THE STRONG FINITE STOCHASTIC REALIZATION PROBLEM - PRELIMINARY RESULTS

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ABSTRACT

The strong finite stochastic realization problem is given a probability space and a finite valued stochastic process, to show existence of and to classify all strong stochastic realizations of the given process that have a finite state space. In this paper the static version of this problem is investigated. Results are given on the classification of finite σ-algebra's that make two given finite σ-algebra's minimal conditional independent.

1. INTRODUCTION

The purpose of this paper is to present preliminary results for the strong finite stochastic realization problem.

What is a stochastic system? In filtering and control problems for dynamic phenomena stochastic models often are appropriate. Markov processes are the most often used models in such cases, whether suitable or not. Stochastic systems theory now proposes to consider stochastic dynamic systems as models for dynamic phenomena. Such a system may loosely be defined as consisting of an input, state and output process satisfying the condition that the future of these processes conditioned on the past depends only on the current state and the future inputs. The importance of a stochastic dynamic system is clearly shown in stochastic filtering and stochastic control theory.

What is the stochastic realization problem? Generally speaking it is the problem of construction of stochastic dynamic systems given the external or input-output behavior. The weak stochastic realization problem for a family of finite dimansional distributions is to show existence and to classify all minimal stochastic systems such that the output process has the same family of finite dimensional distributions as the given process. In contrast with this, the strong stochastic realization problem is given a probability space and a process to show existence of and to classify all minimal stochastic systems such that the output process is a modification of the given process.

What are the available results for this problem? The weak Gaussian stochastic realization problem has been investigated by P. FAURRE [4], while contributions to the strong version have been given by A. LINDQUIST, G. PICCI, and G. RUCKEBUSCH, see [5] for references. The classification of all minimal conditional independent σ-algebra's in case that these σ-algebra's are generated by Gaussian random variables, is given in [10]. For related results see also [11].

The finite stochastic realization problem is the version of the problem where the output and state process are restricted to take values in finite sets. Finite stochastic systems are also known as stochastic automata. This problem has first been posed by BLACKWELL and KOOPMANS [1]. There are many contributions to the weak finite stochastic realization problem, see [7,8] for references. However there are still many open questions, primarily the characterization of minimal realizations. Potential applications of the finite stochastic realization problem are in stochastic models for telecommunication, computer-communication, and engineering problems with jump processes.

The strong finite stochastic realization problem is the topic of this paper. The problem formulation may be found in section 2. Attention is here restricted to a static version of this problem, namely the classification of all finite σ-algebra's that make two given finite σ-algebra's minimal conditional independent. Results for the latter problem are presented in section 4.

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2. PROBLEM FORMULATION

In this section some notation is introduced and the problem defined.

In the paper (Ω,F,P) denotes a complete probability space, consisting of a set Ω, a σ-algebra F, and a probability measure P. Let

$$\underline{F} = \{G \subset F \mid G \text{ a } \sigma\text{-algebra, containing all the null sets of } F\},$$

$$\underline{F}_f = \{G \in \underline{F} \mid G \text{ generated by a finite number of atoms}\},$$

the latter being called the set of finite σ-algebra's. If $F_1,F_2 \in \underline{F}$ then $F_1 \vee F_2$ is the smallest σ-algebra in $\underline{F}$ containing F_1 and F_2. For $G \in \underline{F}$ let

$$L^+(G) = \{x : \Omega \to R_+ \mid x \text{ is } G \text{ measurable}\}.$$

If $x : \Omega \to R^n$ is a random variable, then $F^x \in \underline{F}$ is the σ-algebra generated by x. The notation $(F_1,F_2) \in I$ is used to indicate that F_1,F_2 are independent σ-algebra's.

2.1. <u>DEFINITION</u>. The *conditional independent relation* for a triple of σ-algebra's $F_1,F_2,G \in \underline{F}$ is defined by the condition that

$$E[x_1x_2 \mid G] = E[x_1 \mid G]\, E[x_2 \mid G]$$

for all $x_1 \in L^+(F_1)$, $x_2 \in L^+(F_2)$. One then says that F_1,F_2 are *conditional independent* given G, or that G *splitts* F_1,F_2. Notation $(F_1,G,F_2) \in CI$.

It is a fact that $(F_1,G,F_2) \in CI$ iff $E[x_1 \mid F_2 \vee G] = E[x_1 \mid G]$ for all $x_1 \in L^+(F_1)$, see [3,II.45]. Furthermore, it is easily proven that $(F_1,G,F_2) \in CI$ if $F_1 \subset G$, or if $(F_1,F_2 \vee G) \in I$. Also $(F_1,G,F_2) \in CI$ iff $(F_2,G,F_1) \in CI$.

Let Z denote the integers,

$$Z_+ = \{1,2,3,\ldots\}, \quad N = \{0,1,2,\ldots\},$$

and for $n \in Z_+$

$$Z_n = \{1,2,\ldots,n\}, \quad N_n = \{0,1,2,\ldots,n\}.$$

A definition of a stochastic dynamic system is needed. There are several alternative definitions in the literature. Consider first the following definition. A discrete time stochastic dynamic system, without input, consists of a collection of objects and relations among which are the state process $x : \Omega \times T \to R^n$ and the output process $y : \Omega \times T \to R^k$ such that for all $t \in T$

$$E[\exp(iu^T x_{t+1} + iv^T y_t) \mid F_t^x \vee F_{t-1}^y] = E[\exp(iu^T x_{t+1} + iv^T y_t) \mid F^{x_t}]$$

where $F_t^x = \vee_{s \le t} F^{x_s}$. This object is called a stochastic dynamic system because for all $t \in T$ x_t determines the distribution of (x_{t+1},y_t). By the above alternative characterization of the conditional independence relation the above condition is equivalent to the property that for all $t \in T$

$$(F_t^{x+} \vee F_t^{y+}, F^{x_t}, F_{t-1}^x \vee F_{t-1}^y) \in CI$$

where $F_t^{x+} = \vee_{s \ge t} F^{x_s}$. This property says that a stochastic dynamic system is characterized by the property that past and future of the output and state process are conditional independent given the current state.

Below a definition is given of a finite stochastic dynamic system without inputs and in discrete time.

2.2. <u>DEFINITION</u>. A *finite stochastic system* is a collection

$$\{\Omega,F,P,T,X,B_X,Y,B_Y,x,y\}$$

where $\{\Omega,F,P\}$ is a complete probability space, $T \subset Z$, X, Y are finite sets, B_X, B_Y are the finite σ-algebra's on X respectively Y generated by all subsets, $x : \Omega \times T \to X$, $y : \Omega \times T \to Y$ are stochastic processes, such that for all $t \in T$

$$(F_t^{x+} \vee F_t^{y+}, F^{x_t}, F_{t-1}^x \vee F_{t-1}^y) \in CI$$

Notation $\{\Omega,F,P,T,X,B_X,Y,B_Y,x,y\} \in FS\Sigma$.

2.3. <u>DEFINITION</u>. An *external finite stochastic system* is a collection

$$\{\Omega,F,P,T,Y,B_Y,y\}$$

where $\{\Omega,F,P\}$ is a complete probability space, $T \subset Z$, Y is a finite set, B_Y the finite σ-algebra on Y generated by all subsets, and $y : \Omega \times T \to Y$ a stochastic process. Notation

$$\{\Omega,F,P,T,Y,B_Y,y\} \in EFS\Sigma.$$

2.4. PROBLEM. The *strong finite stochastic realization problem* is given an external finite stochastic system

$$\sigma_e = \{\Omega,F,P,T,Y,B_Y,z\} \in EFS\Sigma$$

to solve the following subproblems.

a. *Does there exist* a finite stochastic system

$$\sigma = \{\Omega,F,P,T,X,B_X,Y,B_Y,x,y\} \in FS\Sigma,$$

on the same probability space as σ_e, such that for all $t \in T$ $y_t = z_t$ a.s. If there exists such a system then one calls σ a *strong finite stochastic realization* of σ_e, notation $\sigma \in SFSR(\sigma_e)$.

b. A *minimal strong finite stochastic realization* of σ_e is a strong stochastic realization $\sigma_1 \in SFSR(\sigma_e)$ such that if $\sigma_2 \in SFSR(\sigma_e)$ is any other realization and for all $t \in T$ $F^{x_2 t} \subset F^{x_1 t}$, then for all $t \in T$ $F^{x_2 t} = F^{x_1 t}$. Notation: $\sigma_1 \in SFSR_{min}(\sigma_e)$. The question is then to *characterize* a minimal strong finite stochastic realization.

c. *Classify* all minimal strong stochastic realizations of σ_e.

d. Provide an *algorithm* that constructs, given σ_e, all minimal strong stochastic realizations.

The strong finite stochastic realization problem has not been resolved. Attention will in the following be restricted to the static case of the problem. Then one supposes to be given a complete probability space, finite sets Y^+, Y^-, random variables $y^+ : \Omega \to Y^+$, $y^- : \Omega \to Y^-$, and one is asked to construct a σ-algebra $G \in \underline{F}$ such that

$$(F^{y+}, G, F^{y-}) \in CI \text{ and } G \subset (F^{y+} \vee F^{y-}),$$

which is minimal in a to be specified sense. Then necessarily $G \in \underline{F}_f$, and there exists a finite set X and a random variable $x : \Omega \to X$ such that $G = F^x$. Below a basis free treatment will be given of this problem, thus σ-algebra's are used rather than random variables. Solution of this problem is a first step of the solution of the strong finite stochastic realization problem.

2.5. DEFINITION. The *minimal condition independence relation* for a triple of σ-algebra's $F_1, F_2, G \in \underline{F}$ is defined by the conditions

1. $(F_1, G, F_2) \in CI$;
2. if $H \in \underline{F}$, $H \subset G$, and $(F_1, H, F_2) \in CI$, then $H = G$.

Notation $(F_1, G, F_2) \in CI_{min}$, and one says that F_1, F_2 are *minimal conditional independent* given G, or that G *splitts* F_1, F_2 *minimally*.

2.6. PROBLEM. The *finite σ-algebraic realization problem* is given $\{\Omega, F, P\}$ and $F^+, F^- \in \underline{F}_f$, to solve the following subproblems.

a. Does there exist a $G \in \underline{F}_f$ such that

$$(F^+, G, F^-) \in CI \textit{ and } G \subset (F^+ \vee F^-)?$$

b. Characterize those $G \in \underline{F}_f$ such that

$$(F^+, G, F^-) \in CI_{min} \textit{ and } G \subset (F^+ \vee F^-).$$

c. Classify all elements of

$$\underline{G}_{min} = \{G \in \underline{F}_f \mid (F^+, G, F^-) \in CI_{min},\ G \subset (F^+ \vee F^-)\}.$$

d. Provide an algorithm that, given F^+, F^-, constructs all elements of $\underline{G}_{min}$.

Problem 2.6 has been solved in the case where the σ-algebra's are generated by finite dimensional Gaussian random variables [10].

3. PRELIMINARIES

In this section certain technical results for the conditional independence relation are presented. Due to space limitation the proofs will not be given here, but are refered to a future paper; see also [2,9].

The following concept will play an important role in the discussion.

3.1. DEFINITION. Let $H, G \in \underline{F}$. The *projection* of H on G is defined to be

$$\sigma(H|G) = \sigma(\{E[h|G] \mid \forall h \in L^+(H)\})$$

the σ-algebra generated by the indicated random variables, with the understanding that all null sets of F are adjoined to it, hence $\sigma(H|G) \in \underline{F}$.

The concept of the projection of one σ-algebra on another has been introduced by MCKEAN [6,p.343].

In some of the examples to be discussed in section 4 one has to calculate $\sigma(F_1|F_2)$ when $F_1, F_2 \in \underline{F}_f$. This is done as follows. A partition of Ω is a collection $\{A_i, i \in Z_n\}$ such that for $i \neq j$ $A_i \cap A_j = \phi$ and $\cup_{i \in Z_n} A_i = \Omega$. By definition of $\underline{F}_f$, for any $F_1 \in \underline{F}_f$ there exists a partition $\{A_i, i \in Z_n\}$ such that $F_1 = \sigma(\{A_i, i \in Z_n\})$. Associate with this partition the random variable $y : \Omega \to R^n$, $Y_i = I_{Ai}$. Then $F_1 = F^y$. Let $F_1, F_2 \in \underline{F}_f$ be associated with $\{A_i, i \in Z_{n1}\}$, $F = F^{y1}$,

$\{B_i, i \in Z_{n2}\}$, $F_2 = F^{y2}$. Then $E[y_1|F^{y2}]$ may be calculated by the well known formula

$$E[y_{1i}|F^{y2}] = \sum_{j=1}^{n_2} (E[y_{1i}I_{Bj}] / E[I_{Bj}])\, I_{Bj},$$

and then

$$\sigma(F|F_2) = \sigma(\{E[Y_{1i}|F^{y2}] | \forall i \in Z_{n1}\}).$$

3.2. PROPOSITION. *Let* $F_1,F_2,F_3,G \in \underline{F}$. *Then*

$$(F_1,G,F_2 \vee F_3) \in CI$$

iff $(F_1,G,F_2) \in CI$ *and* $(F_1,G \vee F_2,F_3) \in CI$.

3.3. THEOREM. *Let* $F_1,F_2,F_3,G \in \underline{F}$ *with* $F_2 \subset F_3$. *One has that* $(F_1,G,F_3) \in CI$

iff $(F_1,G,F_2) \in CI$ *and* $(F_1,G \vee F_2,F_3) \in CI$,

iff $(F_1,G,F_2) \in CI$ *and* $\sigma(F_1|F_3 \vee G) \subset F_2 \vee G$.

3.4. COROLLARY. *Let* $F_1,F_2,G \in \underline{F}$. *Then* $(F_1 \vee G,G,G \vee F_2) \in CI$

iff $(F_1,G,F_2) \in CI$.

3.5. THEOREM. *Let* $F_1,F_2,G_1,G_2 \in \underline{F}$, *with* $G_2 \subset G_1$. *One has that*

$$(F_1,G_1,F_2) \in CI \text{ and } \sigma(F_1|G_1) \subset G_2$$

iff $(F_1,G_2,F_2) \in CI$ *and* $\sigma(F_1|F_2 \vee G_1) \subset F_2 \vee G_2$.

3.6. COROLLARY. *Let* $F_1,F_2 \in \underline{F}$. *Then* $(F_1,\sigma(F_1|F_2),F_2) \in CI$.

3.7. PROPOSITION. *Let* $F_1,F_2,G \in \underline{F}$.
a. *Let* $G \subset F_2$. *Then* $(F_1,G,F_2) \in CI$ *iff* $\sigma(F_1|F_2) \subset G$.
b. *If* $(F_1,G,F_2) \in CI$ *then* $(F_1,\sigma(G|F_1),F_2) \in CI$. *Hence* $\sigma(F_2|F_1) \subset \sigma(G|F_1)$.

3.8. PROPOSITION. *Let* $F_1,F_2,G_1,G_2 \in \underline{F}$. *If* $(F_1,G_1,F_2) \in CI$, $G_2 \subset (F_2 \vee G_1)$, *and* $(\sigma(F_1|G_1),G_2,F_2) \in CI$, *then* $(F_1,G_2,F_2) \in CI$.

3.9. PROPOSITION. *Let* $F_1,F_2,F_3,G \in \underline{F}$.
a. *If* $(F_1,G,F_2) \in CI$ *then* $\sigma(F_1|F_2 \vee G) = \sigma(F_1|G)$.
b. $\sigma(F_1|\sigma(F_1|F_2)) = \sigma(F_1|F_2)$.
c. $\sigma(F_1|\sigma(F_2|F_1) \vee \sigma(F_1|F_2)) = \sigma(F_2|F_1)$.
d. *If* $F_2 \subset F_3 \subset F_1 \vee F_2$, *then* $F_3 = F_2 \vee \sigma(F_1|F_3)$.
e. $\sigma(\sigma(F_1|F_2)|\sigma(F_2|F_1)) = \sigma(F_2|F_1)$.

4. THE FINITE σ-ALGEBRAIC REALIZATION PROBLEM

In this section results will be derived for the finite σ-algebraic realization problem. The theory for the realization problem in Hilbert space and for finite dimensional linear systems will be a guideline for the discussion given below.

Let be given $F^+, F^- \in \underline{F}_f$. There always exists a $G \in \underline{F}_f$ such that $(F^+, G, F^-) \in CI$ and $G \subset (F^+ \vee F^-)$. For example $G = F^+$ or $G = F^-$ satisfy this condition. This is easily shown by verifying the definition of the conditional independence relation. This solves subproblem 2.6.a.

The characterization of those $G \in \underline{F}_f$ such that $(F^+, G, F^-) \in CI_{min}$ and $G \subset (F^+ \vee F^-)$ is subproblem 2.6.b. Consider first a special case of this subproblem.

4.1. PROPOSITION. [6]. *Let* $F^+, F^- \in \underline{F}$. *Then* $G \in \underline{F}$, $G \subset F^-$, *and* $(F^+, G, F^-) \in CI_{min}$ *iff* $G = \sigma(F^+ | F^-)$.

Thus within F^- there is an unique σ-algebra making F^+, F^- minimal conditional independent. One calls $\sigma(F^+|F^-)$ the minimal future-induced realization of F^+, F^-. A result as 4.1 with + and − interchanged also holds, and one calls $\sigma(F^-|F^+)$ the past-induced minimal realization of F^+, F^-.

To formulate a characterization of minimal splitting σ-algebra's a condition like stochastic observability is needed. Such a condition is motivated next. Consider F^{y+}, F^{y-}, $F^x \in \underline{F}_f$ with the random variables y^+, y^-, x as defined below 3.1. Stochastic observability is defined by the condition that the map

$$x \to E[y^+ | B^x]$$

is injective on the support of x. The interpretation of this condition is that if one knows the conditional probability measure of y^+ given x, then stochastic observability implies that one can recover the value of the state x. The conditional probability measure of y^+ given x one can in principle recover by performing many observations of y^+ for the same x. The stochastic observability condition is equivalent to $\sigma(F^{y+}|G^x) = G^x$. The following conjecture should then be clear.

4.2. CONJECTURE. Let $F^+, F^-, G \in \underline{F}_f$. One has that $(F^+, G, F^-) \in CI_{min}$ *iff*

1. $(F^+, G, F^-) \in CI$;
2. $\sigma(F^+|G) = G = \sigma(F^-|G)$.

4.3. PROPOSITION. *Let* $F^+, F^-, G \in \underline{F}$. *If* $(F^+, G, F^-) \in CI_{min}$, *then* $\sigma(F^+|G) = G = \sigma(F^-|G)$.

PROOF. By $(F^+, G, F^-) \in CI$ and 3.5 one has that $(F^+, \sigma(F^+|G), F^-) \in CI$. This, $\sigma(F^+|G) \subset G$, the assumption, and the definition of CI_{min} imply that $\sigma(F^+|G) = G$. A symmetric argument yields the other equality. □

However the converse implication of 4.2 does not hold as the following example shows. This example is due to J.C. Willems.

4.4. EXAMPLE. Let $\Omega = Z_9$, $F = 2^\Omega$ the σ-algebra generated by the atoms of F, and $P : F \to [0,1]$ the probability measure that gives equal weight to all the atoms of F. This will be called the uniform measure on $\{\Omega,F\}$. Furthermore let

$$\begin{aligned}
F^+ &= \sigma(\{1,2,3\},\{4,5,6\},\{7,8,9\}),\\
F^- &= \sigma(\{1,4,7\},\{2,5,8\},\{3,6,9\}),\\
G_1 &= \sigma(\{2,3\},\{6,9\},\{7,8\},\{1,4\},\{5\}),\\
G_2 &= \{\phi,\Omega\}.
\end{aligned}$$

Then $(F^+,G_1,F^-) \in CI$, $\sigma(F^+|G_1) = G_1 = \sigma(F^-|G_1)$, $(F^+,G_2,F^-) \in CI_{min}$ and $G_2 \subset G_1$. The proof of these statements is an elementary calculation.

In this and subsequent examples the reader is suggested to draw a picture of the probability space with the atoms of F^+ as horizontal bars and the atoms of F^- as vertical bars.

The reader may be tempted to think that any two minimal splitting σ-algebra's have the same number of non-trivial atoms. This is not true.

4.5. EXAMPLE. Let $\Omega = Z_8$, $F = 2^\Omega$, $P : F \to [0,1]$ the uniform measure,

$$\begin{aligned}
F^+ &= \sigma(\{1\}),\{2,3\},\{4,5,6\},\{7,8\}),\\
F^- &= \sigma(\{1,2,3\},\{3,5,7\},\{6,8\}),\\
G_1 &= \sigma(\{1,2,4\},\{3\},\{5,6,7,8\}),\\
G_2 &= \sigma(\{1\},\{2,3,4,5\},\{6\},\{7,8\}).
\end{aligned}$$

Then $(F^+,G_1,F^-) \in CI_{min}$ and $(F^+,G_2,F^-) \in CI_{min}$. The minimality is proven by verifying that there is no proper sub σ-algebra of G_1,G_2 making F^+,F^- conditional independent.

The characterization of minimal splitting σ-algebra's remains unsolved.

The classification of all $G \in \underline{F}_f$ such that $(F^+,G,F^-) \in CI_{min}$ and $G \subset (F^+ \vee F^-)$ is subproblem 2.6.c. It must be pointed out that $(F^+,G,F^-) \in CI_{min}$ does not imply that $G \subset (F^+ \vee F^-)$. A counterexample is easily given. The above restriction is made to simplify the problem. In realization theory it is usually the case that any minimal realization projected on the past gives the future-induced realization. What is true of this statement for the finite σ-algebraic realization problem?

4.6. CONJECTURE. Let $F^+,F^-,G \in \underline{F}$, and assume that $(F^+,G,F^-) \in CI_{min}$ and $G \subset (F^+ \vee F^-)$. Let $F^{+-} = \sigma(F^+|F^-)$.

a. Then $\sigma(G|F^+) = F^{-+}$ and $\sigma(G|F^-) = F^{+-}$.

b. Then also $G \subset (F^{-+} \vee F^{+-})$.

4.7. EXAMPLE. Let $\Omega = Z_{10}$, $F = 2^\Omega$, $P : F \to [0,1]$ be the uniform measure on F. Furthermore let

$F^+ = \sigma(\{1,4\},\{2,5,8\},\{3,6,9\},\{7,10\})$,
$F^- = \sigma(\{1,2,3\},\{4,5,6,7\},\{8,9,10\})$,
$G = \sigma(\{1,2,4,5\},\{3\},\{6,7,9,10\},\{8\})$.

Then $(F^+,G,F^-) \in CI_{min}$, $\sigma(G|F^-) = F^{+-}$, $\sigma(G|F^+) = F^+ \neq F^{-+} =$
$= \sigma(\{1,4\},\{2,3,5,6,8,9\},\{7,10\})$, $G \not\subset (F^{-+} \vee F^{+-})$.

Conjecture 4.6 is thus false. If one wants to preserve the property that any minimal σ-algebra projected on the past gives the future-induced σ-algebra and symmetrically, then a condition must be imposed. As to how to choose this condition is indicated by the following result

4.8. PROPOSITION. *Let* $F^+,F^-,G \in \underline{F}$, *and assume that* $(F^+,G,F^-) \in CI$ *and* $G \subset (F^+ \vee F^-)$. *One has that* $G \subset (F^{-+} \vee F^{+-})$ *iff* $\sigma(G|F^+) = F^{-+}$ *and* $\sigma(G|F^-) = F^{+-}$.

PROOF. $\sigma(G|F^+ \vee F^-) = G \subset (F^+ \vee F^{+-})$
iff $(F^+ \vee F^-, F^+ \vee F^{+-}, G) \in CI$ by 3.7.a,
iff $(F^-,F^{+-} \vee F^+, G) \in CI$ by reduction (⇒) or 3.4 (⇐),
iff $(F^-,F^{+-},F^+ \vee G) \in CI$ by 3.2 and by 3.6,
iff $(F^-,F^{+-}, G) \in CI$ and $(F^-,F^{+-} \vee G,F^+) \in CI$ by 3.2,
iff $\sigma(G|F^-) \subset F^{+-}$ by 3.7.a and by 3.4,
iff $\sigma(G|F^-) = F^{+-}$ by 3.7.b.
The conclusion then follows with a symmetric argument. □

In the following the classification problem 2.6.c. is restricted to those $G \in \underline{F}_f$ such that $(F^+,G,F^-) \in CI_{min}$ and $G \subset (F^{-+} \vee F^{+-}) := F_0$.

4.9. PROPOSITION. *Let* $F^+,F^-,G \in \underline{F}$ *and assume that* $G \subset F_0$.
a. *Then* $(F^{-+},G,F^{+-}) \in CI$ *iff* $(F^+,G,F^-) \in CI$.
b. *Then also* $(F^{-+},G,F^{+-}) \in CI_{min}$ *iff* $(F^+,G,F^-) \in CI_{min}$.

PROOF. a. ⇒. By 3.6. $(F^+,F_0,F^-) \in CI$. This and 3.4. imply $(F^+,F_0,F^-) \in CI$. With 3.9.c. one concludes that $\sigma(F^-|F_0) = F^{+-}$, hence $(F^{-+},G,F^{+-}) = (F^{-+},G,\sigma(F^-|F_0)) \in CI$. These statements, $G \subset (F^{-+} \vee F_0)$, and 3.8 imply that $(F^{-+},G,F^-) \in CI$. By a symmetric argument $(F^+,G,F^-) \in CI$. ⇐. This is obvious by the definition 2.1.
b. This result follows easily from a. and 2.5. □

4.10. PROPOSITION. *Let* $F^+,F^-,G \in \underline{F}$, *and assume that* $(F^+,G,F^-) \in CI$ *and* $G \subset F_0$. *Then* a. $\sigma(G|F^+) = F^{-+}$, $\sigma(G|F^-) = F^{+-}$; b. $\sigma(G|F^{-+}) = F^{-+}$ *and* $\sigma(G|F^{+-}) = F^{+-}$.

PROOF. a. Apply 4.8. b. Apply 4.9.a, 3.9.e and a. □

The following result is an attempt to obtain the required classification 2.6.c and algorithm 2.6.d. Note the analogy with the classification in the Hilbert space case [5].

4.11. THEOREM. *Let* $F^+,F^- \in \underline{F}$,

$$\underline{G}_1 = \{G \in \underline{F} \mid (F^{-+},G,F^{+-}) \in CI,\ G \subset F_0 \\ \sigma(F^{-+}|G) = G = \sigma(F^{+-}|G)\},$$

$$\underline{H} = \{H \in \underline{F} \mid F^{+-} \subset H \subset F_0, \\ \sigma(F^{-+}|H) = \sigma(F^{+-}|\sigma(F^{-+}|H))\}.$$

Furthermore, define the realization map $r : \underline{H} \to \underline{G}_1$, $r(H) = \sigma(F^{-+}|H)$. Then r is well defined, and a bijection.

PROOF. 1. r is well defined. Let $G = r(H) = \sigma(F^{-+}|H)$. By $F^{+-} \subset H$ $(F^{-+},H,F^{+-}) \in CI$. This and 3.7.b imply that $(F^{-+},G,F^{+-}) = (F^{-+},\sigma(F^{-+}|H),F^{+-}) \in CI$. By $H \in \underline{H} = \sigma(F^{-+}|H) \subset F_0$. Also

$$\sigma(F^{-+}|G) = \sigma(F^{-+}|\sigma(F^{-+}|H)) = \sigma(F^{-+}|H) = G$$

by 3.9.b, while

$$\sigma(F^{+-}|G) = \sigma(F^{+-}|\sigma(F^{-+}|H)) = \sigma(F^{-+}|H) = G,$$

by definition of $\underline{H}$. Thus r is well defined.

2. r is surjective. Let $G \in \underline{G}_1$, $H = F^{+-} \vee G$. Then $F^{+-} \subset H = F^{+-} \vee G \subset F_0$, by $G \subset F_0$. Also

$$\sigma(F^{+-}|\sigma(F^{-+}|H)) = \sigma(F^{+-}|\sigma(F^{-+}|F^{+-} \vee G))$$

$$= \sigma(F^{+-}|\sigma(F^{-+}|G)) \quad \text{by 3.9.a,}$$

$$= \sigma(F^{+-}|G) = G, \quad \text{by } G \in \underline{G}_1,$$

$$= \sigma(F^{-+}|G) = \sigma(F^{-+}|F^{+-} \vee G) = \sigma(F^{-+}|H).$$

Thus $H \in \underline{H}$. As shown above $G = \sigma(F^{-+}|H) = r(H)$.

3. r is injective. Let $H_1,H_2 \in \underline{H}$ be such that $r(H_1) = r(H_2)$. Then $H_1,H_2 \in \underline{H}$ and 3.9.d imply that $H_1 = F^{+-} \vee \sigma(F^{-+}|H_1) = F^{+-} \vee r(H_1) = F^{+-} \vee r(H_2) = H_2$. □

The last condition of $\underline{H}$ cannot be dispensed with.

4.12. EXAMPLE. Let $\Omega = Z_7$, $F = 2^{\Omega}$, $P : F \to [0,1]$ the uniform measure,

$$F^{+} = \sigma(\{1,2\},\{3,4,5\},\{6,7\}),$$
$$F^{-} = \sigma(\{1,3\},\{2,4,6\},\{5,7\}).$$

Then $F^{-+} = F^{+}$ and $F^{+-} = F^{-}$. According to 4.11 all elements of $\underline{G}_1$ are given by: $F^{-+} = F^{+}$, $F^{+-} = F^{-}$,

$$G_1 = \sigma(\{1,2\},\{3\},\{4,5,6,7\}),\ G_2 = \sigma(\{1,3\},\{2\},\{4,5,6,7\}),$$
$$G_3 = \sigma(\{1,2,3,4\},\{5\},\{6,7\}),\ G_4 = \sigma(\{1,2,3,4\},\{5,7\},\{6\}),$$
$$G_5 = \sigma(\{1\},\{2,6\},\{3,4,5\},\{7\}),\ G_6 = \sigma(\{1,3\},\{3,6\},\{4,5\},\{7\}),$$
$$G_7 = \sigma(\{1\},\{2,6\},\{3,4\},\{5,7\}),\ G_8 = \sigma(\{1\},\{2,4,6\},\{3,5\},\{7\}),$$
$$G_9 = \sigma(\{1\},\{2,4\},\{3,5\},\{6,7\}),\ G_{10} = \sigma(\{1,2\},\{3,5\},\{4,6\},\{7\}).$$

In this case these σ-algebra's also satisfy $(F^+,G,F^-) \in CI_{min}$ and $G \subset (F^+ \vee F^-)$. Furthermore the set $\underline{H}$ as defined in 4.11 is not totally ordered.

Another aspect of the classification of all minimal realizations is the relation between these. In realization theory of Hilbert spaces all minimal realizations are equivalent [5].

4.13. CONJECTURE. Let $F_1,F_2,G_1,G_2 \in \underline{F}$. If $(F^+,G_1,F^-) \in CI_{min}$, $(F,G_2,F^- \in CI_{min}$ and $G_1,G_2 \subset F_0$, then $\sigma(G_1|G_2) = G_2$ and $\sigma(G_2|G_1) = G_1$. Unfortunately this conjecture is also false.

4.14. EXAMPLE. Consider example 4.12. Then $\sigma(G_2|G_3) \neq G_3$ and $\sigma(G_3|G_2) \neq G_2$. Let

$$R = \left\{ (G_1,G_2) \in \underline{F}_f \times \underline{F}_f \mid (F^+,G_1,F^-) \in CI_{min},\ (F^+,G_2,F^-) \in CI_{min},\ G_1,G_2 \subset F_0,\ \sigma(G_1|G_2) = G_2,\ \sigma(G_2|G_1) = G_1 \right\}.$$

Then 4.14 shows that R is not an equivalence relation.

To conclude let us summarize the results of the finite σ-algebraic realization problem. The characterization of the minimal conditional independent relation is unsolved. A partial classification of all minimal splitting σ-algebra's is given, although a condition has been imposed. The projection is not an equivalence relation for minimal splitting σ-algebra's. Apparently the results for the finite σ-algebraic realization problem are completely different from the Hilbert space case [5]. Much remains to be done.

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