) = \rho(Q) + 2 \cdot (\text{number of } j \in \mathbb{Z} \text{ such that } Q \text{ separates both } \bar{r}_0^j \text{ and } \bar{s}_0^j, \text{ and } \bar{r}_1^j \text{ and } \bar{s}_1^j) \tag{23}$$

(as Q separates exactly one of the pairs $\bar{r}_0^j$, $\bar{s}_0^j$ and $\bar{r}_1^j$, $\bar{s}_1^j$ if and only if Q separates r_1^j and s_1^j).

If we have proved (21), Case 1 is done, as in the new situation the value of $2|E| - k$ is decreased (in the graph G), and hence there would exist a periodic packing of paths P_i^j connecting $\bar{r}_i^j$ and $\bar{s}_i^j$ ($i = 0, ..., k$; $j \in \mathbb{Z}$). Replacing the paths P_0^j and P_1^j by the path $P_0^j P_1^j$ (for $j \in \mathbb{Z}$), we would obtain a periodic packing of paths for the original situation.

To show (21), let Q be any dual walk in G' from $\{I', O'\}$ to $\{I', O'\}$. If for no $j \in \mathbb{Z}$, Q separates both $\bar{r}_0^j$, $\bar{s}_0^j$ and $\bar{r}_1^j$, $\bar{s}_1^j$, then $\bar{\rho}(Q) = \rho(Q)$, and hence $e(Q) \geqslant \rho(Q) = \bar{\rho}(Q)$.

If for some $j \in \mathbb{Z}$, Q separates both $\bar{r}_0^j$, $\bar{s}_0^j$ and $\bar{r}_1^j$, $\bar{s}_1^j$, then we may assume that $j = 0$ (by translating Q). By (19), for no other value of $j \in \mathbb{Z}$, Q does separate both $\bar{r}_0^j$, $\bar{s}_0^j$ and $\bar{r}_1^j$, $\bar{s}_1^j$. So by (23), $\bar{\rho}(Q) = \rho(Q) + 2$. Moreover, $e(Q) \geqslant \rho(Q) + 2$, since path P_1^0 intersects Q twice (as P_1^0 passes vertex v), while Q does not separate r_1^0 and s_1^0. Hence $e(Q) \geqslant \bar{\rho}(Q)$.

Case 2. $s_1^0 \in \text{bd}(O')$ and $\text{Im}(r_1^0) + 1 < \text{Im}(v) < \text{Im}(s_1^0) - 1$. (See Fig. 2.)

Now let $b := \lceil \operatorname{Im}(v) - \operatorname{Im}(r_1^0) \rceil - 1$ (where $\lceil\ \ \rceil$ denotes upper integer part). Put

$$\begin{aligned}
&\bar{r}_0^j := r_1^j, && \bar{s}_0^j := v + (j-b)\mathbf{i} && (j \in \mathbb{Z}),\\
&\bar{r}_1^j := v + (j-b)\mathbf{i}, && \bar{s}_1^j := s_1^j && (j \in \mathbb{Z}), \qquad (24)\\
&\bar{r}_i^j := r_i^j, && \bar{s}_i^j := s_i^j && (i = 2, \ldots, k;\ j \in \mathbb{Z}).
\end{aligned}$$

We claim that again (21) holds, which would finish this case as before. Note that again (23) holds. Moreover, $\operatorname{Im}(\bar{s}_0^j) \leqslant \operatorname{Im}(\bar{r}_0^j) + 1$.

To show (21) in this case, let Q again be a dual walk in G' from $\{I', O'\}$ to $\{I', O'\}$. If for no $j \in \mathbb{Z}$, Q separates both $\bar{r}_0^j$, $\bar{s}_0^j$ and $\bar{r}_1^j$, $\bar{s}_1^j$, then $\bar{\rho}(Q) = \rho(Q)$, and hence $e(Q) \geqslant \rho(Q) = \bar{\rho}(Q)$. If for some $j \in \mathbb{Z}$, Q separates both $\bar{r}_0^j$, $\bar{s}_0^j$ and $\bar{r}_1^j$, $\bar{s}_1^j$, then again this j is unique (as $\operatorname{Im}(\bar{s}_0^j) \leqslant \operatorname{Im}(\bar{r}_0^j) + 1$). We may assume $j = 0$. So $\bar{\rho}(Q) = \rho(Q) + 2$. Moreover, $e(Q) \geqslant \rho(Q) + 2$. This can be seen as follows.

As Q separates both $\bar{r}_0^0$, $\bar{s}_0^0$ and $\bar{r}_1^0$, $\bar{s}_1^0$, as $v = \bar{s}_0^0 + b\mathbf{i}$, and as $\operatorname{Im}(v) < \operatorname{Im}(\bar{s}_1^0) - 1$, we know that there exists a $t \in \mathbb{Z}$ such that $Q + t\mathbf{i}$ separates both $\bar{r}_0^0$, v and v, $\bar{s}_1^0$. Hence $e(Q + t\mathbf{i}) \geqslant \rho(Q + t\mathbf{i}) + 2$, as $Q + t\mathbf{i}$

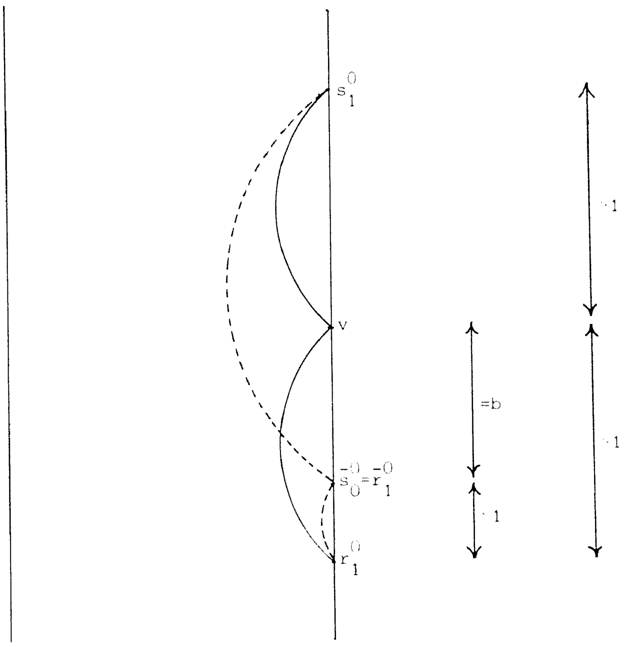

FIGURE 2

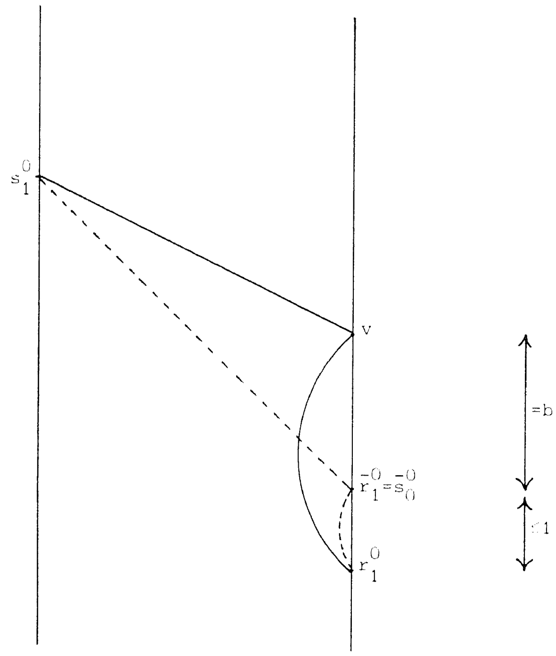

FIGURE 3

intersects P_1^0 twice (as P_1^0 passes v, while Q does not separate r_1^0 and s_1^0). Hence $e(Q) \geqslant \rho(Q) + 2$.

Concluding, we have $e(Q) \geqslant \bar{\rho}(Q)$.

Case 3. $s_1^0 \in \mathrm{bd}(I')$. (See Fig. 3.)
Again let $b := \lceil \mathrm{Im}(v) - \mathrm{Im}(r_1^0) \rceil - 1$, and define as in (24). We claim that again (21) holds, finishing this case as before. Again (23) holds, and $\mathrm{Im}(\bar{s}_0^j) \leqslant \mathrm{Im}(\bar{r}_0^j) + 1$.

Proving (21) in this case is similar to Case 2. Note that again, if Q separates both $\bar{r}_0^0$, $\bar{s}_0^0$ and $\bar{r}_1^0$, $\bar{s}_1^0$, then there exists a t so that $Q + t\mathbf{i}$ separates both $\bar{r}_0^0$, v and v, $\bar{s}_1^0$, implying $e(Q + t\mathbf{i}) \geqslant \rho(Q + t\mathbf{i}) + 2$. ∎

3. POLYNOMIAL-TIME SOLVABILITY

Our theorem characterizes the existence of pairwise edge-disjoint paths of given homotopies (if the parity condition holds). Although our proof is constructive, it does not yield directly a polynomial-time algorithm to find these paths (if they exist), mainly by the very large auxiliary graph G''

(used in proving Claim 2). We will show however that the theorem implies that the paths can be found in polynomial time.

First note that it is not even immediate that our theorem yields a "good characterization", i.e., that the problem of deciding if the paths exist belongs to NP $\cap$ co-NP. However, the following lemma implies that our theorem gives a good characterization.

Again, let $G=(V, E)$ be a planar graph embedded in $\mathbb{C}$. Let O be the unbounded face, and let I be some other face, including 0. Let $C_1, ..., C_k$ be curves in $\mathbb{C}\backslash(I \cup O)$ with end points in $V \cap \mathrm{bd}(I \cup O)$. Consider again the graph G' described in the proof above. I.e., let $\tau(z) := e^{2\pi z}$ for $z \in \mathbb{C}$, and let $G' := \tau^{-1}[G]$, a graph with vertex set $V' := \tau^{-1}[V]$. For $i=1, ..., k$ and $j \in \mathbb{Z}$, let r_i^j be the unique point in $\mathbb{C}$ with $\tau(r_i^j) = C_i(0)$ and $j \leqslant \mathrm{Im}(r_i^j) < j+1$, let C_i^j be the unique curve in $\mathbb{C}$ with $C_i^j(0) = r_i^j$ and $\tau \circ C_i^j = C_i$, and let $s_i^j := C_i^j(1)$. Let $O' := \tau^{-1}[O]$ and $I' := \tau^{-1}[I]$. Trivially, the cut condition (2) is equivalent to

$$e(Q) \geqslant \rho(Q) \qquad \text{for each dual walk } Q \text{ in } G' \text{ from } \{I', O'\} \text{ to } \{I', O'\}, \tag{25}$$

where $\rho(Q) :=$ number of (i, j) for which Q separates r_i^j and s_i^j.

Let R be a shortest dual walk from I' to O' (i.e., with a minimum number of edges). Again, let $R + j\mathbf{i}$ denote the translation of R by $z \mapsto z + j\mathbf{i}$. Note that by the minimality of R, the paths $R + j\mathbf{i}$ do not have faces or edges in common (except for I', O'), provided G is connected.

LEMMA. *Let G be connected. Then* (25) *holds, if and only if $e(Q) \geqslant \rho(Q)$ for each dual walk Q in G' from $\{I', O'\}$ to $\{I', O'\}$ so that Q intersects at most $4|E|$ of the walks $R + j\mathbf{i}$ in faces distinct from I', O'.*

Proof. Necessity being trivial, we only show sufficiency. Suppose $e(Q) < \rho(Q)$ for some dual walk Q from $\{I', O'\}$ to $\{I', O'\}$, and suppose we have chosen this Q so that it intersects the minimum number t of the $R + j\mathbf{i}$. If $t \leqslant 4|E|$ we are done, so assume $t > 4|E|$. In particular $t > 1$, implying

$$e(R) \geqslant \rho(R). \tag{26}$$

By translation, we may assume that Q intersects $R+\mathbf{i}$, $R+2\mathbf{i}$, ..., $R+t\mathbf{i}$. Moreover, we may assume that it first intersects $R+\mathbf{i}$, next $R+2\mathbf{i}$, next $R+3\mathbf{i}$, ..., finally $R+t\mathbf{i}$. Otherwise we would have that, for some j, Q intersects $R+j\mathbf{i}$ (in face F say), next $R+(j+1)\mathbf{i}$, and then $R+j\mathbf{i}$ again (in face F' say). But then we can replace the part of Q between F and F' by the part of $R+j\mathbf{i}$ between F and F'. This does not change $\rho(Q)$, and does not increase $e(Q)$, since R, and hence $R+j\mathbf{i}$ also, is a shortest dual walk from I' to O'.

We next prove that Q contains faces $F', F'' \notin \{I', O'\}$ so that $F'' = F' + \mathbf{i}$. Let

$$R = (I', e_0, F_1, e_1, ..., e_{m-1}, F_m, e_m, O'). \tag{27}$$

Choose for each $j = 1, ..., t$ an element h_j from $\{1, ..., m\}$ so that Q intersects $R + j\mathbf{i}$ in face $F_{h_j} + j\mathbf{i}$. Since $t > 4|E| > m$ (as $\tau[e_0], ..., \tau[e_m]$ are distinct edges of G), there exists a $j \in \{2, ..., t-1\}$ such that either $h_{j-1} \leqslant h_j$ and $h_{j+1} \leqslant h_j$ or $h_{j-1} \geqslant h_j$ and $h_{j+1} \geqslant h_j$. (Indeed, if $h_1 = h_2$ let $j = 2$; if $h_1 < h_2$ let j be the largest value so that $h_1 < h_2 < \cdots < h_j$ (so $j \leqslant m \leqslant t-1$); similarly, if $h_1 > h_2$ let j be the largest value such that $h_1 > h_2 > \cdots > h_j$.) It follows that part $F_{h_{j-1}} + (j-1)\mathbf{i}, ..., F_{h_j} + j\mathbf{i}$ of Q intersects part $F_{h_j} + (j-1)\mathbf{i}, ..., F_{h_{j+1}} + j\mathbf{i}$ of $Q - \mathbf{i}$. So Q and $Q - \mathbf{i}$ have a face in common, implying that Q contains faces $F', F'' \notin \{I', O'\}$ such that $F'' = F' + \mathbf{i}$.

Hence Q can be decomposed as $Q'Q''Q'''$, where Q' connects $\{I', O'\}$ with F', Q'' connects F' and $F'' = F' + \mathbf{i}$, and Q''' connects F'' and $\{I', O'\}$. Now let $Q^\circ := Q''' - \mathbf{i}$. Then $Q'Q^\circ$ is a dual walk from $\{I', O'\}$ to $\{I', O'\}$ satisfying

$$\begin{aligned} e(Q'Q^\circ) &= e(Q) - e(Q''), \\ \rho(Q'Q^\circ) &= \rho(Q) - \text{number of } i = 1, ..., k \text{ for which} \\ &\qquad C_i \text{ connects } \mathrm{bd}(O) \text{ and } \mathrm{bd}(I). \end{aligned} \tag{28}$$

The second equation follows from the fact that Q intersects more than $|E| + 2$ of the $R + j\mathbf{i}$, whereas each pair r_i^j, s_i^j is separated by at most $e(R) \leqslant |E|$ of the $R + j\mathbf{i}$ (by (26)). As $Q'Q^\circ$ intersects $t-1$ of the $R + j\mathbf{i}$, we know that $e(Q'Q^\circ) \geqslant \rho(Q'Q^\circ)$ holds, and hence (as $e(Q) < \rho(Q)$)

$$\begin{aligned} e(Q'') \leqslant (&\text{number of } i = 1, ..., k\text{: } C_i \text{ connects } \mathrm{bd}(O) \text{ and} \\ &\mathrm{bd}(I)) - 1. \end{aligned} \tag{29}$$

Now let L be a shortest dual path in G' from I' to F'. So $e(L) \leqslant |E|$. Consider the dual walk

$$\tilde{Q} := L \cdot Q'' \cdot (Q'' + \mathbf{i}) \cdot (Q'' + 2\mathbf{i}) \cdot \;\cdots\; \cdot (Q'' + (3|E| - 1)\mathbf{i})(L^{-1} + 3|E|\mathbf{i}) \tag{30}$$

(where L^{-1} denotes the path reverse to L). Then

$$\begin{aligned} e(\tilde{Q}) = 2e(L) + 3|E| \cdot e(Q'') &\leqslant 2|E| + 3|E| \cdot [(\text{number of } i = 1, ..., k\text{:} \\ &\qquad C_i \text{ connects } \mathrm{bd}(O) \text{ and } \mathrm{bd}(I)) - 1] \\ &< 3|E|(\text{number of } i = 1, ..., k\text{: } C_i \text{ connects } \mathrm{bd}(O) \text{ and } \mathrm{bd}(I)) \\ &\leqslant \rho(\tilde{Q}). \end{aligned} \tag{31}$$

However, $\tilde{Q}$ intersects at most $3|E|+2 \leqslant 4|E|$ of the $R+j\mathbf{i}$, thus proving the Lemma. ∎

Now consider the finite graph $G''=(V'', E'')$ (analogous to that occurring in the proof of the theorem) obtained from G' by contracting all vertices not "in between of" R and $R+4|E|\mathbf{i}$ to two vertices. Again, by $\bar{r}_i^j$ and $\bar{s}_i^j$ we denote the vertices r_i^j and s_i^j after contraction. Let $K := \{(i, j) \mid \bar{r}_i^j \neq \bar{s}_i^j\}$. Consider the cut condition for G'':

$$d_{G''}(U) \geqslant \rho(U) \qquad \text{for each subset } U \text{ of } V'', \tag{32}$$

where $d_{G''}(U)$ is the number of edges having exactly one of its end points in U, and $\rho(U)$ is the number of pairs (i, j) in K for which U separates $\bar{r}_i^j$ and $\bar{s}_i^j$. By the Lemma we have, if G is connected,

the cut condition holds for G, I, $C_1, ..., C_k$ if and only if (32) holds. (33)

In particular

the theorem gives a good characterization, (34)

since if the cut condition for G is violated we can show this by specifying a violated cut for G'' (if G is not connected, then the cut condition (2) is violated by one of the components of G).

The Lemma also implies

the cut condition (2) can be checked in polynomial time. (35)

Indeed, checking the cut condition (2) reduces to testing if $d_{G''}(U) \geqslant \rho(U)$ holds for each subset U of V'', which can be done easily in polynomial time. (For each pair of edges e', e'' on $\mathrm{bd}(O'')$ (where O'' denotes the unbounded face of G''), we find a shortest dual walk Q'' in G'' from O'' to O'' such that Q'' starts with O'', e' and ends with e'', O''. Then Q'' determines a subset U of V'' such that the only two edges on $\mathrm{bd}(O')$ leaving U are e' and e'', and such that $d_{G''}(U)$ is minimal. Since e', e'' determine $\rho(U)$, the inequality $e(Q) \geqslant \rho(Q)$ is easily checked. If this inequality holds for each pair of edges e', e'' on $\mathrm{bd}(O'')$ then (32) holds, and otherwise not.)

So our theorem together with (35) implies that the problem of deciding if paths as required exist belongs to the complexity class $\mathscr{P}$ (if the parity condition holds). How are we to find these paths in polynomial time when they exist? We describe a brute-force polynomial-time method.

Consider any shortest dual walk R in G from I to O. We may assume that the curves C_i are given as walks in G. The steps of the algorithm are as follows.

Step 1. Check if the cut condition holds. If not, stop (the required paths do not exist). If so, go to step 2 (the required paths exist).

Step 2. Check if there exist a curve C_i and an edge e of G such that C_i is homotopic to e. If so, delete C_i and e, and repeat step 2. (Add $P_i := e$ to the final packing of pairs.) If not, go to step 3.

Step 3. Check if the cut condition is preserved after deleting all edges on the boundary of I. If so, delete all edges on the boundary of I, and repeat step 3. If not, go to step 4.

Step 4. If there is no curve C_i left, stop. If there are curves C_i left, we know that there is a packing of paths as required (as the cut condition holds), and that one of the curves should use an edge on the boundary of I (as we have performed step 3), without being itself homotopic to this edge (as we have performed step 2). Hence some curve C_i can be replaced by two curves C_i', C_i'' such that C_i' and C_i'' are homotopic nontrivial, such that C_i is homotopic to $C_i' \cdot C_i''$, and such that the cut condition is preserved. As the cut condition is preserved, we know moreover that we can take C_i' and C_i'' such that they do not intersect the edges of R more than $e(R)$ times. So we can find for some curve C_i curves C_i' and C_i'' which are homotopic nontrivial, such that C_i is homotopic to $C_i' \cdot C_i''$, such that C_i' and C_i'' intersect R at most $e(R)$ times, and such that the cut condition is preserved after replacing C_i by $C_i' \cdot C_i''$. There are at most $|V| \cdot e(R)$ paths C_i' to consider (up to homotopy); similarly for C_i''. Replace C_i by C_i' and C_i'', and go to step 2. (In the final packing, replace paths $P_i' \sim C_i'$ and $P_i'' \sim C_i''$ by $P_i' \cdot P_i'' \sim C_i$.)

The polynomially bounded running time of this algorithm follows from the facts that the cut condition can be checked in polynomial time, that steps 2 and 3 are performed at most $|E|$ times, and that step 4 is performed at most $|E| - k = \frac{1}{2} \sum_{v \in V} (\deg_G(v) - \deg_{C_1, \ldots, C_k}(v))$ times (as by splitting C_i into C_i' and C_i'' this last sum is decreased by 1).

4. Further Remarks

The parity condition (1) cannot be deleted in the theorem, as is shown by Fig. 4, (in which dotted lines represent curves).

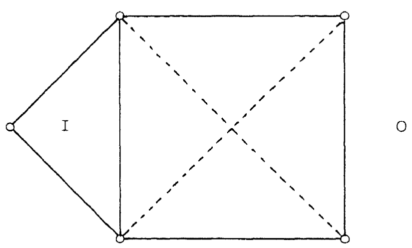

Figure 4

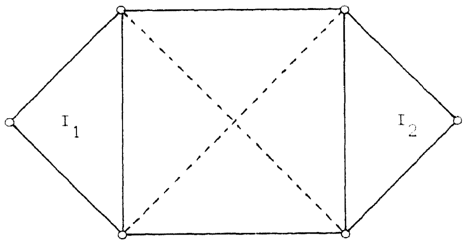

FIGURE 5

The obvious extension of our theorem to more than one "hole" does not hold, as is shown by the example in Fig. 5. Kaufmann and Mehlhorn [1] showed that an extension to arbitrarily many holes holds in the case of so-called grid graphs. See [5] for a generalization.

There is another extension of the Okamura–Seymour theorem, due to Okamura [3], which resembles our theorem, but which is different: Let $G=(V, E)$ be a planar graph embedded in the plane $\mathbb{C}$, let O be the interior of the unbounded face, let I be the interior of some other face, let $r_1, ..., r_m, s_1, ..., s_m \in V \cap \text{bd}(O)$, and let $r_{m+1}, ..., r_k, s_{m+1}, ..., s_k \in V \cap \text{bd}(I)$, so that the parity condition (6) holds. Then there exist pairwise edge-disjoint paths $P_1, ..., P_k$ such that P_i connects r_i and s_i $(i=1, ..., k)$ if and only if the cut condition (7) holds.

We did not see an implication, one way or the other, between our theorem and Okamura's.

ACKNOWLEDGMENTS

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