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GRAPHS WHOSE NEIGHBORHOODS HAVE NO SPECIAL CYCLES

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To a graph G is canonically associated its neighborhood-hypergraph, $\mathcal{N}(G)$, formed by the closed neighborhoods of the vertices of G. We characterize the graphs G such that (i) $\mathcal{N}(G)$ has no induced cycle, or (ii) $\mathcal{N}(G)$ is a balanced hypergraph or (iii) $\mathcal{N}(G)$ is triangle free. (i) is another short proof of a result by Farber; (ii) answers a problem asked by C. Berge. The case of strict neighborhoods is also solved.

Introduction

The *balanced hypergraphs* constitute a natural generalization of bipartite graphs and of unimodular hypergraphs (see [2]). Berge in [3] asked for a characterization of graphs for which the neighborhoods of the vertices form a balanced hypergraph.

We give here such a characterization (Theorem 2). Our method of proof also yields a characterization of those graphs whose neighborhoods have no induced triangle (Theorem 3) and a new proof of a result of Farber [8] characterizing those graphs whose neighborhoods have no induced cycle (Theorem 1). Analogous characterizations for the case of strict neighborhoods are also given (Section 4).

1. Preliminaries

For the general terminology concerning graphs and hypergraphs, we refer to [2]. Our graphs or hypergraphs are loopless but may be infinite and contain multiple edges.

By a (hypergraphic) *cycle* we mean a finite alternating sequence of distinct vertices and distinct edges $v_1E_1 \cdots v_pE_pv_1$ $(p \geq 3)$ such that $\{x_i, x_{i+1}\} \in E_i$ for $1 \leq i \leq p \pmod p$. For graphs, the (graphic) cycle will be identified with the sequence of its vertices $v_1 \cdots v_pv_1$; an edge of the form v_iv_j with $|i-j| \neq 1$ is called a *chord* of the cycle. As usual C_p denotes the cycle (with no chords) on p vertices.

Given a hypergraph $H=(X, \mathscr{E})$ and a subset $A \subset X$, $H_A=(A, \{E \cap A;\ E \in \mathscr{E}\})$ denotes the *subhypergraph induced* by A. A *partial induced subhypergraph* of H will be a hypergraph of the form H'_A for some $A \subset X$ and some $H'=(X, \mathscr{E}')$ with $\mathscr{E}' \subset \mathscr{E}'$.

The graphs not containing any C_p (for $p \geq 4$) as *induced subgraph* are usually named *triangulated graphs* or *chordal graphs* (see [7] for a recent survey on these graphs).

We say that a hypergraph has an *induced* C_p $(p \geq 3)$ if it has a partial induced hypergraph which is isomorphic to C_p. A hypergraph is *triangle-free* if it has no induced C_3 (see [1]). It is *balanced* (cf. [2]) if it has no induced odd cycle $(C_{2q+1}, q \geq 1)$; hypergraphs *with no induced cycle* $(C_p, p \geq 3)$ were also called *totally balanced.*

Given a graph G, we denote by $N(v)$ the (closed) *neighborhood* of a vertex v, i.e. the set formed by v and all the vertices adjacent to v. The hypergraph having as vertices the vertices of G and as edges the neighborhoods of these vertices is called the *neighborhood-hypergraph* of G and is denoted by $\mathcal{N}(G)$.

An (incomplete) *sun* of order $p \geq 3$ is a graph S with the following properties:

(1.1) S is triangulated.

(1.2) S has a cycle of length p: $a_1 \cdots a_pa_1$.

(1.3) S has exactly $2p$ vertices: $a_1, \ldots, a_p, b_1, \ldots, b_p$.

(1.4) Every vertex b_i has only two neighbours in S: a_i and a_{i+1} (i is taken modulo p).

The a_i's form the *central* set of S. The b_i's form the *stable* set of S. When the central set is a clique, S is the *complete sun* of order p, denoted by S_p. By an *odd* (*even*) *sun*, we mean a sun of odd (even) order; Fig. 1 exhibits a sun of order 9 that does not strictly contain any other odd sun.

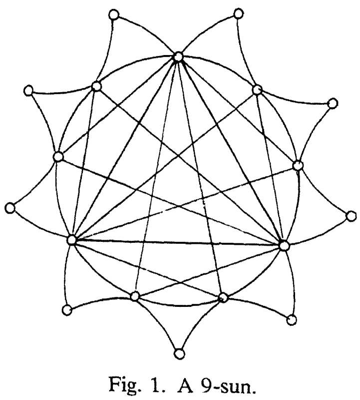

Fig. 1. A 9-sun.

2. Main results

Let G be a graph and $\mathcal{N}(G)$ its neighborhood-hypergraph.

Theorem 1 (Farber [8]). *$\mathcal{N}(G)$ has no induced cycle if and only if G is a triangulated graph with no induced complete sun.*

These graphs were called 'strongly chordal' by Farber, they have nice properties relative to elimination schemes (see [8, 10]).

Theorem 2. *$\mathcal{N}(G)$ is balanced if and only if G is a triangulated graph with no induced odd sun.*

As in the first version of this result [4, 6], the condition 'no induced odd sun' can be replaced, for triangulated graphs by the following condition:

(2.1) In *every cycle of length $4k+2 \geqslant 6$ there are at least $2k+2$ vertices belonging to some chords of the cycle.*

Theorem 3. *$\mathcal{N}(G)$ is triangle-free if and only if G does not contain C_4, C_5, C_6 or S_3 as induced subgraph.*

3. Proofs.

The theorems are simple applications of the following lemma:

Lemma 3.1. *Let $p \geqslant 3$ be an integer and suppose G is a graph in which every cycle of length k, for $4 \leqslant k \leqslant 2p$, possesses a chord. Then, $\mathcal{N}(G)$ has an induced C_p if and only if G has an induced sun of order p.*

Proof. Clearly, if K is some induced subgraph of $G, \mathcal{N}(K)$ is isomorphic to an induced partial subhypergraph of $\mathcal{N}(G)$. Thus, the 'if' part of the lemma is easy and left to the reader.

The converse is proved by contradiction: Suppose that every cycle of G with length $k, 4 \leqslant k \leqslant 2p$, has a chord and suppose G has no induced p-sun while $\mathcal{N}(G)$ has an induced C_p. By definition, there exists a set A of p vertices $a_1, \ldots, a_p$ and a set B of p vertices $b_1, \ldots, b_p$ with the following properties (during the proof, all indices are modulo p):

(3.2) $a_1 N(b_1) \cdots a_p N(b_p) a_1$ is a hypergraphic cycle of $\mathcal{N}(G)$.

(3.3) $N(b_j) \cap A = \{a_j, a_{j+1}\}$ for every j.

(3.3) is clearly equivalent to:

(3.4) For $j \neq i$ or $i+1$, $a_i \neq b_j$ and $a_i b_j$ is not an edge of G.

Claim 1. *If* $v_1v_2 \cdots v_qv_1$ *is a cycle* C *of* G $(4 \leqslant q \leqslant 2p)$, *then either* v_2v_q *is a chord of* C *or* C *has a chord of the form* v_1v_k *for some* $k, 3 \leqslant k \leqslant q-1$.

The proof is easy by induction on k.

Claim 2. G *contains an edge of the form* a_ia_j $(i \neq j)$. Otherwise, by (3.2), the set of a_i's and the set of b_i's are disjoint. Thus $a_1b_1a_2b_2 \cdots a_pb_pa_1$ is a cycle of length $2p$ in G and our Claim 1 together with (3.4) implies that b_kb_{k+1} is a chord of this cycle for each k $(1 \leqslant k \leqslant p)$. Hence $A \cup B$ induces a triangulated subgraph of G which is a sun of order p: the central set is B, the stable set is A. The contradiction proves our claim.

Claim 3. *If* a_ia_j *is an edge of* G, *then* a_ia_{i+1} *is also an edge of* G.

By symmetry, we may suppose $i = 1$. Let j be the smallest integer for which a_1a_j is an edge of G. If $j > 2$, the vertices $a_1b_1a_2a_j$ are different. $a_2b_2a_3b_3 \cdots a_{j-1}b_{j-1}a_j$ is a walk not passing through a_1 or b_1, by (3.4). This walk induces a minimal path, say P, from a_2 to a_j.

By (3.4) and the definition of j, the cycle $a_1b_1Pa_1$ with length $\geqslant 4$ has no chord containing a_1. Hence b_1a_j must be an edge (Claim 1), in contradiction with (3.4). So, $j = 2$.

Claim 4. $a_1 \cdots a_pa_1$ *is a cycle of* G.

It is an easy consequence of the previous claim.

Claim 5. $a_i \neq b_j$ *for all* i, j.

Otherwise $N(b_j)$ would contain a_{i-1}, a_i and a_{i+1} (Claim 4), in contradiction with (3.4).

Claim 6. G *contains some edge* b_ib_j.

Otherwise, $A \cup B$ would induce a p-sun with central set A and stable set B.

For obtaining the final contradiction, we observe that in the last claim i and j play a symmetrical role. So, we may assume without loss of generality that G contains an edge of the form b_1b_j with $j \neq 2$. Then G has the following cycle:

$$b_1a_2a_3a_4 \cdots a_jb_jb_1$$

and, by Claim 1, some edge b_1a_i $(3 \leqslant i \leqslant j)$ or the edge b_ja_2 must exist, contradicting (3.4) □

Lemma 3.5. *If* G *has an induced* C_p $(p \geqslant 4)$, *then* $\mathcal{N}(G)$ *has an induced* C_k *for each value of* k *between* $[\frac{1}{2}(p+1)]$ *and* $[\frac{2}{3}p]$ *(inclusive)*.

The proof is an easy exercise.

Proof of Theorem 1. The above lemmas imply: $\mathcal{N}(g)$ has no induced cycle iff G is triangulated and has no induced sun. To conclude the proof, we check that every sun contains a complete sun. Although this has been proven previously (see, for example, [8, 10]) we provide a short proof for the sake of completeness.

This is done by induction on the order p of the sun. The cases $p = 3$, 4 are straightforward. Considering a sun S of order $p > 4$, with central cycle $a_1 \cdots a_pa_1$

and stable corresponding set $b_1 \cdots b_p$, and assuming that S has no induced complete sun, we look at the effect of the contraction of the vertices a_i and a_{i+1} into a single vertex a. Deleting b_i, we get a sun of order $p-1$ that contains, by induction hypothesis, a complete q-sun S'. Thus S contains (decontracting a in S' and adding b_i) a sun of order $q+1$. By induction hypothesis again, $q+1=p$.

So, for every i, $\{a_1 \cdots a_p\} \backslash \{a_i, a_{i+1}\}$ induces a complete subgraph; since $p \geq 5$, S is a complete sun. □

Proof of Theorem 2. If $\mathcal{N}(G)$ is balanced, G has to be triangulated by Lemma 3.5. The theorem then follows by Lemma 3.1. □

Proof of Theorem 3. The same easy reasoning applies. □

4. Strict neighborhoods

For a graph G, we define the *strict neighborhood* $N^0(v)$ of a vertex v by $N^0(v) = N(v) \backslash v$, and we denote by $\mathcal{N}^0(G)$ the collection of all strict neighborhoods of vertices of G.

Theorem 4. *$\mathcal{N}^0(G)$ has no induced cycle iff G is a bipartite graph with no induced C_{2q}, for $2q \geq 6$.*

These graphs are known as '*chordal bipartite graphs*' (see [9]).

Theorem 5. *$\mathcal{N}^0(G)$ is balanced iff G is a bipartite graph with no induced C_{4q+2} for $q \geq 1$.*

Theorem 6. *$\mathcal{N}^0(G)$ is triangle free iff G has no C_3 or C_6 as induced subgraph.*

Proofs. Theorem 6 is quite obvious: if $\mathcal{N}^0(G)$ is triangle free, G has no induced C_3 or C_6; the reader will easily check the converse assertion.

For the other theorems, we remark: if $\mathcal{N}^0(G)$ has no induced odd cycle, then G has no induced odd cycle and, since the vertices of an odd cycle always induce some triangle, G is bipartite. Thus, Theorems 4 and 5 easily follow from

Lemma 4.1. *Let G be a bipartite graph and q an integer ≥ 3. Then $\mathcal{N}^0(G)$ has an induced C_q iff G has an induced C_{2q}.*

The easy proof is omitted.

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