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Classification of Minimal Graphs of Given Face-Width on the Torus

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For any graph G embedded on the torus, the *face-width* $r(G)$ of G is the minimum number of intersections of G and C, where C ranges over all nonnullhomotopic closed curves on the torus. We call G *r-minimal* if $r(G) \geqslant r$ and $r(G') < r$ for each proper minor G' of G. We classify the r-minimal graphs by means of certain symmetric integer polygons in the plane $\mathbb{R}^2$. Up to a certain natural equivalence, the number of r-minimal graphs on the torus is equal to $\frac{1}{6}r^3 + \frac{5}{6}r$ if r is odd and to $\frac{1}{6}r^3 + \frac{4}{3}r$ if r is even.

1. INTRODUCTION

Let S be the torus. A closed curve on S is called *nontrivial* if it is not nullhomotopic. For any graph G embedded on S, the *face-width* (or *reresentativity*) $r(G)$ of G is the minimum of $|C \cap G|$, where C ranges over all nontrivial closed curves on S. (We identify a graph G embedded on a surface with its image.)

We call a graph G embedded on S *r-minimal* if $r(G) \geqslant r$ and $r(G') < r$ for each proper embedded minor G' of G. (A graph G' is called a *minor* of graph G if G' arises from G by deleting and contracting edges and by deleting isolated vertices. It is a *proper* minor if $G' \neq G$. If G and G' are embedded on a surface, then G' is called an *embedded minor* if we do not contract nontrivial loops and we maintain the embedding throughout (up to homotopy).)

It is easy to see that, for any fixed r, r-minimality is maintained under the following operations:

(i) replacing G by $\phi(G)$, where $\phi : S \to S$ is a homeomorphism;
(ii) replacing G by its surface dual; (1)
(iii) ΔY-exchange.

Here *ΔY-exchange* means replacing a triangular face by a vertex connected to the three vertices of the triangle, and conversely.

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The operations (1) imply an equivalence relation for r-minimal graphs (which we denote by $\sim$). In this paper we classify the equivalence classes. The classification is based on considering symmetric integer polygons related to graphs on the torus.

A *polygon* in $\mathbb{R}^2$ is the convex hull of a finite nonempty set of points in $\mathbb{R}^2$. (We do not require full dimensionality.) A polygon P in $\mathbb{R}^2$ is *integer* if all its vertices have integer coordinates only. Call P *symmetric* (*about the origin*) if $P = -P$. The *height* height(P) of a polygon P is defined by

$$\text{height}(P) := \min_{c \in \mathbb{Z}^2 \setminus \{(0,0)^T\}} \max\{c^T x \mid x \in P\}. \tag{2}$$

An integer polygon P is *r-minimal* if height$(P) \geqslant r$, while height$(P') < r$ for each integer polygon $P' \neq P$ contained in P. Two polygons P, P' are called *equivalent* (denoted by $P \sim P'$) if there exists a unimodular transformation $U: \mathbb{R}^2 \to \mathbb{R}^2$ such that $UP = P'$. (A *unimodular transformation* is a linear transformation U satisfying $U\mathbb{Z}^2 = \mathbb{Z}^2$. Equivalently, it is a linear transformation $x \to Ax$, where A is an integer matrix with determinant ± 1.) Note that the height of a polytope is invariant under equivalence.

We give, for each $r \geqslant 1$, a one-to-one relation between equivalence classes of r-minimal graphs on the torus and equivalence classes of symmetric r-minimal integer polygons in $\mathbb{R}^2$. For each fixed r there exist only finitely many such classes. We also give a description of the classes, yielding a formula for the number of equivalence classes of r-minimal graphs.

Remark 1. Scott Randby (personal communication) showed that for the projective plane, for each r, there is exactly one equivalence class of r-minimal graphs. We do not know an extension to nonorientable compact surfaces of higher genus.

Remark 2. Integer polygons. Any polygon P in $\mathbb{R}^2$ is fully determined by the function $f: \mathbb{Z}^2 \to \mathbb{R}$ defined by

$$f(c) := \max\{c^T x \mid x \in P\} \qquad \text{for} \quad c \in \mathbb{Z}^2. \tag{3}$$

It can be shown quite easily that P is integer if and only if f takes only integer values (this is a special case of a more general theorem of Hoffman [4]). In fact it is quite standard to see that for any function $f: \mathbb{Z}^2 \to \mathbb{Z}$, there exists a symmetric integer polygon P satisfying (3), if and only if f satisfies the following ("norm-type") conditions:

$$\begin{aligned} &\text{(i)} \quad f(c+c') \leqslant f(c) + f(c') &&\text{for all} \quad c, c' \in \mathbb{Z}^2; \\ &\text{(ii)} \quad f(k \cdot c) = |k| \cdot f(c) &&\text{for all} \quad k \in \mathbb{Z}, c \in \mathbb{Z}^2 \end{aligned} \tag{4}$$

(cf. [8]). If f satisfies (4), the corresponding polygon is:

$$P := \{x \in \mathbb{R}^2 \mid c^T x \leqslant f(c) \text{ for each } c \in \mathbb{Z}^2\}. \tag{5}$$

2. Integer Polygons Obtained from Graphs on the Torus

Since the torus S can be obtained from the plane $\mathbb{R}^2$ by identifying any two vectors x and y whenever $x-y$ is an integer vector, it is not surprising that the plane is of help in studying the torus. In particular, graphs on the torus can be studied with the help of polygons in $\mathbb{R}^2$ (cf. [8]).

We represent the torus S as the product $S^1 \times S^1$ of two copies of the unit circle S^1 in the complex plane $\mathbb{C}$. For $m, n \in \mathbb{Z}$, let $C_{m,n} : S^1 \to S^1 \times S^1$ be the closed curve on S defined by

$$C_{m,n}(z) := (z^m, z^n) \tag{6}$$

for $z \in S^1$. As is well known (cf. Stillwell [10]), the curves $C_{m,n}$ form a system of representatives for the homotopy classes of curves on the torus. We will say that C has *type* (m, n) if C is freely homotopic to $C_{m,n}$. (C is *freely homotopic to* C', in notation $C \sim C'$, if there exists a homotopic shift of C over the torus, bringing C to C', not fixing a "base point.")

For each graph G on the torus, let $f_G : \mathbb{Z}^2 \to \mathbb{Z}$ be defined by

$$f_G(m, n) := \min\{\mathrm{cr}(G, C) \mid C \sim C_{m,n}\} \tag{7}$$

for $(m, n)^T \in \mathbb{Z}^2$. Here $\mathrm{cr}(G, C)$ denotes the number of intersections of G and C, counting multiplicities.

It is not difficult to show that the function f_G satisfies (4). (The inequality in (i) follows from the fact that if $C \sim C_{m,n}$ and $C' \sim C_{m',n'}$ and $(m, n)^T$ and $(m', n')^T$ are linearly independent, then C and C' have a crossing. We can concatenate C and C' at this crossing so as to obtain a closed curve $C'' \sim C_{m+m',n+n'}$ with $\mathrm{cr}(G, C'') = \mathrm{cr}(G, C) + \mathrm{cr}(G, C')$.)

Hence, the set $P(G)$ defined by

$$P(G) := \{x \in \mathbb{R}^2 \mid c^T x \leqslant f_G(c) \text{ for each } c \in \mathbb{Z}^2\} \tag{8}$$

is a symmetric integer polygon. It satisfies

$$f_G(c) = \max\{c^T x \mid x \in P(G)\} \tag{9}$$

for each $c \in \mathbb{Z}^2$.

Note that $P(G)$ is full-dimensional (i.e., not a line segment and not a point) if and only if G is cellularly embedded. (A graph G is *cellularly embedded* if each face is an open disk, or equivalently, if $r(G) > 0$.) Moreover,

$$r(G) = \mathrm{height}(P(G)). \tag{10}$$

This follows from

$$r(G) = \min_{c \in \mathbb{Z}^2 \setminus \{(0,0)^T\}} f_G(c) = \min_{c \in \mathbb{Z}^2 \setminus \{(0,0)^T\}} \max\{c^T x \mid x \in P(G)\} = \text{height}(P) \tag{11}$$

by (7), (9), and (2).

The operation $G \to P(G)$ maintains equivalence:

THEOREM 1. *If graphs G and G' (embedded on the torus) are equivalent, then $P(G)$ and $P(G')$ are equivalent.*

Proof. The polygon $P(G)$ is trivially maintained under the operations (ii) and (iii) in (1) (a f_G is trivially maintained under these operations). Consider now a homeomorphism $\phi : S \to S$. Let $m, n, m', n' \in \mathbb{Z}$ be so that $\phi \circ C_{1,0} \sim C_{m,n}$ and $\phi \circ C_{0,1} \sim C_{m',n'}$. Then $(1,0)^T \mapsto (m,n)^T$, $(0,1)^T \mapsto (m',n')^T$ defines a unimodular transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$, since $\phi \circ C_{1,0}$ and $\phi \circ C_{0,1}$ have exactly one crossing, implying $|\det U| = 1$. Moreover, for each integer vector $c \in \mathbb{Z}^2$ one has $\phi \circ C_c \sim C_{Uc}$. Then for each $c \in \mathbb{Z}^2$: $f_G(c) = f_{\phi(G)}(Uc)$. So $P(\phi(G))$ arises by a unimodular transformation from $P(G)$. Therefore, $P(G)$ and $P(\phi(G))$ are equivalent. ▮

3. KERNELS

In studying r-minimal graphs, the concept of "kernel" introduced in [7] is helpful. A graph G embedded on the torus S is called a *kernel*, if $f_{G'} \neq f_G$ for each proper embedded minor G' of G.

It was shown in [7] that

> a cellularly embedded graph G on the torus is a kernel if and only if the medial graph $M(G)$ of G is the union of a minimally crossing system of simple nontrivial closed curves $D_1, ..., D_k$. (12)

Here we use the following terminology. For any graph G embedded on the torus, "the" *medial graph* $M(G)$ of G is "the" 4-regular graph obtained by putting a vertex on each edge of G, and by joining, for each vertex v of G, the vertices on the edges indicent with v, by edges so as to form a circuit, like the interrupted lines in Fig. 1.

Each 4-regular graph H, cellularly embedded on the torus S so that the faces can be bicolored, is a medial graph of some graph. Note that each two cellularly embedded graphs G and G' with $M(G) = M(G')$ can be

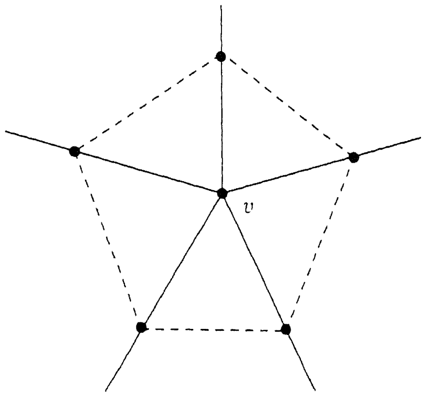

FIGURE 1

obtained from each other by homotopic shifts and taking surface duals. For obtaining the results below it is basic to observe that

$$f_G(m, n) = \tfrac{1}{2}\text{mincr}(M(G), C_{m,n}) \tag{13}$$

for each $c = (m, n)^T \in \mathbb{Z}^2$. Here mincr$(H, C)$ denotes the minimum number of crossings of H and C', where C' ranges over all closed curves freely homotopic to C so that C' does not traverse vertices of H.

Closed curves $D_1, \ldots, D_k$ form a *minimally crossing* system of closed curves if for all $i \neq i'$, D_i and $D_{i'}$ have a minimal number of intersections among all closed curves D and D' freely homotopic to D_i and $D_{i'}$, respectively. (So each intersection of D_i and $D_{i'}$ is a crossing (and not a touching).)

It was also shown in [7] that if G is a kernel and $D_1, \ldots, D_k$ are as in (12), then

$$f_G(m, n) = \tfrac{1}{2} \sum_{i=1}^{k} \text{mincr}(C_{m,n}, D_i) \tag{14}$$

for each $(m, n)^T \in \mathbb{Z}^2$. Here mincr$(C, D)$ denotes the minimum number of crossings (counting multiplicities) of C' and D', where C' and D' range over all closed curves freely homotopic to C and D, respectively.

Clearly, f_G and, hence, $P(G)$ are maintained under the following operations on graphs embedded on the torus:

$$\begin{array}{ll} \text{(i)} & \text{homotopic shifts of the graph over the torus;} \\ \text{(ii)} & \text{taking the surface dual;} \\ \text{(iii)} & \Delta\text{Y-exchange.} \end{array} \tag{15}$$

(We take the surface dual only if the graph is cellularly embedded.)

It was shown in [7] that

$$\text{if } G \text{ and } G' \text{ are kernels with } P(G)=P(G'), \text{ then } G' \text{ can be obtained from } G \text{ by the operations (15).} \tag{16}$$

(The reason is that the closed curve systems making up the medial graphs $M(G)$ and $M(G')$ can be moved to each other, using only "$\Delta\nabla$-exchange" as shown in Fig. 2. This induces ΔY-exchanges bringing G to G' (up to duality).)

This is a special case of a more general result for compact orientable surfaces. On the other hand, for the torus, a stronger statement can be proved. Let G and G' be graphs embedded on the torus. We call a graph G' a *ΔY-minor* of G if G' arises from some embedded minor of G by the operations (15) (maintaining the embedding throughout).

THEOREM 2. *Let G and G' be graphs embedded on the torus, where G is a kernel. Then G is a ΔY-minor of G' if and only if $P(G)\subseteq P(G')$.*

Proof. Necessity of the condition is easy, since $P(G)$ is maintained under the operations (15), while $P(G)\subseteq P(G')$ if G is a minor of G'.

To see sufficiency, assume that $P(G)\subseteq P(G')$. First let G be not cellularly embedded. Without loss of generality, there is a closed curve C of type $(1, 0)$ on the torus not intersecting G. Then each component of G should be a closed curve of type $(1, 0)$. So G consists of just some number k of pairwise disjoint simple closed curves of type $(1, 0)$. Then G is a ΔY-minor of G', if and only if G' contains k pairwise disjoint simple closed curves each freely homotopic to C. In [5] (cf. [9, 1]) it was shown that this last holds if and only if for each closed curve D one has $\mathrm{cr}(G', D)\geqslant k\cdot \mathrm{mincr}(C, D)$. This last is equivalent to $f_{G'}\geqslant f_G$, i.e., to $P(G')\supseteq P(G)$.

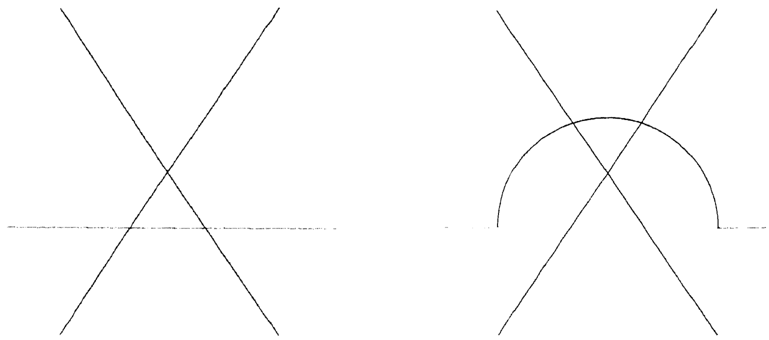

FIGURE 2

Next let G' be cellularly embedded (in particular, connected). Let $M(G)$ and $M(G')$ be the medial graphs of G and G', respectively. Since G is a kernel, by (12) $M(G)$ is the union of a minimally crossing system of simple nontrivial closed curves $D_1, ..., D_k$.

Now $P(G) \subseteq P(G')$ implies that $f_G(c) \leqslant f_{G'}(c)$ for each integer vector c, and hence by (13),

$$\text{mincr}(M(G), C) \leqslant \text{mincr}(M(G'), C) \tag{17}$$

for each closed curve C.

Combining (12), (14), and (17), the choice of the D_i gives

$$\text{mincr}(M(G'), C) \geqslant \sum_{i=1}^{k} \text{mincr}(C, D_i) \tag{18}$$

for each closed curve C on S.

It is shown in [2] that (18) is equivalent to the fact that $M(G')$ contains closed curves $D'_1, ..., D'_k$, so that no edge of $M(G')$ is traversed more than once and so that $D'_i \sim D_i$ for $i = 1, ..., k$. We may assume (cf. [6]) that $D'_1, ..., D'_k$ form a minimally crossing system of nontrivial closed curves, each without self-crossings.

In fact, we can assume that the system $D'_1, ..., D'_k$ traverses each edge of $M(G')$ exactly once. This can be seen as follows. We can decompose the edges of $M(G')$ not used by $D'_1, ..., D'_k$ into pairwise noncrossing, simple closed curves $D'_{k+1}, ..., D'_l$. Any trivial closed curve among $D'_{k+1}, ..., D'_l$ can be inserted in one of the other curves without increasing the total number of crossings. (This can be done since $M(G')$ is connected.) So we may assume that each of $D'_{k+1}, ..., D'_l$ is nontrivial. Since they are simple and pairwise noncrossing, they must be pairwise freely homotopic. Since $M(G')$ is a medial graph, each closed curve not traversing vertices of $M(G')$ has an even number of crossings with $M(G')$. Also, since $M(G)$ is a medial graph, each closed curve has an even number of crossings with $D_1, ..., D_k$, and hence with $D'_1, ..., D'_k$. So each closed curve has an even number of crossings with $D'_{k+1}, ..., D'_l$. So $l-k$ is even. Therefore, if $l > k$ we can insert D'_{k+1} and D'_{k+2} into one of the curves among $D'_1, ..., D'_k$, without changing its homotopy. Repetating this, we find $D'_1, ..., D'_k$ as required.

Now at any "touching" of two D'_i and D'_j (possibly $i = j$), we can "open" the graph as in Fig. 3.

Doing this at each touching we have transformed $M(G')$ to a graph H'' that is the union of a minimally crossing system of simple closed curves $D''_1, ..., D''_k$, with $D''_i \sim D'_i$ for $i = 1, ..., k$. Since openings of $M(G')$ correspond to deleting and contracting edges of G', H'' is the medial graph of some minor G'' of G'.

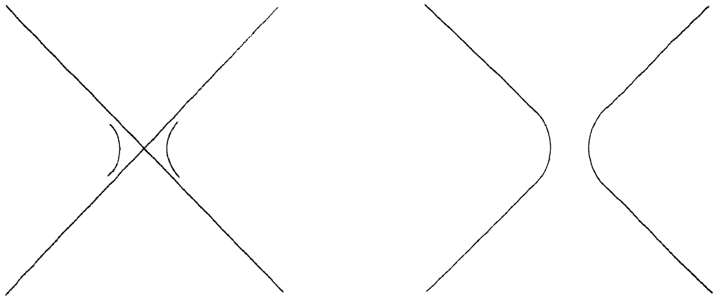

FIGURE 3

By (12), G'' is a kernel, and by (14), $f_{G''} = f_G$. So by (16), G'' arises by the operations (15) from G. So G is ΔY-minor of G'. ∎

This theorem states that for each function $f: \mathbb{Z}^2 \to \mathbb{Z}$ satisfying (4) there exists a unique minor-minimal graph among all graphs G with $f_G \geqslant f$—unique up to the operations (15). This is more general than (16), which states that there exists a unique minor-minimal graph among all graphs G with $f_G = f$.

We give a corollary on "toroidal grids." Let $k \geqslant 3$. The product $C_k \times C_k$ of two copies of the k-circuit C_k is called the *toroidal k-grid*. Clearly, the toroidal k-grid can be embedded on the torus, in fact (if $k \geqslant 4$) in a unique way, up to homeomorphisms of the torus and of the grid. Let H be the embedding of $C_k \times C_k$ on the torus, consisting of k disjoint circuits of type $(1, 0)$ crossed by k disjoint circuits of type $(0, 1)$.

By (16), H is a kernel. Since it is self-dual and does not allow ΔY-exchange (as all vertices have degree four and each face is bounded by four edges), Theorem 2 implies:

COROLLARY 2a. *Let G be a graph embedded on the torus. Then G contains H as an embedded minor, if and only if $P(G)$ contains $(k, 0)^T$ and $(0, k)^T$.*

Proof. The medial graph $M(H)$ of H is the union of a minimally crossing system of k simple closed curves of type $(1, 1)$ and k simple closed curves of type $(1, -1)$. Hence

$$\operatorname{mincr}(M(H), C_{m,n}) = k\,|m+n| + k\,|m-n|. \tag{19}$$

So

$$f_H(m, n) = \tfrac{1}{2}k(|m+n| + |m-n|) = k \cdot \max\{|m|, |n|\}. \tag{20}$$

Hence $P(H)$ is the convex hull of $\pm(k, 0)^T$ and $\pm(0, k)^T$. So $P(H)$ is contained in $P(G)$ if and only if $P(G)$ contains $(k, 0)^T$ and $(0, k)^T$, and hence Theorem 2 implies the corollary. ∎

This directly gives:

COROLLARY 2b. *Let G be a graph embedded on the torus, and let* $k \geqslant 3$. *Then G contains a toroidal k-grid as a minor, if and only if* $(1/k)\, P(G)$ *contains two linearly independent integer vectors.*

Proof. Directly from Corollary 2a. ∎

In [3] we derive from this result that every graph G embedded on the torus contains a toroidal $\lfloor \frac{2}{3} r(G) \rfloor$-grid minor.

4. KERNELS OBTAINED FROM SYMMETRIC INTEGER POLYGONS

Above we saw that each graph G on the torus gives a symmetric integer polygon $P(G)$ in $\mathbb{R}^2$. We now show conversely that for each symmetric integer polygon P in $\mathbb{R}^2$ there exists a graph G such that $P(G) = P$. So there exists a kernel G with $P(G) = P$, which should be unique by (16). We give a construction.

Let P be a symmetric integer polygon in $\mathbb{R}^2$. We first construct an infinite graph Γ_P embedded in $\mathbb{R}^2$ as follows.

Let $v_1, ..., v_{2k}$ be the vertices of P, in counterclockwise order. (So $v_{j+k} = -v_j$ for $j = 1, ..., k$.) If P just consists of the origin, we take $k = 0$. For each $i = 1, ..., k$ let

$$L_i := \{x \in \mathbb{R}^2 \mid (v_{i+1} - v_i)^T x \in \mathbb{Z}\}. \tag{21}$$

So L_i consists of a collection of parallel lines, each orthogonal to the vector $v_{i+1} - v_i$. Define

$$\Gamma_P := L_1 \cup \cdots \cup L_k. \tag{22}$$

Then Γ_P is a graph in $\mathbb{R}^2$, generally with an infinite set of vertices. The vertex set is the set of crossings of different classes L_i.

We can obtain the torus S from $\mathbb{R}^2$ by identifying any two vectors x, y whenever $x - y$ is an integer vector. Since Γ_P is invariant under translations by an integer vector, this identification makes Γ_P to a graph, denoted by H_P, embedded in the torus S.

The faces of H_P can be colored black and white so that adjacent faces have different colors. This follows from the fact that we can color the faces of Γ_P black and white so that adjacent faces have different colors and so that the coloring is invariant under translations by an integer vector. (Color $x \in \mathbb{R}^2 \backslash \Gamma_P$ black if $\sum_{i=1}^{k} \lfloor (v_{i+1} - v_i)^T x \rfloor$ is even, and white if this sum is odd. Here $\lfloor \quad \rfloor$ denotes lower integer part.)

Let H'_P arise from H_P by "rerouting" slightly the "curves" traversing any vertex of H_P of degree larger than four, in such a way that each point of S is traversed by not more than two of the curves, not introducing any new crossings. E.g., a vertex of degree 10 can be changed as in Fig. 4. Note that also after rerouting, the faces of the graph can be bicolored. Hence it is a medial graph again.

If P is full-dimensional, then H'_P is 4-regular and cellularly embedded. Then we define G_P as some (arbitrary) graph satisfying $M(G_P) = H'_P$.

If P is not full-dimensional, H_P consists of a number $2t$ of pairwise disjoint nontrivial closed curves on S, each freely homotopic to some curve C, say. In this case G_P will be a graph consisting of t pairwise disjoint nontrivial closed curves each freely homotopic to C. In fact, if P has vertices v_1 and v_2 with $v_2 = -v_1$ then we can take $C = C_{m,n}$, where $(m, n)^T$ is any integer vector orthogonal to v_1 with m and n relative prime. If P only consists of the origin, then H_P and G_P are empty.

It can be derived from (12) that G_P indeed is a kernel (as H'_P consists of a system of closed curves that are minimally crossing). In fact:

THEOREM 3. *G_P is a kernel with $P(G_P) = P$.*

Proof. We must show $P(G_P) = P$, or equivalently,

$$f_{G_P}(c) = \max\{c^T x \mid x \in P\} \tag{23}$$

for all $c \in \mathbb{Z}^2$.

Choose $c = (m, n)^T \in \mathbb{Z}^2$. By symmetry we may assume that $\max\{c^T x \mid x \in P\}$ is attained at vertex v_1. So $c^T v_1 \geqslant c^T v_2 \geqslant \cdots \geqslant c^T v_{k+1}$.

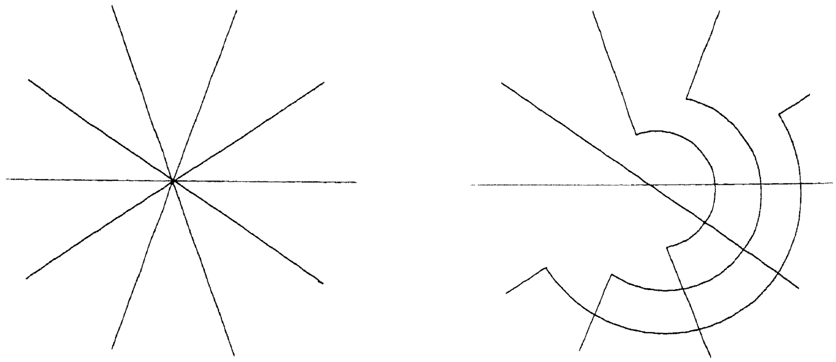

FIGURE 4

Let B be any curve in $\mathbb{R}^2$ connecting vectors y and y' with $y'-y=c$, in such a way that B does not traverse any vertex of Γ_P and has end points not in Γ_P. Then by the construction of Γ_P, B should cross at least

$$\sum_{i=1}^{k} |c^T(v_{i+1}-v_i)| = \sum_{i=1}^{k} c^T(v_i - v_{i+1}) = c^T v_1 - c^T v_{k+1} = 2c^T v_1 \quad (24)$$

edges of Γ_P. This follows from the fact that for each $i=1, ..., k$ there are $|c^T(v_{i+1}-v_i)|$ integer values between $(v_{i+1}-v_i)^T y$ and $(v_{i+1}-v_i)^T y'$. Hence there are $|c^T(v_{i+1}-v_i)|$ lines in L_i separating y and y'.

So the projection of B onto the torus S (under the quotient map) should cross at least $2c^T v_1$ edges of H_P. As this minimum can be attained by taking for B a straight line segment, we know that $\text{mincr}(H_P, C_{m,n}) = 2c^T v_1$. Hence by (13), $f_G(c) = \frac{1}{2}\text{mincr}(H_P, C_{m,n}) = c^T v_1$. ∎

Remark 3. One can show that for any symmetric integer polygon P, the number of edges of the kernel G_P is equal to the area of P. To see this, let $v_1, ..., v_{2k}$ be the vertices of P in counterclockwise order and let L_i again be defined as in (21). For $i=1, ..., k$ let $w_i := v_{i+1} - v_i$. Define $R := [0, 1) \times [0, 1)$. Then for any i, j satisfying $1 \leqslant i < j \leqslant k$ one has

$$\text{the number of crossings of } L_i \text{ and } L_j \text{ in } R \text{ is equal to } \det[w_i, w_j]. \quad (25)$$

This can be seen by observing that the set $L_i \cap L_j$ forms the dual lattice of the lattice Λ generated by w_i and w_j. (Note that $\det[w_i, w_j] > 0$ since we have chosen $v_1, ..., v_{2k}$ in counterclockwise order.) Standard lattice theory then implies that $|L_i \cap L_j \cap R|$ equals the determinant of Λ, which is equal to $\det[w_i, w_j]$.

This shows (25). It implies that the number of vertices of H'_P is equal to

$$\sum_{1 \leqslant i < j \leqslant k} \det[w_i, w_j]. \quad (26)$$

As each vertex of H'_P gives an edge of G_P, the number of edges of G_P is equal to (26).

On the other hand, by elementary geometry and induction on k one shows that the area $\text{area}(P)$ of P is also equal to (26). Indeed, let P' be the polygon with vertices $v_1, ..., v_{k-1}, -v_1, ..., -v_{k-1}$. By induction we know

$$\text{area}(P') = \sum_{1 \leqslant i < j \leqslant k-2} \det[w_i, w_j] + \sum_{1 \leqslant i \leqslant k-2} \det[w_i, w_{k-1} + w_k]. \quad (27)$$

Now polygon P' arises from P by splitting off two triangles of area $\frac{1}{2}\det[w_{k-1}, w_k]$ each. Hence

$$\begin{aligned}\operatorname{area}(P) &= \operatorname{area}(P') + \det[w_{k-1}, w_k] \\ &= \sum_{1 \leqslant i < j \leqslant k-2} \det[w_i, w_j] \\ &\quad + \sum_{1 \leqslant i \leqslant k-2} \det[w_i, w_{k-1} + w_k] + \det[w_{k-1}, w_k] \\ &= \sum_{1 \leqslant i < j \leqslant k} \det[w_i, w_j]. \end{aligned} \tag{28}$$

This gives the required equality. ∎

5. Equivalence of Polygons and of Graphs

The equivalence relation of graphs on the torus is strongly related to the equivalence relation of symmetric integer polygons:

Theorem 4. *Two symmetric integer polygons P and P' are equivalent, if and only if the graphs G_P and $G_{P'}$ are equivalent.*

Proof. Let P and P' be two equivalent symmetric integer polygons. Let U be a unimodular transformation bringing P to P'. Then it is not difficult to check that there exists a homeomorphism $\phi: S \to S$ bringing G_P to $G_{P'}$.

Conversely, if G_P and $G_{P'}$ are equivalent, then by Theorem 1 $P(G_P)$ and $P(G_{P'})$ are equivalent. Since $P = P(G_P)$ and $P' = P(G_{P'})$ it follows that P and P' are equivalent. ∎

Theorem 4 implies that there exists a one-to-one relation between equivalence classes of kernels on the torus and equivalence classes of symmetric integer polygons in $\mathbb{R}^2$, given by

$$\begin{aligned} &\text{(i)} \quad \langle G \rangle \mapsto \langle P(G) \rangle, && \text{where } G \text{ is a kernel;} \\ &\text{(ii)} \quad \langle P \rangle \mapsto \langle G_P \rangle, && \text{where } P \text{ is a symmetric integer polygon.} \end{aligned} \tag{29}$$

Here $\langle ... \rangle$ denotes the equivalence class of ....

This brings us to the classification of equivalence classes of r-minimal graphs. Let $\mathscr{P}_r$ denote the collection of all symmetric r-minimal integer polygons.

Theorem 5. *For each $P \in \mathscr{P}_r$ the graph G_P is r-minimal. Each r-minimal graph is equivalent to G_P for some $P \in \mathscr{P}_r$.*

Proof. Let $P \in \mathscr{P}_r$. Then by (10),

$$r(G_P) = \text{height}(P(G_P)) = \text{height}(P) \geqslant r. \tag{30}$$

For each proper embedded minor G' of G_P one has $P(G') \neq P$, implying that

$$r(G') = \text{height}(P(G')) < \text{height}(P) = r. \tag{31}$$

So G_P is r-minimal.

Let G be an r-minimal graph. Then $P(G)$ is r-minimal. For suppose not. Then $P(G)$ contains a symmetric integer polygon $P' \neq P(G)$ with $\text{height}(P') = r$. By Theorem 2, G contains a minor G' that arises by the operations (15) from $G_{P'}$. Since $P(G') = P(G_{P'}) = P' \neq P(G)$, G' is a proper minor of G. However,

$$r(G') = \text{height}(P(G')) = \text{height}(P') = r, \tag{32}$$

contradicting the r-minimality of G. ∎

6. r-Minimal Integer Polygons

Fix $r \geqslant 1$. We give a construction of symmetric r-minimal integer polygons. Each of them is either a quadrangle or a hexagon. For any choice of integers α, β satisfying $0 \leqslant \alpha < r$ and $0 \leqslant \beta < r$, let $Q_{\alpha,\beta}$ be the convex hull of the four points $\pm(r, \alpha)^T$, $\pm(-\beta, r)^T$. For any choice of integers α, β, γ satisfying $0 < \alpha < r$, $0 < \beta < r$, and $0 < \gamma < r$, let $H_{\alpha,\beta,\gamma}$ be the convex hull of the six points $\pm(r, \alpha)^T$, $\pm(r-\beta, r)^T$, $\pm(-\gamma, r-y)^T$. (Note that the definitions of $Q_{\alpha,\beta}$ and $H_{\alpha,\beta,\gamma}$ depend on r.)

THEOREM 6. *Each $Q_{\alpha,\beta}$ belongs to $\mathscr{P}_r$.*

Proof. To show that $\text{height}(Q_{\alpha,\beta}) \geqslant r$, let $(c, d)^T$ be a nonzero integer vector. We show that

$$\max\{(c, d)(x, y)^T \mid (x, y)^T \in Q_{\alpha,\beta}\} \geqslant r. \tag{33}$$

We may assume that the last nonzero component in (c, d) is positive. If $c \geqslant 1$ then $d \geqslant 0$, implying that $(c, d)(r, \alpha)^T \geqslant cr \geqslant r$. If $c \leqslant 0$ then $d \geqslant 1$, implying that $(c, d)(-\beta, r)^T \geqslant dr \geqslant r$.

Since for $(c, d) := (1, 0)$ and $(c, d) := (0, 1)$, the maximum is r, which is uniquely attained at $(r, \alpha)^T$ and $(-\beta, r)^T$, respectively, $Q_{\alpha,\beta}$ is r-minimal. ∎

THEOREM 7. *Each $H_{\alpha,\beta,\gamma}$ belongs to $\mathscr{P}_r$.*

Proof. To show that height$(H_{\alpha,\beta,\gamma}) \geqslant r$, let (c, d) be a nonzero integer vector. We show

$$\max\{(c, d)(x, y)^T \mid (x, y)^T \in H_{\alpha,\beta,\gamma}\} \geqslant r. \tag{34}$$

We may assume that the last nonzero component in $(c, d, c+d)$ is positive. If $c+d \geqslant 1$ and $c \geqslant 1$ then

$$(c, d)(r, \alpha)^T \geqslant (c, 1-c)(r, \alpha)^T = (c-1)(r-\alpha) + r \geqslant r. \tag{35}$$

If $c+d \geqslant 1$ and $d \geqslant 1$ then

$$(c, d)(r-\beta, r)^T \geqslant (1-d, d)(r-\beta, r)^T = (d-1)\beta + r \geqslant r. \tag{36}$$

If $c+d=0$ then $d \geqslant 1$, implying that

$$(c, d)(-\gamma, r-\gamma)^T = dr \geqslant r. \tag{37}$$

Since for $(c, d) := (1, 0)$, $(c, d) := (0, 1)$, and $(c, d) := (-1, 1)$ the maximum is r, which is uniquely attained at $(r, \alpha)^T$, $(r-\beta, r)^T$, and $(-\gamma, r-\gamma)^T$, respectively, $H_{\alpha,\beta,\gamma}$ is r-minimal. ∎

THEOREM 8. *Each polygon in $\mathscr{P}_r$ is equivalent to at least one of the $Q_{\alpha,\beta}$, $H_{\alpha,\beta,\gamma}$.*

Proof. I. Let $P \in \mathscr{P}_r$. We first show that for each vertex v of P

there exists a nonzero integer vector c such that $c^T v = r$ and $c^T x \leqslant r-1$ for each integer vector $x \neq v$ in P. (38)

Indeed, by the r-minimality of P there exists a nonzero integer vector d such that $d^T v = r' \geqslant r$ and $d^T x \leqslant r-1$ for each integer vector $x \neq v$ in P. If $r' = r$ we are done, so suppose $r' > r$. We may assume that the components of d are relatively prime (otherwise we could divide d by the g.c.d. of the components), and therefore we may assume that $d = (1, 0)^T$. (There exists a unimodular transformation bringing any integer vector with relatively prime components to $(1, 0)^T$. Note that (38) is preserved under any unimodular transformation.) So $v = (r', \lambda)^T$ for some λ.

If there does not exist an $i \in \{1, ..., r'-r\}$ such that $i\lambda - \lfloor i\lambda/r' \rfloor r' \leqslant r$, then by the pigeonhole principle there exist $i < j$ in $\{1, ..., r'-r\}$ such that $i\lambda - \lfloor i\lambda/r' \rfloor r' = j\lambda - \lfloor j\lambda/r' \rfloor r'$ (since each $i\lambda - \lfloor i\lambda/r' \rfloor r'$ would be in $\{r+1, ..., r'-1\}$). Then

$$x := \begin{pmatrix} r'-j+i \\ \lambda - \lfloor j\lambda/r' \rfloor + \lfloor i\lambda/r' \rfloor \end{pmatrix} = \frac{r'-j+i}{r'} \begin{pmatrix} r' \\ \lambda \end{pmatrix} = \frac{r'-j+i}{r'} v \tag{39}$$

would be an integer vector $x \neq v$ in P. Since $j-i \leqslant r'-r-1$, it follows that

$$d^T x = r' - j_i \geqslant r' - (r'-r-1) = r+1 > r, \tag{40}$$

contradicting our assumption.

If there does exist an $i \in \{1, ..., r'-r\}$ such that $i\lambda - \lfloor i\lambda/r' \rfloor r' \leqslant r$, then let

$$c := \begin{pmatrix} -\lfloor i\lambda/r' \rfloor \\ i \end{pmatrix}. \tag{41}$$

So $c^T v \leqslant r$. We show that $c^T x \leqslant r-1$ for each integer vector $x \neq v$ in P, thus proving (38).

Suppose $x \neq v$ is an integer vector in P with $c^T x \geqslant r$. Let x' be the point on the line segment connecting x and the origin such that $c^T x' = c^T v$. Now consider the point

$$u := v - \begin{pmatrix} i \\ \lfloor i\lambda/r' \rfloor \end{pmatrix}. \tag{42}$$

Then $c^T u = c^T v$. So u, v, and x' are on a line. Now $d^T v = r' > r$, $r \leqslant d^T u < r'$ (since $d^T u = r' - i$) and $d^T x' \leqslant r-1$ (since if $d^T x' \geqslant 0$ then $d^T x' \leqslant d^T x \leqslant r-1$ and if $d^T x' < 0$ then $d^T x' < 0 \leqslant r-1$). So u is on the line segment connecting v and x'. This implies that u belongs to P, contraditing the fact that u is an integer vector with $u \neq v$ and $d^T u \geqslant r$.

II. We next show the theorem. Let $v_1, ..., v_{2k}$ be the vertices of P, in counterclockwise order (so $v_{j+k} = -v_j$ for $j = 1, ..., k$). Write $v_j = (v'_j, v''_j)^T$ for $j = 1, ..., 2k$.

By (38), for each $j = 1, ..., 2k$, there exists an integer vector c_j satisfying $c_j^T v_j = r$ and $c_j^T x < r$ for all $x \neq v_j$ in P. We may assume that $c_{j+k} = -c_j$.

Then for each two distinct j, j' from $\{1, ..., k\}$ one has $\det(c_j, c_{j'}) = \pm 1$. Otherwise, the triangle with vertices $c_j, c_{j'}, 0$ would contain a nonzero integer vector d with $d \neq c_j$ and $d \neq c_{j'}$. Then we would have $d^T x < r$ for each vector x in P. This contradicts the fact that height$(P) \geqslant r$.

We may assume that $c_1 = (1, 0)^T$ and $c_2 = (0, 1)^T$. So $v'_1 = r$, $|v''_1| < r$ and $v''_2 = r$, $|v'_2| < r$. Moreover, each c_j with $3 \leqslant j \leqslant k$ should be equal to $(\pm 1, \pm 1)^T$, since $\det(c_1, c_j) = \pm 1$ and $\det(c_2, c_j) = \pm 1$. Hence $k \leqslant 3$, since if $k \geqslant 4$ then $\det(c_3, c_4) = \pm 2$.

If $k = 2$, then $v'_2 v''_1 \leqslant 0$. For suppose $v'_2 v''_1 > 0$. If $v'_2 < 0$ and $v''_1 < 0$ then $\max\{x' + x'' \mid x \in P\} < r$, as this maximum is attained at v_1 or at v_2, while $v'_1 = v''_2 = r$. This contradicts the fact that height$(P) \geqslant r$. Similarly, if $v'_2 > 0$ and $v''_1 > 0$ then $-v'_2 < 0$ and $-v''_1 < 0$, and, hence, $\max\{-x' + x'' \mid x \in P\} < r$, as this maximum is attained at v_2 or at v_3, while $v''_2 = -v'_3 = r$.

This shows $v_2'v_1'' \leqslant 0$. By symmetry we may assume $v_1'' \geqslant 0$ and $v_2' \leqslant 0$. So $P = Q_{\alpha,\beta}$ for $\alpha := v_1''$ and $\beta := -v_2'$.

If $k=3$, then we may assume that $c_3 = (-1, 1)^T$. Then $v_1' = r$, $|v_1''| < r$, $|v_1'' - v_1'| < r$, $v_2'' = r$, $|v_2'| < r$, $|v_2'' - v_2'| < r$, and $v_3'' - v_3' = r$, $|v_3'| < r$, $|v_3''| < r$. So $P = H_{\alpha,\beta,\gamma}$ for $\alpha := v_1''$, $\beta := r - v_2'$, $\gamma := -v_3'$. ∎

7. Counting Equivalence Classes

As above, two polygons Q, Q' are called *equivalent* (denoted by $Q \sim Q'$) if there exists a unimodular transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$ such that $UQ = Q'$. It follows directly from Theorem 8 that:

THEOREM 9. *For each r, the number of equivalence classes in $\mathscr{P}_r$ is finite.*

Proof. Directly from Theorem 8, since the number of $Q_{\alpha,\beta}$ and $H_{\alpha,\beta,\gamma}$ is finite. ∎

In fact, an explicit formula for the number of equivalence classes can be given. First we count those containing quadrangles. To this end we first note:

THEOREM 10. *For any $0 \leqslant \alpha, \alpha' < r$ and $0 \leqslant \beta, \beta' < r$, $Q_{\alpha,\beta}$ is equivalent to $Q_{\alpha',\beta'}$ if and only if*

(i) $\{\alpha, \beta\} = \{\alpha', \beta'\}$, *or*

(ii) $\{\alpha, \beta\} = \{0, \gamma\}$, $\{\alpha', \beta'\} = \{0, r-\gamma\}$ *for some* γ.

Proof. Sufficiency is easy. (Note that $Q_{0,\gamma}$ goes to $Q_{0,r-\gamma}$ by the unimodular transformation $(x, y)^T \to (x+y, -y)^T$.)

To see necessity, first observe that the unimodular transformation $T_1 : (x, y)^T \to (-y, x)^T$ brings

$$\begin{pmatrix} r \\ \alpha \end{pmatrix}, \begin{pmatrix} -\beta \\ r \end{pmatrix}, \begin{pmatrix} -r \\ -\alpha \end{pmatrix}, \begin{pmatrix} \beta \\ -r \end{pmatrix}, \tag{43}$$

to

$$\begin{pmatrix} -\alpha \\ r \end{pmatrix}, \begin{pmatrix} -r \\ -\beta \end{pmatrix}, \begin{pmatrix} \alpha \\ -r \end{pmatrix}, \begin{pmatrix} r \\ \beta \end{pmatrix}, \tag{44}$$

respectively. In particular, it brings $Q_{\alpha,\beta}$ to $Q_{\beta,\alpha}$.

Moreover, the unimodular transformation $T_2 : (x, y)^T \to (x+y, -y)^T$ brings $Q_{0,\gamma}$ to $Q_{0,r-\gamma}$.

Next let $Q_{\alpha,\beta}$ be equivalent to $Q_{\alpha',\beta'}$. Let U be a unimodular matrix bringing $Q_{\alpha,\beta}$ to $Q_{\alpha',\beta'}$. We may assume that U brings $(r,\alpha)^T$ to $(r,\alpha')^T$. (Otherwise, replace U by T_1U or $-T_1U$.) Then U brings $(-\beta, r)^T$ either to $(-\beta', r)^T$ or to $(\beta', -r)^T$.

If U brings $(-\beta, r)^T$ to $(-\beta', r)^T$, then the matrix corresponding to U is

$$\begin{pmatrix} r & -\beta' \\ \alpha' & r \end{pmatrix}\begin{pmatrix} r & -\beta \\ \alpha & r \end{pmatrix}^{-1} = \frac{1}{r^2+\alpha\beta}\begin{pmatrix} r^2+\alpha\beta' & r(\beta-\beta') \\ r(\alpha'-\alpha) & r^2+\alpha'\beta \end{pmatrix}. \tag{45}$$

Since this is an integer matrix, $r^2+\alpha\beta$ should divide both $r(\alpha'-\alpha)$ and $r(\beta-\beta')$. So $\alpha=\alpha'$ and $\beta=\beta'$.

If U brings $(-\beta, r)^T$ to $(\beta', -r)^T$, then the matrix corresponding to U is

$$\begin{pmatrix} r & \beta' \\ \alpha' & -r \end{pmatrix}\begin{pmatrix} r & -\beta \\ \alpha & r \end{pmatrix}^{-1} = \frac{1}{r^2+\alpha\beta}\begin{pmatrix} r^2-\alpha\beta' & r(\beta+\beta') \\ r(\alpha'+\alpha) & -r^2+\alpha'\beta \end{pmatrix}. \tag{46}$$

Again since this is an integer matrix, $r^2+\alpha\beta$ should divide both $r(\alpha+\alpha')$ and $r(\beta+\beta')$. So $\alpha\beta=0$, and both $\alpha+\alpha'$ and $\beta+\beta'$ belong to $\{0, r\}$. This yields (ii) or (if $\alpha=\beta=\alpha'=\beta'=0$) (i). ∎

This implies:

THEOREM 11. *For each fixed r, the number of equivalence classes in $\mathscr{P}_r$ consisting of quadrangles is equal to $\frac{1}{2}r^2+\frac{1}{2}$ if r is odd and to $\frac{1}{2}r^2+1$ if r is even.*

Proof. From Theorem 10 it follows that the number of classes is equal to the number of sets $\{\alpha, \beta\}$ with $\alpha, \beta \in \{1, ..., r-1\}$ (possibly $\alpha=\beta$), plus the number of sets $\{0, \beta\}$ with $\beta \in \{0, ..., \lfloor r/2 \rfloor\}$. This number is equal to $r-1+\frac{1}{2}(r-1)(r-2)+1+\lfloor r/2 \rfloor$, which equals the values given in the theorem. ∎

Next we count classes containing hexagons. We first show:

THEOREM 12. *For any $0<\alpha, \alpha', \beta, \beta', \gamma, \gamma'<r$, $H_{\alpha,\beta,\gamma}$ is equivalent to $H_{\alpha',\beta',\gamma'}$ if and only if $(\alpha', \beta', \gamma')$ is a cyclic permutation of (α, β, γ) or of $(r-\gamma, r-\beta, r-\alpha)$.*

Proof. First observe that the unimodular transformation $T_3 : (x, y)^T \to (x-y, x)^T$ brings

$$\begin{pmatrix} r \\ \alpha \end{pmatrix}, \begin{pmatrix} r-\beta \\ r \end{pmatrix}, \begin{pmatrix} -\gamma \\ r-\gamma \end{pmatrix}, \begin{pmatrix} -r \\ -\alpha \end{pmatrix}, \begin{pmatrix} \beta-r \\ -r \end{pmatrix}, \begin{pmatrix} \gamma \\ \gamma-r \end{pmatrix} \tag{47}$$

to

$$\begin{pmatrix} r-\alpha \\ r \end{pmatrix}, \begin{pmatrix} -\beta \\ r-\beta \end{pmatrix}, \begin{pmatrix} -r \\ -\gamma \end{pmatrix}, \begin{pmatrix} \alpha-r \\ -r \end{pmatrix}, \begin{pmatrix} \beta \\ \beta-r \end{pmatrix}, \begin{pmatrix} r \\ \gamma \end{pmatrix}, \tag{48}$$

respectively. In particular, it brings $H_{\alpha,\beta,\gamma}$ to $H_{\gamma,\alpha,\beta}$.

Moreover, the transformation $T_4 : (x, y)^T \to (y, x)^T$ brings

$$\begin{pmatrix} r \\ \alpha \end{pmatrix}, \begin{pmatrix} r-\beta \\ r \end{pmatrix}, \begin{pmatrix} -\gamma \\ r-\gamma \end{pmatrix}, \begin{pmatrix} -r \\ -\alpha \end{pmatrix}, \begin{pmatrix} \beta-r \\ -r \end{pmatrix}, \begin{pmatrix} \gamma \\ \gamma-r \end{pmatrix} \tag{49}$$

to

$$\begin{pmatrix} \alpha \\ r \end{pmatrix}, \begin{pmatrix} r \\ r-\beta \end{pmatrix}, \begin{pmatrix} r-\gamma \\ -\gamma \end{pmatrix}, \begin{pmatrix} -\alpha \\ -r \end{pmatrix}, \begin{pmatrix} -r \\ \beta-r \end{pmatrix}, \begin{pmatrix} \gamma-r \\ \gamma \end{pmatrix}, \tag{50}$$

respectively. In particular, it brings $H_{\alpha,\beta,\gamma}$ to $H_{r-\beta,r-\alpha,r-\gamma}$. This shows sufficiency.

To see necessity, let U be a unimodular transformation bringing $H_{\alpha,\beta,\gamma}$ to $H_{\alpha',\beta',\gamma'}$. We may assume that it brings

$$\begin{pmatrix} r \\ \alpha \end{pmatrix}, \begin{pmatrix} r-\beta \\ r \end{pmatrix} \text{to} \begin{pmatrix} r \\ \alpha' \end{pmatrix}, \begin{pmatrix} r-\beta' \\ r \end{pmatrix}, \qquad \text{respectively.} \tag{51}$$

(Otherwise, multiply U by an appropriate combination of T_3 and T_4.) Hence the matrix corresponding to U is

$$\begin{pmatrix} r & r-\beta' \\ \alpha' & r \end{pmatrix} \begin{pmatrix} r & r-\beta \\ \alpha & r \end{pmatrix}^{-1} = \frac{1}{r^2 - r\alpha + \alpha\beta} \begin{pmatrix} r^2 - r\alpha + \alpha\beta' & r(\beta-\beta') \\ r(\alpha'-\alpha) & r^2 - r\alpha' + \alpha'\beta \end{pmatrix}. \tag{52}$$

As this is an integral matrix and as $r^2 - r\alpha + \alpha\beta' > 0$ we know that $r^2 - r\alpha + \alpha\beta' \geqslant r^2 - r\alpha + \alpha\beta$. Hence $\beta' \geqslant \beta$.

Similarly, as $r^2 - r\alpha' + \alpha'\beta > 0$, we know that $r^2 - r\alpha' + \alpha'\beta \geqslant r^2 - r\alpha + \alpha\beta$, that is, $(r-\beta)\alpha' \leqslant (r-\beta)\alpha$. So $\alpha' \leqslant \alpha$.

On the other hand, $r^2 - r\alpha' + \alpha'\beta' = r^2 - r\alpha + \alpha\beta$ (since matrix (52) has determinant ± 1). So $(r-\beta')\alpha' = (r-\beta)\alpha$. Since $r-\beta' \leqslant r-\beta$ and $\alpha' \leqslant \alpha$, we have $\alpha = \alpha'$ and $\beta = \beta'$. So U is the identity transformation. ∎

This implies:

THEOREM 13. *For each fixed r, the number of classes in* $\mathscr{P}_r$ *consisting of hexagons is equal to* $\frac{1}{6}r^3 - \frac{1}{2}r^2 + \frac{5}{6}r - \frac{1}{2}$ *if r is odd and to* $\frac{1}{6}r^3 - \frac{1}{2}r^2 + \frac{4}{3}r - 1$ *if r is even.*

Proof. We use Theorem 12. If α, β, γ are distinct and $\{\alpha, \beta, \gamma\} \neq \{r-\alpha, r-\beta, r-\gamma\}$ then there exist six triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha',\beta',\gamma'} \sim Q_{\alpha,\beta,\gamma}$. The number of such triples (α, β, γ) is equal to $(r-1)(r-2)(r-3)$ if r is odd and to $(r-1)(r-2)(r-3) - 3(r-2)$ if r is even.

If α, β, γ are distinct and $\{\alpha, \beta, \gamma\} = \{r-\alpha,\ r-\beta,\ r-\gamma\}$ then there exist three triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha', \beta', \gamma'} \sim Q_{\alpha, \beta, \gamma}$. The number of such triples (α, β, γ) is equal to zero if r is odd and to $3(r-2)$ if r is even.

If $|\{\alpha, \beta, \gamma\}| = 2$, then there exist six triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha', \beta', \gamma'} \sim Q_{\alpha, \beta, \gamma}$. The number of such (α, β, γ) is equal to $3(r-1)(r-2)$.

If $|\{\alpha; \beta, \gamma\}| = 1$ and $\alpha \neq r - \alpha$, then there exist two triples $(\alpha', \beta', \gamma')$ such that $Q_{\alpha', \beta', \gamma'} \sim Q_{\alpha, \beta, \gamma}$. The number of such (α, β, γ) is equal to $r-1$ if r is odd and to $r-2$ if r is even.

If $|\{\alpha, \beta, \gamma\}| = 1$ and $\alpha = r - \alpha$, then there exists one triple $(\alpha', \beta', \gamma')$ such that $Q_{\alpha', \beta', \gamma'} \sim Q_{\alpha, \beta, \gamma}$. The number of such (α, β, γ) is equal to zero if r is odd, and to one if r is even.

This all gives that if r is odd, the number of equivalence classes is equal to

$$\begin{aligned} &\tfrac{1}{6}(r-1)(r-2)(r-3) + 0 + \tfrac{1}{6}(3(r-1)(r-2)) + \tfrac{1}{2}(r-1) + 0 \\ &\quad = \tfrac{1}{6}r^3 - \tfrac{1}{2}r^2 + \tfrac{5}{6}r - \tfrac{1}{2}. \end{aligned} \tag{53}$$

If r is even, it is equal to

$$\begin{aligned} &\tfrac{1}{6}((r-1)(r-2)(r-3) - 3(r-2)) \\ &\qquad + \tfrac{1}{3}(3(r-2)) + \tfrac{1}{6}(3(r-1)(r-2)) + \tfrac{1}{2}(r-2) + 1 \\ &\quad = \tfrac{1}{6}r^3 - \tfrac{1}{2}r^2 + \tfrac{4}{3}r - 1. \quad \blacksquare \end{aligned} \tag{54}$$

Combining Theorems 11 and 13 gives

THEOREM 14. *The number of equivalence classes of $\mathscr{P}_r$, and hence of equivalence classes of r-minimal graphs on the torus, is equal to $\frac{1}{6}r^3 + \frac{5}{6}r$ if r is odd and to $\frac{1}{6}r^3 + \frac{4}{3}r$ if r is even.*

Proof. Directly from Theorems 11 and 13. ∎

Remark 4. Calculating we see that for $r = 1, 2, 3, 4, 5, 6, 7$ the number of equivalence clases of r-minimal graphs on the torus is equal to $1, 4, 7, 16, 25, 44, 63$, respectively. It follows from Remark 3 that for any r-minimal graph G, if $P(G) = Q_{\alpha, \beta}$ then G has $2r^2 + 2\alpha\beta$ edges, and if $P(G) = H_{\alpha, \beta, \gamma}$ then G has $3r^2 + \alpha\beta + \alpha\gamma + \beta\gamma - r(\alpha + \beta + \gamma)$ edges.

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