Characterizing partition functions of the vertex model by rank growth

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Abstract. We characterize which graph invariants are partition functions of a vertex model over $\mathbb{C}$, in terms of the rank growth of associated 'connection matrices'.

1. Introduction

Let $\mathcal{G}$ denote the collection of all undirected graphs, two of them being the same if they are isomorphic. In this paper, all graphs are finite and may have loops and multiple edges. Let $k \in \mathbb{N}$ and let $\mathbb{F}$ be a commutative ring. Following de la Harpe and Jones [4], call any function $y : \mathbb{N}^k \to \mathbb{F}$ a (k-*color*) *vertex model* (*over* $\mathbb{F}$).[2] The *partition function* of y is the function $p_y : \mathcal{G} \to \mathbb{F}$ defined for any graph $G = (V, E)$ by

(1) $$p_y(G) := \sum_{\kappa : E \to [k]} \prod_{v \in V} y_{\kappa(\delta(v))}.$$

Here $\delta(v)$ is the set of edges incident with v. Then $\kappa(\delta(v))$ is a multisubset of $[k]$, which we identify with its incidence vector in $\mathbb{N}^k$. Moreover, we use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and for $n \in \mathbb{N}$,

(2) $$[n] := \{1, \ldots, n\}.$$

We can visualize κ as a coloring of the edges of G and $\kappa(\delta(v))$ as the multiset of colors 'seen' from v. The vertex model was considered by de la Harpe and Jones [4] as a physical model, where vertices serve as particles, edges as interactions between particles, and colors as states or energy levels. It extends the Ising-Potts model. Several graph parameters are partition functions of some vertex model, like the number of matchings. There are real-valued graph parameters that are partition functions of a vertex model over $\mathbb{C}$, but not over $\mathbb{R}$. (A simple example is $(-1)^{|E(G)|}$.)

In this paper, we characterize which functions $f : \mathcal{G} \to \mathbb{C}$ are the partition function of a vertex model over $\mathbb{C}$. The characterization differs from an earlier characterization given in [2] (which our present characterization uses) in that it is based on the rank growth of associated 'connection matrices'.

To describe it, we need the notion of a k-fragment. For $k \in \mathbb{N}$, a k-*fragment* is an undirected graph $G = (V, E)$ together with an injective 'label' function $\lambda : [k] \to V$, where $\lambda(i)$ is a vertex of degree 1, for each $i \in [k]$. (You may alternatively view these degree-1 vertices as ends of 'half-edges'.)

If G and H are k-fragments, the graph $G \cdot H$ is obtained from the disjoint union of G and H by identifying equally labeled vertices and by ignoring each of the k identified points

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[2] In [7] it is called an *edge coloring model*. Colors are also called *states*.

as vertex, joining its two incident edges into one edge. (A good way to imagine this is to see a graph as a topological 1-complex.) Note that it requires that we also should consider the 'vertexless loop' as possible edge of a graph, as we may create it in $G \cdot H$. We denote this vertexless loop by O. Observe that if y is a vertex model over $\mathbb{C}$ with n colors, then $p_y(O) = n$.

Let $\mathcal{G}_k$ denote the collections of k-fragments. For any $f : \mathcal{G} \to \mathbb{C}$ and $k \in \mathbb{N}$, the *kth connection matrix* is the $\mathcal{G}_k \times \mathcal{G}_k$ matrix $C_{f,k}$ defined by

(3) $$(C_{f,k})_{G,H} := f(G \cdot H)$$

for $G, H \in \mathcal{G}_k$.

Now we can formulate our characterization:

Theorem 1. *A function $f : \mathcal{G} \to \mathbb{C}$ is the partition function of a vertex model over $\mathbb{C}$ if and only if $f(\emptyset) = 1$, $f(O) \in \mathbb{R}$, and*

(4) $$\text{rank}(C_{f,k}) \leq f(O)^k$$

for each $k \in \mathbb{N}$.

Let us relate this to Szegedy's theorem [7], which characterizes the partition functions of vertex models over $\mathbb{R}$. Call a function $f : \mathcal{G} \to \mathbb{C}$ *multiplicative* if $f(\emptyset) = 1$ and $f(G \dot{\cup} H) = f(G)f(H)$ for all graphs G and H, where $G \dot{\cup} H$ denotes the disjoint union of G and H. Then Szegedy's theorem reads:

(5) A function $f : \mathcal{G} \to \mathbb{R}$ is the partition function of a vertex model over $\mathbb{R}$ if and only if f is multiplicative and $C_{f,k}$ is positive semidefinite for each k.

For related results for the 'spin model' see [3] and [6].

Our proof of Theorem 1 is based on some elementary results from the representation theory of the symmetric group, and on the following alternative characterization of partition functions of vertex models given in [2], which uses the Nullstellensatz.

For any graph $G = (V, E)$, any $U \subseteq V$, and any $s : U \to V$, define

(6) $$E_s := \{us(u) \mid u \in U\} \text{ and } G_s := (V, E \cup E_s)$$

(adding multiple edges if E_s intersects E). Let S_U be the group of permutations of U. Then:

(7) A function $f : \mathcal{G} \to \mathbb{C}$ is the partition function of some k-color vertex model over $\mathbb{C}$ if and only if f is multiplicative and for each graph $G = (V, E)$, each $U \subseteq V$ with $|U| = k + 1$, and each $s : U \to V$:

$$\sum_{\pi \in S_U} \text{sgn}(\pi) f(G_{s \circ \pi}) = 0.$$

2. Some results on the symmetric group

In our proof we need Proposition 3 below, which we prove in a number of steps. (The result might be known, and must not be a difficult exercise for those familiar with the representation theory of the symmetric group, but we could not find an explicit reference.)

We recall a few standard results from the representation theory of the symmetric group S_n (cf. James and Kerber [5]). Basis is the one-to-one relation between the partitions λ of n and the irreducible representations r_λ of S_n. Here a *partition* λ of n is a finite nonincreasing sequence $(\lambda_1, \ldots, \lambda_t)$ of positive integers with sum n. One puts $\lambda \vdash n$ if λ is a partition of n. The number t of terms of λ is called the *height* of λ, denoted by $\text{height}(\lambda)$. Denote by f^λ the dimension of representation r_λ, and by χ_λ the character of r_λ.

For any $\lambda \vdash n$, the *Young shape* Y_λ of $\lambda = (\lambda_1, \ldots, \lambda_t)$ is the following subset of $\mathbb{N}^2$:

$$Y_\lambda := \{(i,j) \mid i \in [t], j \in [\lambda(i)]\}. \tag{8}$$

For any $\pi \in S_n$, let $o(\pi)$ denote the number of orbits of π.

Proposition 1. *For any $n \in \mathbb{Z}_+$, $\lambda \vdash n$, and $d \in \mathbb{C}$:*

$$\sum_{\pi \in S_n} \chi_\lambda(\pi) d^{o(\pi)} = f^\lambda \prod_{(i,j) \in Y_\lambda} (d + j - i). \tag{9}$$

Proof. As both sides of (9) are polynomials in d, we can assume that $d \in \mathbb{Z}_+$. Consider the natural representation r of S_n on $(\mathbb{C}^d)^{\otimes n}$. Note that its character χ satisfies $\chi(\pi) = d^{o(\pi)}$ for each $\pi \in S_n$.

For any $\alpha \vdash n$, let μ_α be the multiplicity of r_α in r. Then

$$\begin{aligned} \sum_{\pi \in S_n} \chi_\lambda(\pi) d^{o(\pi)} = \sum_{\pi \in S_n} \chi_\lambda(\pi)\chi(\pi) = \sum_{\pi \in S_n} \chi_\lambda(\pi) \sum_{\alpha \vdash n} \mu_\alpha \chi_\alpha(\pi) = \\ \sum_{\alpha \vdash n} \mu_\alpha \sum_{\pi \in S_n} \chi_\alpha(\pi)\chi_\lambda(\pi) = \sum_{\alpha \vdash n} \mu_\alpha n! \delta_{\alpha,\lambda} = n! \mu_\lambda = f^\lambda \prod_{(i,j) \in Y_\lambda} (d + j - i). \end{aligned} \tag{10}$$

The last equality follows from the fact that μ_λ is (by Schur duality) equal to the dimension of the irreducible representation of $\text{GL}(d, \mathbb{C})$ corresponding to λ (cf. [1] eq. 9.28). This shows (9). ∎

For any $n \in \mathbb{Z}_+$ and $d \in \mathbb{C}$, let $M_n(d)$ be the $S_n \times S_n$ matrix with

$$(M_n(d))_{\rho,\sigma} := d^{o(\rho\sigma^{-1})} \tag{11}$$

for $\rho, \sigma \in S_n$.

Proposition 2. *For any $n \in \mathbb{Z}_+$ and $d \in \mathbb{C}$:*

$$\text{(12)} \qquad \operatorname{rank}(M_n(d)) = \begin{cases} n! & \textit{if } d \notin \mathbb{Z}, \\ \sum((f^\lambda)^2 \mid \lambda \vdash n, \operatorname{height}(\lambda) \le |d|) & \textit{if } d \in \mathbb{Z}. \end{cases}$$

Proof. First, we have

$$\text{(13)} \qquad \operatorname{rank}(M_n(-d)) = \operatorname{rank}(M_n(d)).$$

Indeed, note that $M_n(-d) = (-1)^n \Delta_{\text{sgn}} M_n(d) \Delta_{\text{sgn}}$, where Δ_{sgn} is the $S_n \times S_n$ diagonal matrix with $(\Delta_{\text{sgn}})_{\pi,\pi} = \operatorname{sgn}(\pi)$ for $\pi \in S_n$. (This because $\operatorname{sgn}(\pi) = (-1)^{n-o(\pi)}$ for all π.) This gives (13).

Let R be the regular representation of S_n. So, for any $\pi \in S_n$, $R(\pi)$ is the $S_n \times S_n$ matrix with

$$\text{(14)} \qquad R(\pi)_{\rho,\sigma} = \begin{cases} 1 & \text{if } \rho = \pi\sigma, \\ 0 & \text{otherwise} \end{cases}$$

for $\rho, \sigma \in S_n$. Then

$$\text{(15)} \qquad M_n(d) = \sum_{\pi \in S_n} d^{o(\pi)} R(\pi).$$

As $M_n(d)$ commutes with each $R(\pi)$,

$$\text{(16)} \qquad \operatorname{rank}(M_n(d)) = \sum((f^\lambda)^2 \mid \lambda \vdash n, \sum_{\pi \in S_n} \chi_\lambda(\pi) d^{o(\pi)} \neq 0)$$
$$= \sum((f^\lambda)^2 \mid \lambda \vdash n, d \notin \{i - j \mid (i,j) \in Y_\lambda\}).$$

The last equality follows from (9).

Now if $d \notin \mathbb{Z}$, then for all $\lambda \vdash n$: $d \neq i - j$ for all $(i,j) \in Y_\lambda$. So $\operatorname{rank}(M_{n,d}) = n!$. If $d \in \mathbb{Z}$, then by (13) we can assume $d \in \mathbb{Z}_+$. Then for all $\lambda \vdash n$: $d \notin \{i - j \mid (i,j) \in Y_\lambda\}$ if and only if $\operatorname{height}(\lambda) \le d$. This proves (12). ∎

Proposition 3. *For any* $d \in \mathbb{C}$*:*

$$\text{(17)} \qquad \sup_{n \in \mathbb{Z}_+} (\operatorname{rank}(M_n(d)))^{1/n} = \begin{cases} \infty & \textit{if } d \notin \mathbb{Z}, \\ d^2 & \textit{if } d \in \mathbb{Z}. \end{cases}$$

Proof. If $d \notin \mathbb{Z}$, the result follows directly from (12), as $\sup_n n!^{1/n} = \infty$.

If $d \in \mathbb{Z}$, we can assume $d \in \mathbb{Z}_+$. Then $\operatorname{rank}(M_n(d)) \le (d^2)^n$. Indeed, let χ be the character of the natural representation r of S_n on $(\mathbb{C}^d)^{\otimes n}$. Then $d^{o(\pi)} = \chi(\pi)$ for all $\pi \in S_n$. Hence $d^{o(\rho\sigma^{-1})} = \chi(\rho\sigma^{-1})$. So $d^{o(\rho\sigma^{-1})}$ is the trace of the product of the $d^n \times d^n$ matrices $r(\rho)$ and $r(\sigma^{-1})$. Hence $\operatorname{rank}(M_n(d)) \le (d^n)^2 = (d^2)^n$. This proves $\le$ in (17).

To prove the reverse inequality, consider for any $m \in \mathbb{Z}_+$, the partition $\lambda_m = (m, \ldots, m)$ of $n := dm$, with $\text{height}(\lambda_m) = d$. By the hook formula,

$$f^{\lambda_m} = n!/\prod_{i=1}^{d}\prod_{j=1}^{m}(i+j-1) = \frac{(dm)!0!1!\cdots(d-1)!}{m!(m+1)!\cdots(m+d-1)!} = \frac{(dm)!}{m!^d p(m)}, \tag{18}$$

where (fixing d), $p(m)$ is a polynomial in m (namely $p(m) = \prod_{i=0}^{d-1}\binom{m+i}{i}$). So, by Stirling's formula, $\lim_{m\to\infty}(f^{\lambda_m})^{1/dm} = d$. By (12), we have for each m, since $\lambda_m \vdash dm$ and $\text{height}(\lambda_m) = d$,

$$\text{rank}(M_{dm}(d)) \geq (f^{\lambda_m})^2. \tag{19}$$

This gives the required inequality. ∎

3. Proof of Theorem 1

Necessity being easy, we show sufficiency. As $f(\emptyset) = 1$ and $\text{rank}(C_{f,0}) \leq f(O)^0 = 1$, we know that f is multiplicative. Moreover, as $\text{rank}(C_{f,1}) \leq f(O)$, we know $f(O) \geq 0$.

We develop some straightforward algebra. Let $k \in \mathbb{N}$. For $G, H \in \mathcal{G}_{2k}$, define the product GH as the $2k$-fragment obtained from the disjoint union of G and H by identifying vertex labeled $k+i$ in G with vertex labeled i in H, and ignoring this vertex as vertex (for $i = 1, \ldots, k$); the vertices of G labeled $1, \ldots, k$ and those of H labeled $k+1, \ldots, 2k$ make GH to a $2k$-labeled graph again.

Geometrically, one may imagine that the $2k$-fragments have the labels $1, \ldots, k$ vertically at the left and the labels $k+1, \ldots, 2k$ vertically at the right. Then GH arises by drawing G at the left from H and connecting the right-side labels of G with the left-side labels of H, in order.

Clearly, this product is associative. Moreover, there is a unit, denoted by $\mathbf{1}_k$, consisting of k disjoint edges $e_1, \ldots, e_k$, where the ends of e_i are labeled i and $k+i$ ($i = 1, \ldots, k$).

Let $\mathbb{C}\mathcal{G}_{2k}$ be the collection of formal $\mathbb{C}$-linear combinations of elements of G_{2k}. Extend the product $G \cdot H$ and GH bilinearly to $\mathbb{C}\mathcal{G}_{2k}$. The latter product makes $\mathbb{C}\mathcal{G}_{2k}$ to a $\mathbb{C}$-algebra.

Let $\mathcal{I}_{2k}$ be the kernel of the matrix $C_{f,2k}$, which we may consider as subset of $\mathbb{C}\mathcal{G}_{2k}$. Then $\mathcal{I}_{2k}$ is an ideal in the algebra $\mathbb{C}\mathcal{G}_{2k}$, and the quotient

$$\mathcal{A}_k := \mathbb{C}\mathcal{G}_{2k}/\mathcal{I}_{2k} \tag{20}$$

is an algebra of dimension $\text{rank}(C_{f,2k})$. We will indicate elements of $\mathcal{A}_k$ by representatives in $\mathbb{C}\mathcal{G}_{2k}$.

Define $\tau : \mathcal{A}_k \to \mathbb{C}$ by

$$\tau(x) := f(x \cdot \mathbf{1}_k). \tag{21}$$

Then $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{A}_k$.

Consider any $k, m \in \mathbb{N}$. For $x \in \mathcal{A}_k$, let $x^{\otimes m}$ be the element of $\mathcal{A}_{km}$ obtained by taking m disjoint copies $x^{(1)}, \ldots, x^{(m)}$ of x, and relabeling in copy $x^{(j)}$ label i to $i + (j-1)k$ and label $k+i$ to $km + i + (j-1)k$, for each $i = 1, \ldots, k$.

Geometrically, one may imagine this of putting m copies of x above each other, and renumbering the labels at the left and right side accordingly in order.

For $\pi \in S_m$, let $P_{k,\pi}$ be the $2km$-fragment consisting of km disjont edges $e_{i,j}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k$, where $e_{i,j}$ connects the vertices labeled $i + (j-1)m$ and $km + \pi(i) + (j-1)m$. Then for any $\rho, \sigma \in S_m$ one has

$$f(x^{\otimes m} P_{k,\rho} \cdot P_{k,\sigma}) = \prod_c \tau(x^{|c|}), \tag{22}$$

where c ranges over the orbits of $\rho\sigma^{-1}$.

Proposition 4. *If x is a nilpotent element of $\mathcal{A}_k$, then $\tau(x) = 0$.*

Proof. Suppose $\tau(x) \neq 0$ and x is nilpotent. Then there is a largest t with $\tau(x^t) \neq 0$. Let $y := x^t$. So $\tau(y) \neq 0$ and $\tau(y^s) = 0$ for each $s \geq 2$. By scaling we can assume that $\tau(y) = 1$.

Choose m such that $m! > f(O)^{2km}$. By (22) we have, for any $\rho, \sigma \in S_m$,

$$f(y^{\otimes m} P_{k,\rho} \cdot P_{k,\sigma}) = \delta_{\rho,\sigma}. \tag{23}$$

So $\text{rank}(C_{f,2km}) \geq m!$, contradicting the fact that $\text{rank}(C_{f,2km}) \leq f(O)^{2km} < m!$. ∎

The following is a direct consequence of Proposition 4:

Proposition 5. *$\mathcal{A}_k$ is semisimple.*

Proof. As $\mathcal{A}_k$ is finite-dimensional, it suffices to show that for each nonzero element x of $\mathcal{A}_k$ there is a y with xy not nilpotent. As $x \notin \mathcal{I}_{2k}$, we know that $f(x \cdot z) \neq 0$ for some $z \in \mathcal{A}_k$. So $\tau(xy) \neq 0$ for some $y \in \mathcal{A}_k$, and hence, by Proposition 4, xy is not nilpotent. ∎

Proposition 6. *If x is a nonzero idempotent in $\mathcal{A}_k$, then $\tau(x)$ is a positive integer.*

Proof. Let x be any idempotent. Then for each $m \in \mathbb{Z}_+$ and $\rho, \sigma \in S_m$, by (22):

$$f(x^{\otimes m} P_{k,\rho} \cdot P_{k,\sigma}) = \tau(x)^{o(\rho\sigma^{-1})}. \tag{24}$$

So for each m:

$$\text{rank}(M_m(\tau(x))) \leq \text{rank}(C_{f,km}) \leq f(O)^{2km}. \tag{25}$$

Hence

$$\sup_m (\text{rank}(M_m(\tau(x))))^{1/m} \leq f(O)^{2k}. \tag{26}$$

By Proposition 3 this implies $\tau(x) \in \mathbb{Z}$ and $\tau(x) \leq f(O)^k$. As $\mathbf{1}_k - x$ also is an idempotent in $\mathbb{C}\mathcal{G}_{2k}$ and as $\tau(\mathbf{1}_k) = f(O)^k$, we have

$$f(O)^k \geq \tau(\mathbf{1}_k - x) = f(O)^k - \tau(x). \tag{27}$$

So $\tau(x) \geq 0$.

Suppose finally that x is nonzero while $\tau(x) = 0$. As $\tau(y) \geq 0$ for each idempotent y, we may assume that x is a minimal nonzero idempotent. Let J be the two-sided ideal generated by x. As $\mathcal{A}_k$ is semisimple, $J \cong \mathbb{C}^{m\times m}$ for some m. As τ is linear, there exists an $a \in J$ such that $\tau(z) = \mathrm{tr}(za)$ for each $z \in J$. As $\tau(z) = 0$ for each nilpotent z, we know that a is a diagonal matrix. As $\tau(yz) = \tau(zy)$ for all $y, z \in J$, a is in fact equal to a scalar multiple of the identity matrix.

As $x \neq 0$, $f(x \cdot z) \neq 0$ for some $z \in \mathcal{A}_k$. So $\tau(xy) \neq 0$ for some y. Hence $a \neq 0$, and so $\tau(x) \neq 0$, contradicting our assumption. ∎

As $\mathbf{1}_1$ is an idempotent, we know that $\tau(\mathbf{1}_1)$ is a nonnegative integer, say n. So $f(O) = n$. Let $k := n+1$. For $\pi \in S_k$ let r_π be the $2k$-fragment consisting of k disjoint edges $e_1, \ldots, e_k$, where the ends of e_i are labeled i and $k+\pi(i)$, for $i = 1, \ldots, k$. (In fact, $r_\pi = P_{1,\pi}$ as defined above.) We define the following element q of $\mathbb{C}\mathcal{G}_{2k}$:

$$q := \sum_{\pi \in S_k} \mathrm{sgn}(\pi) r_\pi. \tag{28}$$

By (7) it suffices to show that $q \in \mathcal{I}_{2k}$, that is, $q = 0$ in $\mathcal{A}_k$.

Now $k!^{-1}q$ is an idempotent in $\mathbb{C}\mathcal{G}_{2k}$. Moreover,

$$\tau(q) = \sum_{\pi \in S_k} \mathrm{sgn}(\pi) n^{o(\pi)} = \sum_{\pi \in S_k} \mathrm{sgn}(\pi) \sum_{\substack{\phi:[k]\to[n] \\ \phi\circ\pi=\phi}} 1 = \sum_{\phi:[k]\to[n]} \sum_{\substack{\pi\in S_k \\ \phi\circ\pi=\phi}} \mathrm{sgn}(\pi) = 0, \tag{29}$$

since no $\phi : [k] \to [n]$ is injective. So by Proposition 6, $q = 0$ in $\mathcal{A}_k$, as required.

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