Control for Coordination of Linear Systems

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Abstract

Control theory for distributed systems is motivated by control of large-scale engineering systems such as electrical power networks, motorway networks, underwater vehicles, and electric-mechanic machines. The control objectives of a control problem for such a system may be such that a tight interaction between the subsystems is necessary. The concept of a coordinated linear system is defined which contains a coordinator and two or more linear subsystems. The controller takes care of the coordination between the subsystems. A geometric condition, conditional linear independence of linear subspaces given another subspace and an invariance condition characterize coordinated linear systems. In case a distributed control system admits a particular representation then control synthesis separates into control synthesis for the coordinator and for each of the local subsystems.

1 Introduction

The problem of this paper is control of linear systems with a modular or distributed structure.

The motivation of this research is the frequent occurence of control problems for large-scale linear systems with a clear modular or distributed structure. A linear system is said to be a *distributed linear system* if it consists of an interconnection of a number of linear subsystems and if control is based on partial observations of the subsystems.

If the control objectives of a control problem for a distributed systems require a tight interaction of the subsystems then a local control synthesis for each subsystem may not meet the control objectives and a coordination control is necessary.

The concept of a coordinated linear system will be defined. The coordinator subsystem is in its dynamics not affected by the other local subsystems. The dynamics of each subsystem (different from the coordinator) is affected only by its own state and by the state of the coordinator. An equivalent condition for a coordinated linear system is that of conditional independent subspaces of the state space given a coordinator subspace, combined with an invariance condition. The properties of conditionally linear independent subspaces are investigated. Finally, for control synthesis of distributed systems which admit a coordinated linear control system representation, the control objectives may be attained by carrying out first control synthesis of the coordinator and subsequently control synthesis of each of the subsystems.

The equivalent conditions for the control problem are geometric in character hence are formulated in terms of linear subspaces of the state space. The geometric approach to linear systems was primarily developed by Murray W. Wonham, see the books [5, 6]. The geometric analysis of vector spaces is based on the book [2]. Control of coordination of large-scale and hierarchical systems is treated for example in the book [1].

A description of the contents of the paper follows. The next section contains the problem formulation. Conditional linear independence of linear subspaces is discussed in Section 3. Coordinated linear subsystems are characterized in Section 4. Control synthesis of a coordinated linear control system is treated in Section 5.

2 Problem formulation

Problem 2.1 *Consider a linear system (without outputs),*

$$dx(t) = Ax(t) + Bu(t), \; x(t_0) = x_0. \tag{1}$$

Determine a linear control law $g(x) = Fx$ such that, possibly after a state space and an input space transformation, the closed-loop system has the representation

$$\begin{aligned} dx(t)/dt &= (A+BF)x(t) = \begin{pmatrix} A_{11} & 0 & A_{1,c} \\ 0 & A_{22} & A_{2,c} \\ 0 & 0 & A_{c,c} \end{pmatrix} x(t), \\ & x(t_0) = x_0, \; n_1, n_2, n_c \in \mathbb{N}, \; n_1 + n_2 + n_3 = n, \\ & A_{i,j} \in \mathbb{R}^{n_i \times n_j}, \; \forall i, j \in 1, 2, c. \end{aligned} \tag{2}$$

The usefulness of the above representation will become clear in the Sections 4 and 5.

3 Conditionally-independent linear subspaces

The reader is assumed to be familiar with the set of the real numbers and with the algebraic structure of vector spaces. Below follow notation and concepts for these mathematical structures.

The set of the *integers* is denoted by $\mathbb{Z}$, the set of the *positive integers* is denoted by $\mathbb{Z}_+ = \{1, 2, \ldots\}$, and the set of the *natural numbers* by $\mathbb{N} = \{0, 1, 2, \ldots\} \subset \mathbb{Z}$. For any $n \in \mathbb{Z}_+$ denote $\mathbb{Z}_n = \{1, 2, \ldots, n\}$.

Denote the set of the *real numbers* by $\mathbb{R}$. The reader is assumed to be familiar with a *linear space*, consisting of a field F of scalars and of a set of vectors V also called a *vector space*. An example of a linear space is $\mathbb{R}^n$), which denotes the set of n-tuples of real numbers for any $n \in \mathbb{Z}_+$. Call two subspaces $X_1, X_2 \subseteq X$ *linear independent subspaces* if $X_1 \cap X_2 = \{0\}$.

In this section the concept of conditionally-independent linear subspaces will be explored. Consider a linear space X. Denote the set of linear subspaces of X by $\text{LinearSubspaces}(X)$.

Definition 3.1 *Consider a linear space X and subspaces $X_1, X_2, X_c \in \text{LinearSubspaces}(X)$. Call X_1, X_2* conditionally linear independent given *X_c if there exists orthogonal complements $X_{i\backslash c} \subseteq X_i$ of $X_i \cap X_c$, equivalently,*

$$X_i = (X_i \cap X_c) \oplus X_{i\backslash c}, \; i = 1, 2, \tag{3}$$

such that $X_{1\backslash c}$ and $X_{2\backslash c}$ are linear independent in X. The notation $(X_1, X_2 | X_c) \in \text{CILinearSubspaces}$ denotes that the linear subspaces X_1, X_2 are conditionally independent given X_c. Call X_c the coordinator subspace *for X_1 and X_2.*

Lemma 3.2 *Consider a linear space X and two subspaces $X_1, X_2 \in \text{LinearSubspaces}(X)$. Define*

$$\begin{aligned} X_c &= X_1 \cap X_2, \\ X_i &= (X_i \cap X_c) \oplus X_{i\setminus c},\ i = 1,2. \end{aligned}$$

Then $X_{1\setminus c}$ and $X_{2\setminus c}$ are linearly independent subspaces.

Proof 3.3 *It is to be proven that $X_{1\setminus c} \cap X_{2\setminus c} = \{0\}$. Let $x \in X_{1\setminus c} \cap X_{2\setminus c}$. Then $x \in X_{1\setminus c}$ implies that x is orthogonal to $X_1 \cap X_c = X_1 \cap X_2 = X_c$. But $x \in X_{1\setminus c} \subseteq X_1$ and $x \in X_{2\setminus c} \subseteq X_2$ imply that $x \in X_1 \cap X_2 = X_c$. Now $x \in X_c$ and being orthogonal to X_c implies that $x = 0$.*

Lemma 3.4 *Consider a linear subspace X and $X_1, X_2, X_c \in \text{LinearSubspaces}$. If $(X_1, X_2 | X_c) \in \text{CILinearSubspaces}$ then $X_1 \cap X_2 \subseteq X_c$.*

Theorem 3.5 *Consider a linear subspace X and $X_1, X_2, X_c \in \text{LinearSubspaces}(X)$.*

(a) $(X_1, X_2 | X_1 \cap X_2) \in \text{CILinearSubspaces}$.

(b) If $(X_1, X_2 | X_c) \in \text{CILinearSubspaces}$ then $X_1 \cap X_2 \subseteq X_c$.

(c) $X_1 \cap X_2$ is the minimal subspace X_c in X such that $(X_1, X_2 | X_c) \in \text{CILinearSubspaces}$.

Proof 3.6 *(a) This follows from Lemma 3.2. (b) This follows from Lemma 3.4. (c) This follows from (a) and (b).*

4 Coordination of linear systems

Definition 4.1 *A time-invariant linear system is a dynamic system as understood in system theory, see [3], with representation*

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t),\ ix(t_0) = x_0, \\ y(t) &= Cx(t) + Du(t), \\ & \textit{where } X = \mathbb{R}^n \textit{ is the state space},\ U = \mathbb{R}^m \textit{ is the input space}, \\ & Y = \mathbb{R}^p \textit{ is the output space,} \\ & A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}, \\ & x : T \to X,\ u : T \to U,\ y : T \to Y. \end{aligned}$$

Definition 4.2 *A linear system is said to be a* coordinated linear system *(without input) if it has a representation of the form*

$$\begin{aligned} dx(t)/dt &= \begin{pmatrix} A_{11} & 0 & A_{1,c} \\ 0 & A_{22} & A_{2,c} \\ 0 & 0 & A_{c,c} \end{pmatrix} x(t),\ x(t_0) = x_0, \qquad (4) \\ & n_1, n_2, n_c \in \mathbb{N},\ n_1 + n_2 + n_c = n, \\ & A_{i,j} \in \mathbb{R}^{n_i \times n_j},\ \forall i,j \in 1,2,c. \end{aligned}$$

Theorem 4.3 *Consider a linear system (without input) with representation*

$$dx(t)/dt = Ax(t),\ x(t_0) = x_0,$$

with a finite-dimensional linear space X*, a linear map* $A : X \to X$*, time index set* $T = \{t_0, t_0+1, \ldots, \} \subset \mathbb{Z}$*, and state trajectory* $x : T \to X$*. Consider linear subspaces* $X_1, X_2 \in \text{LinearSubspaces}(X)$ *and define,*

$$\begin{aligned} X_c &= X_1 \cap X_2, & (5)\\ X_1 &= X_c \oplus X_{1\setminus c},\ X_2 = X_c \oplus X_{2\setminus c}, \text{ where } \oplus \text{ denotes orthogonal complement.} & (6) \end{aligned}$$

Then it follows from Theorem 3.5 that $(X_1, X_2 | X_c) \in \text{CILinearSubspaces}$.

There exists a basis of X *such that with respect to this basis the linear system has the coordinated linear system representation,*

$$\bar{x}(t+1) = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} \bar{x}(t),\ \bar{x}(t_0) = \bar{x}_0, \tag{7}$$

if and only if the following invariance conditions hold,

$$AX_{1\setminus c} \subseteq X_{1\setminus c},\ AX_{2\setminus c} \subseteq X_{2\setminus c}. \tag{8}$$

Proof 4.4 *(*$\Leftarrow$*) Because* $X_c = X_1 \cap X_2$ *there exist orthogonal decompositions*

$$X_1 = X_c \oplus X_{1\setminus c},\ \ X_2 = X_c \oplus X_{2\setminus c}.$$

Because by Theorem 3.5 $(X_1, X_2 | X_c) \in \text{CILinearSubspaces}$, $X_{1\setminus c} \cap X_{2\setminus c} = \{0\}$*. Choose a subspace* $X_3 \in \text{LinearSubspaces}(X)$ *such that*

$$X = X_{1\setminus c} \oplus X_{2\setminus c} \oplus X_3,$$

and choose a basis of X *compatible with this decomposition. By assumption* $AX_{1\setminus c} \subseteq X_{1\setminus c}$ *and* $AX_{2\setminus c} \subseteq X_{2\setminus c}$*. Then the representation of Equation (4) follows because*

$$\begin{pmatrix} A_{11}x_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix},\ \textit{etc.}$$

$(\Rightarrow)$ *The invariance of the subspaces* $X_{1\setminus c}$ *and* $X_{2\setminus c}$ *with respect to the linear map A follows directly from the representation.*

Consider a distributed linear system which consists of an interconnection of two subsystems. Denoted the relevant state spaces of these subsystems by X_1 and X_2; suppose that $X = X_1 + X_2$. Denote $X_c = X_1 \cap X_2$. From Theorem 3.5 follows that $(X_1, X_2 | X_c) \in X_c) \in$ CILinearSubspaces. If the invariance condition of Theorem 4.3 holds then one can choose a basis of X such that the system has a representation as a coordinated linear system. In case the invariance condition Equation (8) does not hold it is suggested to extend the coordinator subspace $X_c \subseteq X$ till the invariance condition holds. The subspace $X_1 + X_2$ is a coordinator subspace but there may be smaller subspaces in the range

$$X_1 \cap X_2 \subseteq X_c \subseteq X_1 + X_2.$$

5 Control synthesis for coordination of linear systems

Definition 5.1 *A linear control system with representation*

$$x(t+1) = Ax(t) + Bu(t),\ x(t_0) = x_0, \tag{9}$$

is said to be a coordinated linear control system *if there exists a basis for the state space X and for the input space U such that with respect to those bases it has the representation*

$$\begin{aligned} x(t+1) &= \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix} u(t), \\ & x(t_0) = x_0. \end{aligned} \tag{10}$$

Problem 5.2 *Consider a linear control system with representation*

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t),\ \bar{x}(t_0) = \bar{x}_0. \tag{11}$$

Determine necessary and sufficient conditions for the existence of a linear control law,

$$g(x) = Fx + Gu_s,\ \ F \in \mathbb{R}^{n\times m},\ G \in \mathbb{R}^{n\times m_s},$$

of a nonsingular state space transformation $S \in \mathbb{R}^{n\times n}$, and of a nonsingular input space transformation $S_u \in \mathbb{R}^{m_s\times m_s}$, such that after the closing of the control loop and the state space and input space transformations the system has a representation as a coordinated linear control system.

It is conjectured that for the above problem a simple geometric condition is necessary and sufficient.

As argued in Section 4 a decomposition of the system matrix A as in Equation (10) exists if an invariance condition holds. By linear row operations it has then to be determined whether there exists a transformation of the input space which transforms the input matrix B to the form displayed in Equation (10). It is expected that this transformation requires a condition. Note that one or two of the block rows may be missing from the B matrix.

Proposition 5.3 *Consider a coordinated linear control system with representation (10). For any symmetric subset of the complex numbers, $S_{specification} \subset \mathbb{C}$ there exist control laws*

$$\begin{aligned} g_1(x) &= F_{11}x_1 + F_{13}x_3,\ g_2(x) = F_{22}x_2 + F_{23}x_3,\ g_3(x) = F_{33}x_3, \\ & \text{such that the inputs,} \\ u_1(t) &= g_1(x(t)),\ u_2(t) = g_2(x(t)),\ u_3(t) = g_3(x(t)), \end{aligned}$$

yield a closed-loop linear system with eigenvalues of the system matrix in $S_{specification}$ if and only if

$$(A_{11}, B_{11}),\ (A_{22}, B_{22}),\ (A_{33}, B_{33}), \tag{12}$$

are controllable pairs.

The reader can now easily formulate the result corresponding to the above proposition for which only exponential stability of the closed-loop linear system is required in terms of stabilizability.

Proof 5.4 *($\Leftarrow$) Note that the closed-loop linear system is equivalent to the following three systems,*

$$\begin{aligned} x_3(t+1) &= (A_{33} + B_{33}F_{33})x_3(t),\ x_3(t_0) = x_{3,0}; \\ x_1(t+1) &= (A_{11} + B_{11}F_{11})x_1(t) + \\ & \quad +(A_{13} + B_{11}F_{13} + B_{13}F_{33})x_3(t),\ x_1(t_0) = x_{1,0}; \\ x_2(t+1) &= (A_{22} + B_{22}F_{22})x_2(t) + \\ & \quad +(A_{23} + B_{22}F_{23} + B_{23}F_{33})x_3(t),\ x_2(t_0) = x_{2,0}. \end{aligned}$$

Because by assumption (A_{33}, B_{33}) *is a controllable pair there exists a matrix* $F_{33} \in \mathbb{R}^{m_3 \times n_3}$ *such that* $\text{spec}(A_{33} + B_{33}F_{33}) \subset S_{specification}$. *Similarly there exist* $F_{11} \in \mathbb{R}^{m_2 \times n_3}$ *and* $F_{22} \in \mathbb{R}^{m_2 \times n_2}$ *such that* $\text{spec}(A_{11} + B_{11}F_{11}) \subset S_{specification}$ *The choice of* F_{13} *and the* F_{23} *are arbitrary because these do not affect the spectrum of the transition matrix. The result then follows from [4].*
(⇒) This follows directly from the decomposition of the system matrix of the full system, $(A + BF)$*, and the corresponding result for ordinary linear systems.*

Note that the control synthesis of a coordinated linear control system proceeds by carrying out first control synthesis of the coordinator and subsequently carrying out independently control synthesis of each of the other subsystems.

6 Concluding remarks

The problem of coordination control of distributed linear systems has been discussed. The concept of two subspaces being conditionally linear independent given another subspace has been formulated and the minimal subspace equals the intersection of the two subspaces. The concept of a coordinated linear system has been proposed. A linear system admits a decomposition as a coordinated linear system if an invariance condition for two subspaces holds. Control synthesis for such a system can be carried out by first doing control synthesis for the coordinator and then for the two subsystems.

Further research is needed on coordination control with partial observations for distributed systems. Another direction is the abstraction of the coordinator subsystem.

References

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