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BAHADUR EFFICIENCY AND SMALL-SAMPLE EFFICIENCY: A NUMERICAL STUDY

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Bahadur efficiency and small-sample efficiency: a numerical study

by

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SUMMARY

The asymptotic (relative) efficiencies of Bahadur and Pitman can be interpreted as limits of a natural small-sample (relative) efficiency concept. In a number of examples the accuracy of the approximation of small-sample efficiencies by asymptotic efficiencies is investigated numerically, paying special attention to Bahadur asymptotic efficiencies. The more recently introduced (asymptotic) deficiencies are also considered.

KEY WORDS & PHRASES: *Bahadur efficiency, Pitman efficiency, small-sample efficiency, deficiency*

1. INTRODUCTION

Let $X_1, X_2, \ldots$ be i.i.d. random variables with a distribution depending on a parameter $\theta \in \Theta$ and suppose that the hypothesis $H_0 : \theta \in \Theta_0$ is to be tested against $\theta \in \Theta_1 = \Theta \setminus \Theta_0$. Let $\{T_n^{(j)}\}$, $j = 1,2$, be two sequences of test statistics for this testing problem. Here $T_n^{(j)}$ only depends on the sample $X_1, \ldots, X_n$. We want to compare the two tests ($j = 1,2$) which reject H_0 for large values of these statistics.

The number of observations required to achieve a fixed power is the criterion of test performance considered in this paper. This criterion immediately leads to the comparison of the sample sizes of two tests achieving equal power at alternatives of interest. Intuitively one may prefer a direct comparison of the powers of two tests at equal sample sizes. However, the present approach is well suited to provide a unified description of the asymptotic efficiencies of Bahadur and Pitman and to relate these asymptotic concepts to a similar small sample concept.

Define $N_j(\alpha,\beta,\theta)$ to be the minimal sample size such that a level-α test based on the statistics $\{T_n^{(j)}\}$ has at least power β at $\theta \in \Theta_1$ $(0<\alpha<\beta<1)$, $j = 1,2$. The *relative efficiency* of $\{T_n^{(2)}\}$ with respect to $\{T_n^{(1)}\}$ at (α,β,θ) is defined as

$$(1) \qquad \mathrm{eff}(T^{(2)},T^{(1)};\alpha,\beta,\theta) = N_1(\alpha,\beta,\theta)/N_2(\alpha,\beta,\theta), \quad \theta \in \Theta_1.$$

A large value (>1) indicates that $\{T_n^{(2)}\}$ is superior to $\{T_n^{(1)}\}$ since less observations are required to achieve the same power. This efficiency notion seems to be a good measure of the relative performance of both sequences of test statistics. If for fixed α and θ the value of β is not too large, N_1 and N_2 remain moderately small and hence (1) is typically concerned with the comparison of tests for small and moderate sample sizes.

Keeping β fixed and either letting θ tend to a value in the boundary $\partial\Theta_0$ of Θ_0 or letting α tend to zero, N_1 and N_2 as a rule tend to infinity and (1) turns into a measure of asymptotic efficiency. In fact

$$(2) \qquad e^P(T^{(2)},T^{(1)};\alpha,\beta,\theta_0) = \lim_{\theta\to\theta_0} \mathrm{eff}(T^{(2)},T^{(1)};\alpha,\beta,\theta), \quad \theta_0 \in \partial\Theta_0,$$

is the *asymptotic (relative) efficiency* of Pitman (*Pitman-ARE*) and

$$(3) \qquad e^B(T^{(2)},T^{(1)};\beta,\theta) = \lim_{\alpha\downarrow 0} \mathrm{eff}(T^{(2)},T^{(1)};\alpha,\beta,\theta), \quad \theta \in \Theta_1,$$

is the asymptotic relative efficiency of Bahadur (*Bahadur-ARE*), provided these limits exist. The representation (3) of the Bahadur-ARE is discussed in BAHADUR (1967a), RAGHAVACHARI (1970) and GROENEBOOM & OOSTERHOFF (1977). The better known equivalent definition in terms of attained levels is less appropriate for our purposes.

The Bahadur-ARE is often computed as a ratio of two (exact) slopes. The *slope* $c_j(\theta)$ of a test based on $\{T_n^{(j)}\}$ is defined as

$$(4) \qquad c_j(\theta) = -2 \lim_{\alpha\downarrow 0} (\log \alpha)/N_j(\alpha,\beta,\theta), \quad \theta \in \Theta_1,$$

provided the limit exists and is independent of β ($j = 1,2$). From (1) and (3) $e^B(T^{(2)},T^{(1)};\beta,\theta) = c_2(\theta)/c_1(\theta)$.

The merit of asymptotic efficiency measures depends on how well they reflect the performance of the tests involved for small or moderate sample sizes. In view of the defining relations (2) and (3) the operational meaning of both the Bahadur-ARE and the Pitman-ARE for moderate sample sizes seems to depend strongly on how accurately the ARE's approximate $\mathrm{eff}(T^{(2)},T^{(1)})$ when N_1 and N_2 are moderately small. Hence, from a practical point of view, the speed of convergence in (2) and (3) largely determines the usefulness of the asymptotic efficiencies of Pitman and Bahadur. The Pitman-ARE is reputed to be a fairly good approximation to $\mathrm{eff}(T^{(2)},T^{(1)})$ for moderate sample sizes in many testing problems, but very little is known about the accuracy of the approximation by the Bahadur-ARE. In the sequel this problem will be examined by a numerical analysis of a few simple examples.

If $e^P(T^{(2)},T^{(1)}) = 1$ (or $e^B(T^{(2)},T^{(1)}) = 1$), the Pitman-ARE (or the Bahadur-ARE) fails to discriminate between the tests based on $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$. To deal with such cases HODGES & LEHMANN (1970) proposed to study the asumptotic behavior of the *deficiency*

(5) $\qquad \mathrm{def}(T^{(2)},T^{(1)};\alpha,\beta,\theta) = N_2(\alpha,\beta,\theta) - N_1(\alpha,\beta,\theta), \qquad \theta \in \Theta_1$
$(0<\alpha<\beta<1).$

The deficiency is a very delicate tool for comparing the performance of two tests.

Asymptotic approximations to deficiencies have been obtained in two distinct ways: expansion of $\mathrm{def}(T^{(2)},T^{(1)})$ in powers of N_1 as $\theta \to \theta_0 \in \partial\Theta_0$ (cf. HODGES & LEHMANN (1970) and ALBERS (1974)) or as $\alpha \downarrow 0$ (cf. KALLENBERG (1978) and GROENEBOOM (1980)), corresponding to Pitman's approach and Bahadur's approach, respectively. Recall that both $\theta \to \theta_0$ and $\alpha \downarrow 0$ imply $N_1 \to \infty$ in most applications. A particular case arises if (5) has a finite limit as $\theta \to \theta_0$ or as $\alpha \downarrow 0$; these limits are called the Pitman asymptotic deficiency $d^P(T^{(2)},T^{(1)})$ and the Bahadur asymptotic deficiency $d^B(T^{(2)},T^{(1)})$, respectively.

Again the practical value of the asymptotic expansions and the asymptotic deficiencies strongly depends on how well the deficiency (5) is approximated for moderate values of N_1 and N_2.

In most applications Bahadur-ARE's do not depend on the fixed power β. The additional fraction of observations needed with the second test to achieve the same power as the first test (for a given $\theta \in \Theta_1$) can then be determined without further knowledge of the power functions of any of the two tests, provided the Bahadur-ARE is known and agrees well with the relative efficiency (1). Similarly, if the leading terms in the expansion of the deficiency do not depend on β and constitute a good estimate of the deficiency (5), these terms immediately yield the additional number of observations required with the second test to achieve the same power as the first test based on n observations (where n is arbitrary). However, if β does appear in the relevant expressions, the power function of at least one of the tests must be known. Moreover, the dependence of the results on β makes an assessment of the relative performance of both tests more difficult. Hence the usefulness of Bahadur-ARE's (or deficiency expansions) is less obvious in such cases.

In section 2 we consider a few simple testing problems and investigate numerically how well the asymptotic efficiencies and deficiency expansions in the sense of Bahadur approximate the small-sample concepts defined in

(1) and (5). For comparison the corresponding asymptotic efficiency and deficiency results in the Pitman sense are also given. Technical proofs are relegated to the appendix.

The numerical results indicate that in some cases the Bahadur-ARE is a better approximation to the relative efficiency (1) than the Pitman-ARE while in several other cases the Bahadur-ARE is a very unsatisfactory approximation indeed, especially for rather large values of α, say $\alpha = .05$. Disappointing limiting behavior of the Pitman-ARE is due to slow convergence of the distributions of the test statistics to a limiting distribution; explanation of unsatisfactory behavior of the Bahadur-ARE is less straightforward but involves the dependence of the shapes of the critical regions on α.

To approximate the deficiency, asymptotic expansions in the sense of Bahadur may sometimes be useful, but it appears that quite a few terms are necessary to ensure a satisfactory approximation. The leading term alone may be very misleading.

2. RESULTS

EXAMPLE 1. Let $X_1, X_2, \ldots$ be independent and normally $N(\theta,1)$ distributed, $\theta \in \mathbb{R}$. Consider the hypothesis $H_0 : \theta \le 0$ against $\theta > 0$. Let $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ and compare the one-sided and two-sided $\bar{X}$-test based on the statistics $\bar{X}_n$ and $|\bar{X}_n|$, respectively. The relative efficiency (or deficiency) indicates how much we loose when we incorrectly apply the two-sided instead of the one-sided test.

This testing problem can be generalized to higher dimension. Let $X_i = (Y_{i,1},\ldots,Y_{i,p})$ have a p-variate normal $N(\theta,I)$ distribution, $\theta = (\theta_1,\ldots,\theta_p) \in \mathbb{R}^p$. Consider the hypothesis $H_0 : \theta = 0$ against $\theta \neq 0$. In a sample of size n the likelihood ratio (LR) test rejects H_0 for large values of the χ^2-statistic $n \sum_{j=1}^{p} \bar{Y}_j^2$ where $\bar{Y}_j = n^{-1}\sum_{i=1}^{n} Y_{i,j}$. Fix an alternative $\theta \neq 0$. The most powerful (MP) test of H_0 against θ rejects H_0 for large values of the linear statistic $n^{\frac{1}{2}}\|\theta\|^{-1} \sum_{j=1}^{p} \theta_j \bar{Y}_j$ where $\|\theta\|$ is the euclidean norm of θ. Here the relative efficiency (deficiency) indicates how much the χ^2-test falls short of the best possible test. Note that the case p = 1 reduces to the previous problem.

The results of BAHADUR (1967b, 1971) on LR tests imply that $e^B(\chi^2\text{-test}, \text{MP test}; \beta,\theta) = 1$ for all $\beta \in (0,1)$ and $\theta \neq 0$. KALLENBERG (1978) has shown that the deficiency is of order $\mathcal{O}(1)$ for p = 1 and of order $\mathcal{O}(\log N_{MP})$ for p > 1, as $\alpha \downarrow 0$. A more detailed expansion is as follows.

PROPOSITION 1. *Let* $X_1, X_2, \ldots$ *have* p-*variate normal* $N(\theta, I)$ *distributions* $(p \geq 1)$. *Let* $a_p = \log(2^{3-p}\pi) - 2\log\Gamma(\tfrac{1}{2}p)$, $0 < \beta < 1$ *and* $u_\beta = \Phi^{-1}(1-\beta)$ *where* Φ *is the standard normal* cdf. *Then, as* $\alpha \downarrow 0$, *the deficiency of the* χ^2-test *w.r.t. the* MP *test has the expansion*

$$N_{\chi^2}(\alpha,\beta,\theta) - N_{MP}(\alpha,\beta,\theta) =$$

$$(p-1)\|\theta\|^{-2} \log N_{MP} \tag{6}$$

$$+ \|\theta\|^{-2} \{2(p-1)\log\|\theta\| - (p-1) + a_p\} \tag{7}$$

$$- (p-1)\|\theta\|^{-3} u_\beta N_{MP}^{-\frac{1}{2}} \log N_{MP} \tag{8}$$

$$+ \|\theta\|^{-3} u_\beta \{-2(p-1)\log\|\theta\| + \tfrac{5}{2}(p-1) - a_p\} N_{MP}^{-\frac{1}{2}} \tag{9}$$

$$+ (p-1)\|\theta\|^{-4} (u_\beta^2+p-2) N_{MP}^{-1} \log N_{MP} \tag{10}$$

$$+ \|\theta\|^{-4} \{(u_\beta^2+p-2)(2(p-1)\log\|\theta\|+a_p) - \tfrac{1}{6}(p-1)(20u_\beta^2 - 11)\} N_{MP}^{-1} \tag{11}$$

$$+ \mathcal{O}(N_{MP}^{-3/2} \log N_{MP}).$$

The proof is given in the appendix. Note that the power β does not enter in the first two terms of the expansion but that subsequent terms heavily depend on β. For $\beta = \tfrac{1}{2}$ the terms (8) and (9) vanish altogether. For p = 1 the Bahadur asymptotic deficiency exists and is equal to $\theta^{-2}\log 4$; for p > 1 it is infinite.

Numerical comparisons of the one-sided and two-sided $\bar{X}$-test and of the chisquare test and the linear MP test are made in Tables 1 and 2. For $\alpha = .05, .01, .001$, for $\beta = .5, .9$ and for $\theta = .25, .5, .75, 1$ the sample sizes

$N_{\overline{X}}$ and $N_{|\overline{X}|}$ are given in the first two columns of Table 1, followed by the relative efficiency $N_{\overline{X}} / N_{|\overline{X}|}$, the deficiency $N_{|\overline{X}|} - N_{\overline{X}}$, the Pitman-ARE e^P, the Bahadur-ARE e^B and the leading terms of the deficiency expansion $d_2^B = (6) + (7)$, $d_4^B = (6)+...+(9)$ and $d_6^B = (6)+...+(11)$. In Table 2 similar quantities are tabulated for $p = 2,5$, but N_{MP}, N_{χ^2} and $\|\theta\|$ replace $N_{\overline{X}}$, $N_{|\overline{X}|}$ and θ.

To achieve exact size α and exact power β at θ, the sample sizes N_1 and N_2 are randomized. Randomizing between two size-α tests based on $\{T_n^{(j)}\}$ with power β_n at n observations and power β_{n+1} at n+1 observations, $\beta_n \leq \beta < \beta_{n+1}$, yields $N_j = n+(\beta-\beta_n)/(\beta_{n+1}-\beta_n)$, $j = 1,2$. Here $(\beta-\beta_n)/(\beta_{n+1}-\beta_n)$ denotes the probability of an additional observation after a sample of size n has been taken. By this randomization procedure rounding-off effects on the entries in the tables are avoided.

Inspection of Tables 1 and 2 shows that the Bahadur-ARE (one) is an unsatisfactory approximation to the relative efficiency, especially for $\alpha = .05$ and $.01$ and $p > 1$. The Bahadur deficiency expansion yields fair approximations to the actual deficiencies, but the first term alone may be far off and even inclusion of several terms does not guarantee good accuracy. The Pitman-ARE is given by $e^P(\alpha,\beta) = (u_\beta-u_\alpha)^2/\gamma$ where γ is the noncentrality parameter of the χ^2-distribution to be solved from the equation $\chi^2_{p,\alpha} = \chi^2_{p,\beta}(\gamma)$; here $\chi^2_{p,\alpha}$ is the upper α-point of the χ^2-distribution with p d.f. (cf. ROTHE (1978)). Apart from the effect of randomization, e^P is equal to the small-sample efficiency, explaining the excellent agreement in Tables 1 and 2.

EXAMPLE 2. To compare the one-sided $\overline{X}$-test and the one-sided t-test, let $X_1, X_2, \ldots$ be independent and $N(\theta,1)$ distributed and suppose $H_0 : \theta \leq 0$ has to be tested against $\theta > 0$. The $\overline{X}$-test and the t-test reject H_0 for large values of the statistics $\overline{X}_n$ and $n^{\frac{1}{2}}\, \overline{X}_n/S_n$ respectively, where S_n^2 is the sample variance. The comparison shows how much we loose by not knowing the variance of the observations. In this case the Pitman-ARE is one and HODGES & LEHMANN (1970) therefore decided to study the Pitman asymptotic deficiency $d^P(\alpha,\beta) = \frac{1}{2}\Phi^{-1}(1-\alpha)^2$. The question arises whether the Bahadur-ARE $e^B(\beta,\theta) = \theta^{-2}\log(1+\theta^2)$ is a better approximation to the relative efficiency than the Pitman-ARE. Table 3 contains the relevant information for the same values of α,β and θ as Table 1. It is seen that for $\beta = .5$ the Bahadur-ARE is quite a good estimate of the relative efficiency, but that for $\beta = .9$ the Bahadur-ARE seriously underestimates the actual relative efficiencies. The Bahadur-ARE does not depend on β but Table 3 shows that the relative efficiencies do!

EXAMPLE 3. Consider independent random variables $X_1, X_2, \ldots$ with a double exponential distribution with density $\frac{1}{2}\exp(-|x-\theta|)$, $x \in \mathbb{R}$. Suppose H_0 : $\theta \leq 0$ has to be tested against $\theta > 0$. The sign test is both locally and asymptotically most powerful for this problem. We compare the sign test to the test based on the sample median defined as the $[\frac{1}{2}n+1]$th order statistic in a sample of size n. The power of this test (to be called the M-test) may decrease if an odd sample size is increased by one, but is otherwise an increasing function of n.

The present definition of the relative efficiency is based on the smallest sample size required to achieve power β with level α. An alternative definition uses the smallest sample size such that for all larger sizes the power is at least β, cf. BAHADUR (1967a), WIEAND (1976). This definition would sometimes lead to different relative efficiencies (N_M can differ at most one unit from the tabulated value). For the other examples both definitions coincide.

It is well known that the Pitman-ARE of the M-test w.r.t. the sign test is one and that their Bahadur-ARE is equal to (cf. SIEVERS (1969))

(12) $$e^B(\text{M-test, sign test}; \beta,\theta)$$

$$= \{-\tfrac{1}{2}\log(4q_\theta(1-q_\theta))\}/\{\log 2 + q_\theta \log q_\theta + (1-q_\theta)\log(1-q_\theta)\}$$

where $q_\theta = \frac{1}{2}e^{-\theta}$, $\theta > 0$. It is easily verified that the Bahadur-ARE is larger than one for all $\theta > 0$, indicating asymptotic superiority of the M-test. The asymptotic behavior of the deficiency for local alternatives is described in the following proposition (the proof is given in the appendix).

<u>PROPOSITION 2</u>. *Let* $X_1, X_2, \ldots$ *have a double exponential distribution with shift parameter* θ. *Then, as* $\theta \downarrow 0$, *the deficiency of the* M-*test w.r.t. the sign test is equal to*

$$
(13) \qquad N_M(\alpha,\beta,\theta) - N_S(\alpha,\beta,\theta)
$$

$$
= \begin{cases} -2(u_\alpha + u_{1-\beta})^{-1} u_\alpha \, u_{1-\beta} \, N_S^{\frac{1}{2}} + \mathcal{O}(1) & \text{if } \tfrac{1}{2} < \beta < 1 \\ -\frac{1}{3}u_\alpha^2 + \gamma_{N_S} + o(1) \quad (\text{with } 0 \le \gamma_{N_S} \le 1) & \text{if } \beta - \tfrac{1}{2} \\ -2u_{1-\beta} \, N_S^{\frac{1}{2}} + \mathcal{O}(1) & \text{if } \alpha < \beta < \tfrac{1}{2}. \end{cases}
$$

Since $u_{1-\beta} > 0$ if $\beta > \frac{1}{2}$ and $u_{1-\beta} < 0$ if $\beta < \frac{1}{2}$, the proposition implies that asymptotically the M-test is superior for $\beta \ge \frac{1}{2}$ and the sign test is superior for $\alpha < \beta < \frac{1}{2}$, a curious result indeed. In concordance with these asymptotic results Table 4 reveals that for moderate sample sizes the M-test is more efficient for $\beta \ge \frac{1}{2}$ and less efficient for $\beta < \frac{1}{2}$ compared to the sign test. For $\beta = \frac{1}{2}$ the Bahadur-ARE is generally in good agreement with the relative efficiencies, but for $\beta \neq \frac{1}{2}$ the approximation is unsatisfactory (cf. Example 2). In most cases the leading term of the Pitman deficiency expansion is a fair approximation to the actual deficiencies; it gives a better over-all description of the qualitative behavior of the two tests than the Bahadur-ARE.

<u>EXAMPLE 4</u>. Four tests for the multivariate linear hypothesis are considered. Let Y be a m×p random matrix whose row vectors Y_i are distributed independently according to p-variate normal $N(\mu_i, \Sigma)$ distributions. Suppose that

$$
EY = XB,
$$

where X is an m×q design matrix of rank $q < m$ and B is an unknown q×p matrix of regression coefficients. The multivariate linear hypothesis has the form

$$
H_0 : AB = 0,
$$

where A is an $n_1 \times q$ matrix of rank $n_1 \leq q$. The matrices of sums of squares and cross-products due to the hypothesis and due to error are respectively

$$S_{h,m} = Y'X(X'X)^{-1}A'\{A(X'X)^{-1}A'\}^{-1}A(X'X)^{-1}X'Y$$

and

$$S_{e,m} = Y'(I_m - X(X'X)^{-1}X')Y,$$

where I_m denotes the m×m identity matrix. $S_{e,m}$ has a central Wishart distribution $W_p(n,\Sigma)$ with $n = m-q$ and $S_{h,m}$ has a non-central Wishart distribution $W_p(n_1,\Sigma,m\Omega)$ where

$$\Omega = m^{-1}B'A'\{A(X'X)^{-1}A'\}^{-1}AB\Sigma^{-1}.$$

As a particular case consider the one-way multivariate analysis of variance set-up with q groups of k observations each (m = qk) where the p-dimensional observations in the i-th group have normal $N(\mu_i,\Sigma)$ distributions. The hypothesis to be tested is $\mu_1 = \ldots = \mu_q$, while $n_1 = q-1$ and $\Omega = q^{-1}\sum_{i=1}^{q}(\mu_i-\mu)'(\mu_i-\mu)\Sigma^{-1}$ with $\mu = q^{-1}\sum_{i=1}^{q}\mu_i$. Note that Ω is fixed for fixed alternatives $(\mu_1',\ldots,\mu_q')$.

We consider the following test statistics

$$T_m^{LR} = -\log|S_{e,m}(S_{e,m} + S_{h,m})^{-1}|,$$

$$T_m^{P} = \mathrm{tr}\, S_{h,m}(S_{h,m} + S_{e,m})^{-1},$$

$$T_m^{H} = \mathrm{tr}\, S_{h,m}\, S_{e,m}^{-1},$$

$$T_m^{R} = \text{largest eigenvalue of } S_{h,m}\, S_{e,m}^{-1},$$

where $|A|$ denotes the determinant of A. The tests based on these statistics are generally called the likelihood ratio (LR) test, Pillai's test, Hotelling's T^2 test and Roy's largest root test, respectively.

Let Θ be the diagonal matrix $\mathrm{diag}(\theta_1,\ldots,\theta_p)$ with $\theta_1 \geq \ldots \geq \theta_p \geq 0$ where the θ_i's are the eigenvalues of Ω. The distributions of the four test statistics only depend on the alternative through Θ. It is shown in HSIEH

(1979) (cf. also GROENEBOOM (1980)) that the exact slopes at Θ of the four tests are given by

$$c_{LR}(\Theta) = \log|I_p + \Theta|,$$

$$c_P\ (\Theta) = -p\ \log\{p^{-1}\mathrm{tr}((I_p + \Theta)^{-1})\},$$

$$c_H\ (\Theta) = \log(1 + \mathrm{tr}\ \Theta),$$

$$c_R\ (\Theta) = \log(1 + \theta_1).$$

The Bahadur-ARE's immediately follow as ratio's of the slopes, cf. section 1.

For all Θ the LR test has optimal slope (HSIEH (1979)). The slope of Pillai's test is optimal iff $\theta_1 = \ldots = \theta_p$, the slopes of Hotelling's test and Roy's test are optimal iff only one eigenvalue (θ_1) is positive. In the cases of equal slopes the leading terms of the Bahadur deficiency expansions have been obtained in GROENEBOOM (1980):

(14) $$d_1^B(P,LR;\beta,\mathrm{diag}(\theta_1,\ldots,\theta_1)) = -(\tfrac{1}{2}p(p+1)-1)\{p\ \log(1+\theta_1)\}^{-1}\log N_{LR},$$

(15) $$d_1^B(H,LR;\beta,\mathrm{diag}(\theta_1,0,\ldots,0)) = -(p-1)(n_1-1)\{\log(1+\theta_1)\}^{-1}\log N_{LR},$$

(16) $$d_1^B(R,LR;\beta,\mathrm{diag}(\theta_1,0,\ldots,0)) = -(p-1)(n_1-1)\{\log(1+\theta_1)\}^{-1}\log N_{LR},$$

(17) $$d^B(H,R;\beta,\ \mathrm{diag}(\theta_1,0,\ldots,0)) = (p-1)(n_1-1)\{\log(1+\theta_1)\}^{-1}.$$
$$.\{\log(\theta_1^{-1}(1+\theta_1)) - (1+\theta_1)^{-1}\}.$$

Note that asymptotically Pillai's test, Hotelling's test and Roy's test are superior to the LR test on the lines where they have optimal slope. Also note that the Bahadur asymptotic deficiency of Hotelling's test w.r.t. Roy's test is finite for $\Theta = \mathrm{diag}(\theta_1,0,\ldots,0)$, Roy's test being slightly better for such Θ.

The Pitman-ARE's of the LR test, Pitman's test and Hotelling's test w.r.t. one another are one; the Pitman-ARE of Roy's test w.r.t. the other

approximation of these terms to the small-sample deficiencies is less outrageous than for the leading terms of the Bahadur deficiency expansions, it remains quite unsatisfactory.

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TABLE 1

One-sided and two-sided $\overline{X}$-test

α	β	θ	$N_{\overline{X}}$	$N_{\|\overline{X}\|}$	$N_{\overline{X}}/N_{\|\overline{X}\|}$	$N_{\|\overline{X}\|}-N_{\overline{X}}$	e^P	e^B	d_2^B	d_4^B	d_6^B
.05	.5	.25	43.29	61.46	.704	18.17	.704	1	22.18	22.18	13.98
		.5	10.83	15.37	.704	4.54	.704	1	5.55	5.55	3.49
		.75	4.82	6.83	.705	2.02	.704	1	2.46	2.46	1.55
		1	2.72	3.85	.708	1.12	.704	1	1.39	1.39	.86
	.9	.25	137.02	168.12	.815	31.10	.815	1	22.18	31.89	33.56
		.5	34.26	42.03	.815	7.77	.815	1	5.55	7.97	8.39
		.75	15.24	18.70	.815	3.46	.815	1	2.46	3.54	3.73
		1	8.60	10.54	.816	1.94	.815	1	1.39	1.99	2.10
.01	.5	.25	86.59	106.16	.816	19.57	.816	1	22.18	22.18	18.08
		.5	21.65	26.54	.816	4.89	.816	1	5.55	5.55	4.52
		.75	9.63	11.80	.816	2.17	.816	1	2.46	2.46	2.01
		1	5.42	6.64	.816	1.22	.816	1	1.39	1.39	1.13
	.9	.25	208.27	238.07	.875	29.80	.875	1	22.18	30.06	31.15
		.5	52.07	59.52	.875	7.45	.875	1	5.55	7.51	7.79
		.75	23.15	26.47	.875	3.32	.875	1	2.46	3.34	3.46
		1	13.02	14.89	.874	1.87	.875	1	1.39	1.88	1.95
.001	.5	.25	152.79	173.24	.882	20.45	.882	1	22.18	22.18	19.86
		.5	38.20	43.31	.882	5.11	.882	1	5.55	5.55	4.96
		.75	16.98	19.25	.882	2.27	.882	1	2.46	2.46	2.21
		1	9.56	10.83	.882	1.27	.882	1	1.39	1.39	1.24
	.9	.25	305.80	334.46	.914	28.66	.914	1	22.18	28.68	29.43
		.5	76.46	83.62	.914	7.17	.914	1	5.55	7.17	7.36
		.75	33.98	37.17	.914	3.19	.914	1	2.46	3.19	3.27
		1	19.12	20.91	.914	1.79	.914	1	1.39	1.79	1.84

TABLE 2

Chi-square test and $\bar{X}$-test for multivariate location

Dimension p = 2

α	β	$\|\theta\|$	N_{MP}	$N_{\chi 2}$	$N_{MP}/N_{\chi 2}$	$N_{\chi 2}-N_{MP}$	e^P	e^B	d_2^B	d_4^B	d_6^B
.05	.5	.25	43.29	79.31	.546	36.02	.546	1	29.33	29.33	40.17
		.5	10.83	19.83	.546	9.00	.546	1	7.33	7.33	10.04
		.75	4.82	8.82	.547	4.00	.546	1	3.25	3.25	4.47
		1	2.72	4.96	.549	2.23	.546	1	1.82	1.82	2.52
	.9	.25	137.02	202.46	.677	65.44	.677	1	47.77	58.17	63.60
		.5	34.26	50.62	.677	16.36	.677	1	11.94	14.54	15.90
		.75	15.24	22.51	.677	7.27	.677	1	5.31	6.46	7.07
		1	8.60	12.68	.678	4.08	.677	1	2.98	3.63	3.97
.01	.5	.25	86.59	131.04	.661	44.44	.661	1	40.42	40.42	45.84
		.5	21.65	32.76	.661	11.11	.661	1	10.11	10.11	11.46
		.75	9.63	14.56	.661	4.94	.661	1	4.49	4.49	5.09
		1	5.42	8.19	.662	2.77	.661	1	2.52	2.52	2.86
	.9	.25	208.27	278.83	.747	70.56	.747	1	54.47	65.29	69.70
		.5	52.07	69.71	.747	17.64	.747	1	13.62	16.32	17.43
		.75	23.15	30.98	.747	7.83	.747	1	6.05	7.25	7.74
		1	13.02	17.45	.746	4.43	.747	1	3.40	4.08	4.36
.001	.5	.25	152.79	204.84	.746	52.05	.746	1	49.51	49.51	52.58
		.5	38.20	51.21	.746	13.01	.746	1	12.38	12.38	13.15
		.75	16.98	22.76	.746	5.78	.746	1	5.50	5.50	5.84
		1	9.56	12.81	.746	3.25	.746	1	3.09	3.09	3.29
	.9	.25	305.80	381.08	.802	75.28	.802	1	60.61	71.34	74.88
		.5	76.46	95.27	.803	18.82	.802	1	15.15	17.84	18.72
		.75	33.98	42.35	.802	8.37	.802	1	6.73	7.93	8.32
		1	19.12	23.83	.802	4.71	.802	1	3.79	4.46	4.68

TABLE 2 (continued)

Chi-square test and $\bar{X}$-test for multivariate location

Dimension p = 5

α	β	$\lVert\theta\rVert$	N_{MP}	$N_{\chi 2}$	$N_{MP}/M_{\chi 2}$	$N_{\chi 2}-N_{MP}$	e^P	e^B	d_2^B	d_4^B	d_6^B
.05	.5	.25	43.29	111.86	.387	68.57	.387	1	-13.28	-13.28	86.33
		.5	10.83	27.97	.387	17.14	.387	1	- 3.33	- 3.33	21.57
		.75	4.82	12.43	.387	7.62	.387	1	- 1.50	- 1.50	9.56
		1	2.72	6.99	.390	4.27	.387	1	- .88	- .88	5.31
	.9	.25	137.02	263.51	.520	126.49	.520	1	60.47	44.91	85.17
		.5	34.26	65.88	.520	31.62	.520	1	15.12	11.22	21.29
		.75	15.24	29.29	.520	14.05	.520	1	6.72	4.98	9.46
		1	8.60	16.49	.521	7.89	.520	1	3.77	2.79	5.31
.01	.5	.25	86.59	176.52	.491	89.93	.491	1	31.09	31.09	105.49
		.5	21.65	44.13	.491	22.48	.491	1	7.77	7.77	26.37
		.75	9.63	19.62	.491	9.99	.491	1	3.45	3.45	11.72
		1	5.42	11.03	.491	5.61	.491	1	1.93	1.93	6.59
	.9	.25	208.27	352.44	.591	144.17	.591	1	87.27	84.16	120.21
		.5	52.07	88.11	.591	36.04	.591	1	21.82	21.04	30.05
		.75	23.15	39.17	.591	16.02	.591	1	9.70	9.35	13.35
		1	13.02	22.03	.591	9.01	.591	1	5.45	5.26	7.51
.001	.5	.25	152.79	263.64	.580	110.85	.580	1	67.44	67.44	121.02
		.5	38.20	65.91	.580	27.71	.580	1	16.86	16.86	30.25
		.75	16.98	29.30	.580	12.32	.580	1	7.49	7.49	13.45
		1	9.56	16.48	.580	6.93	.580	1	4.21	4.21	7.56
	.9	.25	305.80	467.29	.654	161.49	.654	1	111.85	116.49	147.01
		.5	76.46	116.83	.654	40.37	.654	1	27.96	29.12	36.75
		.75	33.98	51.92	.654	17.95	.654	1	12.43	12.94	16.33
		1	19.12	29.22	.654	10.10	.654	1	6.99	7.28	9.19

TABLE 3

$\overline{X}$-test and t-test

α	β	θ	$N_{\overline{X}}$	N_t	$N_{\overline{X}}/N_t$	$N_t-N_{\overline{X}}$	e^P	e^B	d^P
.05	.5	.25	43.29	44.67	.969	1.38	1	.970	1.35
		.5	10.83	12.27	.882	1.44	1	.893	1.35
		.75	4.82	6.35	.759	1.53	1	.793	1.35
		1	2.72	4.33	.629	1.61	1	.693	1.35
	.9	.25	137.02	138.39	.990	1.37	1	.970	1.35
		.5	34.26	35.66	.961	1.40	1	.893	1.35
		.75	15.24	16.69	.913	1.45	1	.793	1.35
		1	8.60	10.09	.852	1.49	1	.693	1.35
.01	.5	.25	86.59	89.31	.970	2.72	1	.970	2.71
		.5	21.65	24.39	.888	2.74	1	.893	2.71
		.75	9.63	12.39	.777	2.76	1	.793	2.71
		1	5.42	8.19	.662	2.77	1	.693	2.71
	.9	.25	208.27	210.99	.987	2.72	1	.970	2.71
		.5	52.07	54.82	.950	2.75	1	.893	2.71
		.75	23.15	25.94	.893	2.79	1	.793	2.71
		1	13.02	15.87	.820	2.85	1	.693	2.71
.001	.5	.25	152.79	157.56	.970	4.76	1	.970	4.77
		.5	38.20	42.93	.890	4.73	1	.893	4.77
		.75	16.98	21.65	.784	4.67	1	.793	4.77
		1	9.56	14.15	.676	4.59	1	.693	4.77
	.9	.25	305.80	310.58	.985	4.78	1	.970	4.77
		.5	76.46	81.24	.941	4.79	1	.893	4.77
		.75	33.98	38.79	.876	4.81	1	.793	4.77
		1	19.12	23.92	.799	4.80	1	.693	4.77

TABLE 4

Sign test and M-test

α	β	θ	N_S	N_M	N_S/N_M	N_M-N_S	$e^P(M,S)$	$e^B(M,S)$	$d_1^P(M,S)$
.05	.25	.25	19.68	24.27	.811	4.58	1	1.017	5.99
		.5	6.60	8.32	.794	1.71	1	1.057	3.47
		.75	3.76	4.46	.843	.70	1	1.114	2.62
		1	3.24	2.67	1.210	- .56	1	1.182	2.43
	.5	.25	54.80	54.43	1.007	- .38	1	1.017	- .90
		.5	17.28	16.45	1.051	- .83	1	1.057	- .90
		.75	9.48	8.50	1.115	- .98	1	1.114	- .90
		1	6.63	4.89	1.356	- 1.74	1	1.182	- .90
	.9	.25	172.25	154.04	1.118	-18.21	1	1.017	-18.91
		.5	52.38	42.50	1.232	- 9.88	1	1.057	-10.43
		.75	27.67	20.59	1.344	- 7.08	1	1.114	- 7.58
		1	17.85	12.56	1.422	- 5.30	1	1.182	- 6.09
.01	.25	.25	55.98	62.72	.892	6.74	1	1.017	10.09
		.5	18.16	20.48	.887	2.32	1	1.057	5.75
		.75	9.82	10.79	.912	.95	1	1.114	4.23
		1	6.91	6.88	1.003	- .02	1	1.182	3.55
	.5	.25	110.07	108.41	1.015	- 1.66	1	1.017	- 1.80
		.5	34.23	32.48	1.054	- 1.75	1	1.057	- 1.80
		.75	18.52	16.62	1.114	- 1.90	1	1.114	- 1.80
		1	12.68	10.61	1.194	- 2.06	1	1.182	- 1.80
	.9	.25	262.12	236.21	1.110	-25.92	1	1.017	-26.76
		.5	79.78	66.04	1.208	-13.73	1	1.057	-14.76
		.75	41.92	32.31	1.298	- 9.61	1	1.114	-10.70
		1	28.37	20.01	1.418	- 8.36	1	1.182	- 8.80
.001	.25	.25	119.52	128.61	.929	9.09	1	1.017	14.75
		.5	37.71	40.63	.928	2.92	1	1.057	8.28
		.75	20.98	22.25	.943	1.27	1	1.114	6.18
		1	15.08	14.44	1.044	- .64	1	1.182	5.24
	.5	.25	193.59	190.52	1.016	- 3.06	1	1.017	- 3.18
		.5	60.29	56.73	1.063	- 3.56	1	1.057	- 3.18
		.75	32.96	28.89	1.141	- 4.07	1	1.114	- 3.18
		1	22.57	18.67	1.209	- 3.91	1	1.182	- 3.18
	.9	.25	385.13	350.20	1.100	-34.93	1	1.017	-35.56
		.5	117.78	98.50	1.196	-19.28	1	1.057	-19.66
		.75	62.73	48.53	1.293	-14.20	1	1.114	-14.35
		1	42.05	30.19	1.393	-11.85	1	1.182	-11.75

TABLE 5

Efficiencies of tests for the bivariate linear hypothesis

$n_1 = 3$ (i.e. 4 levels in one-way MANOVA)

α	β	θ_1	θ_2	N_{LR}	N_P	N_H	N_R	$\frac{N_{LR}}{N_P}$	$\frac{N_{LR}}{N_H}$	$\frac{N_{LR}}{N_R}$	$e^B_{P,LR}$	$e^B_{H,LR}$	$e^B_{R,LR}$
.05	.5	.1	0	80.55	81.12	80.08	78.99	.993	1.006	1.020	.976	1	1
		.1	.1	41.69	41.06	42.35	49.46	1.015	.984	.843	1	.956	.5
		.25	0	35.66	36.35	35.26	35.03	.981	1.011	1.018	.944	1	1
		.25	.1	25.95	25.49	26.52	29.80	1.018	.979	.871	.987	.942	.701
		.25	.25	19.34	18.52	20.18	23.27	1.044	.958	.831	1	.909	.5
		.5	0	20.82	21.71	20.51	20.52	.959	1.015	1.015	.899	1	1
		.5	.1	17.52	17.42	17.91	19.23	1.006	.978	.911	.952	.939	.810
		.5	.25	14.60	13.83	15.43	17.27	1.056	.946	.846	.987	.890	.645
		.5	.5	12.04	10.95	13.05	14.66	1.099	.923	.821	1	.855	.5
		1	0	13.54	14.80	13.34	13.41	.915	1.015	1.009	.830	1	1
		1	.1	12.33	12.75	12.61	13.08	.967	.978	.943	.888	.941	.879
		1	.25	11.09	10.66	11.77	12.57	1.040	.942	.882	.940	.885	.756
		1	.5	9.83	8.83	10.79	11.74	1.114	.911	.837	.981	.834	.631
		1	1	8.59	7.40	9.65	10.46	1.161	.889	.821	1	.792	.5
	.9	.1	0	179.94	180.49	179.37	174.56	.997	1.003	1.031	.976	1	1
		.1	.1	90.99	90.41	91.59	109.12	1.006	.993	.834	1	.956	.5
		.25	0	75.49	76.14	74.96	73.20	.992	1.007	1.031	.944	1	1
		.25	.1	54.07	53.69	54.51	62.36	1.007	.992	.867	.987	.942	.701
		.25	.25	38.85	38.19	39.54	47.06	1.017	.982	.826	1	.909	.5
		.5	0	40.81	41.58	40.30	39.54	.981	1.013	1.032	.899	1	1
		.5	.1	33.94	33.92	34.13	37.04	1.000	.995	.916	.952	.939	.810
		.5	.25	27.54	26.98	28.16	32.67	1.021	.978	.843	.987	.890	.645
		.5	.5	21.61	20.80	22.43	26.53	1.039	.963	.815	1	.855	.5
		1	0	23.65	24.67	23.16	22.89	.959	1.021	1.033	.830	1	1
		1	.1	21.40	21.86	21.38	22.32	.979	1.001	.959	.888	.941	.879
		1	.25	18.89	18.79	19.33	21.28	1.006	.978	.888	.940	.885	.756
		1	.5	16.15	15.49	16.93	19.45	1.042	.954	.830	.981	.834	.631
		1	1	13.18	12.13	14.16	16.43	1.086	.930	.802	1	.792	.5
.01	.5	.1	0	125.05	126.10	124.17	120.61	.992	1.007	1.037	.976	1	1
		.1	.1	63.94	62.88	65.08	80.08	1.017	.983	.779	1	.956	.5
		.25	0	54.60	55.83	53.81	52.73	.978	1.015	1.036	.944	1	1
		.25	.1	39.22	38.50	40.12	46.62	1.019	.978	.841	.987	.942	.701
		.25	.25	28.76	27.53	30.09	36.50	1.045	.956	.788	1	.909	.5
		.5	0	31.17	32.67	30.49	30.16	.954	1.022	1.033	.899	1	1
		.5	.1	25.88	25.79	26.42	28.80	1.003	.980	.899	.952	.939	.810
		.5	.25	21.22	20.18	22.47	26.17	1.052	.945	.811	.987	.890	.645
		.5	.5	17.11	15.73	18.63	22.00	1.088	.918	.778	1	.855	.5
		1	0	19.49	21.50	18.98	18.93	.906	1.027	1.029	.830	1	1
		1	.1	17.55	18.30	17.83	18.61	.959	.984	.943	.888	.941	.879
		1	.25	15.54	15.15	16.49	18.01	1.026	.943	.863	.940	.885	.756
		1	.5	13.49	12.35	14.91	16.86	1.093	.905	.800	.981	.834	.631
		1	1	11.40	10.03	13.01	14.75	1.136	.877	.773	1	.792	.5

TABLE 5 (continued)

Efficiencies of tests for the bivariate linear hypothesis

n_1 = 3 (i.e. 4 levels in one-way MANOVA)

α	β	θ_1	θ_2	N_{LR}	N_P	N_H	N_R	$\frac{N_{LR}}{N_P}$	$\frac{N_{LR}}{N_H}$	$\frac{N_{LR}}{N_R}$	$e^B_{P,LR}$	$e^B_{H,LR}$	$e^B_{R,LR}$
.01	.9	.1	0	239.60	240.55	238.71	230.44	.996	1.004	1.040	.976	1	1
		.1	.1	120.89	119.93	121.87	151.55	1.008	.992	.798	1	.956	.5
		.25	0	100.57	101.63	99.71	96.68	.990	1.009	1.040	.944	1	1
		.25	.1	71.77	71.16	72.47	85.33	1.009	.990	.841	.987	.942	.701
		.25	.25	51.42	50.38	52.52	65.12	1.021	.979	.790	1	.909	.5
		.5	0	54.31	55.56	53.49	52.17	.978	1.015	1.041	.899	1	1
		.5	.1	44.97	44.96	45.27	49.81	1.000	.993	.903	.952	.939	.810
		.5	.25	36.34	35.51	37.31	44.68	1.023	.974	.813	.987	.890	.645
		.5	.5	28.36	27.19	29.62	36.39	1.043	.957	.779	1	.855	.5
		1	0	31.29	32.89	30.54	30.03	.952	1.025	1.042	.830	1	1
		1	.1	28.18	28.91	28.15	29.53	.975	1.001	.954	.888	.941	.879
		1	.25	24.77	24.63	25.41	28.46	1.006	.975	.870	.940	.885	.756
		1	.5	21.01	20.10	22.19	26.25	1.045	.947	.800	.981	.834	.631
		1	1	16.95	15.67	18.42	22.10	1.082	.920	.767	1	.792	.5
.001	.5	.1	0	184.61	186.43	183.08	176.27	.990	1.008	1.047	.976	1	1
		.1	.1	93.67	91.98	95.52	123.84	1.018	.981	.756	1	.956	.5
		.25	0	80.02	81.93	78.61	76.47	.977	1.018	1.046	.944	1	1
		.25	.1	57.02	55.88	58.41	70.01	1.020	.976	.814	.987	.942	.701
		.25	.25	41.41	39.64	43.46	55.39	1.045	.953	.748	1	.909	.5
		.5	0	45.07	47.50	43.92	43.17	.949	1.026	1.044	.899	1	1
		.5	.1	37.15	37.08	37.91	41.88	1.002	.980	.887	.952	.939	.810
		.5	.25	30.18	28.77	32.05	38.71	1.049	.942	.780	.987	.890	.645
		.5	.5	24.00	22.23	26.26	32.51	1.079	.914	.738	1	.855	.5
		1	0	27.55	30.66	26.66	26.48	.899	1.033	1.041	.830	1	1
		1	.1	24.64	25.84	24.96	26.17	.953	.987	.942	.888	.941	.879
		1	.25	21.63	21.25	22.97	25.55	1.018	.942	.846	.940	.885	.756
		1	.5	18.52	17.18	20.60	24.13	1.078	.899	.767	.981	.834	.631
		1	1	15.33	13.74	17.70	20.97	1.116	.866	.731	1	.792	.5
	.9	.1	0	316.42	317.87	315.05	-	.995	1.004	-	.976	1	1
		.1	.1	159.32	157.82	160.96	208.85	1.010	.990	.763	1	.956	.5
		.25	0	132.91	134.60	131.60	127.16	.987	1.010	1.045	.944	1	1
		.25	.1	94.54	93.61	95.66	115.79	1.010	.988	.816	.987	.942	.701
		.25	.25	67.63	66.07	69.30	89.56	1.024	.976	.755	1	.909	.5
		.5	0	71.77	73.74	70.53	68.61	.973	1.018	1.046	.899	1	1
		.5	.1	59.22	59.27	59.71	66.52	.999	.992	.890	.952	.939	.810
		.5	.25	47.69	46.51	49.21	60.78	1.025	.969	.785	.987	.890	.645
		.5	.5	37.10	35.49	38.97	49.78	1.045	.952	.745	1	.855	.5
		1	0	41.24	43.71	40.11	39.44	.994	1.028	1.046	.830	1	1
		1	.1	37.01	38.16	36.97	38.98	.970	1.001	.950	.888	.941	.879
		1	.25	32.40	32.24	33.36	37.92	1.005	.971	.854	.940	.885	.756
		1	.5	27.36	26.16	29.09	35.41	1.046	.941	.773	.981	.834	.631
		1	1	21.91	20.31	24.01	29.85	1.079	.913	.734	1	.792	.5

TABLE 5 (continued)

Efficiencies of tests for the bivariate linear hypothesis

n_1 = 7 (i.e. 8 levels in one-way MANOVA)

α	β	θ_1	θ_2	N_{LR}	N_P	N_H	N_R	$\frac{N_{LR}}{N_P}$	$\frac{N_{LR}}{N_H}$	$\frac{N_{LR}}{N_R}$	$e^B_{P,LR}$	$e^B_{H,LR}$	$e^B_{R,LR}$
.05	.5	.1	0	114.74	116.27	113.49	110.10	.987	1.011	1.042	.976	1	1
		.1	.1	59.85	59.29	60.60	71.58	1.010	.988	.836	1	.956	.5
		.25	0	51.58	53.35	50.51	49.72	.967	1.021	1.037	.944	1	1
		.25	.1	37.78	37.55	38.37	43.34	1.006	.985	.872	.987	.942	.701
		.25	.25	28.49	27.77	29.53	34.36	1.026	.965	.829	1	.909	.5
		.5	0	30.70	32.82	29.87	29.80	.936	1.028	1.030	.899	1	1
		.5	.1	25.97	26.37	26.32	28.27	.985	.987	.919	.952	.939	.810
		.5	.25	21.87	21.32	22.89	25.75	1.026	.955	.850	.987	.890	.645
		.5	.5	18.33	17.45	19.61	22.14	1.050	.935	.828	1	.855	.5
		1	0	20.43	22.99	19.91	20.04	.889	1.026	1.020	.830	1	1
		1	.1	18.73	19.76	18.93	19.66	.948	.989	.952	.888	.941	.879
		1	.25	17.00	16.89	17.81	19.03	1.006	.954	.893	.940	.885	.756
		1	.5	15.28	14.51	16.49	17.96	1.053	.926	.851	.981	.834	.631
		1	1	13.59	12.59	14.93	16.18	1.079	.910	.840	1	.792	.5
	.9	.1	0	240.01	241.63	238.49	225.94	.993	1.006	1.062	.976	1	1
		.1	.1	121.91	121.36	122.54	146.41	1.005	.995	.833	1	.956	.5
		.25	0	102.00	103.85	100.55	95.97	.982	1.014	1.063	.944	1	1
		.25	.1	73.24	73.11	73.57	83.79	1.002	.996	.874	.987	.942	.701
		.25	.25	53.05	52.44	53.84	64.23	1.012	.985	.826	1	.909	.5
		.5	0	56.19	58.37	54.80	52.86	.963	1.025	1.063	.899	1	1
		.5	.1	46.80	47.43	46.70	50.18	.987	1.002	.933	.952	.939	.810
		.5	.25	38.20	37.85	38.87	44.97	1.009	.983	.849	.987	.890	.645
		.5	.5	30.35	29.62	31.34	37.03	1.024	.968	.820	1	.855	.5
		1	0	33.51	36.33	32.25	31.57	.923	1.039	1.061	.830	1	1
		1	.1	30.35	31.98	29.92	30.95	.949	1.014	.981	.888	.941	.879
		1	.25	26.92	27.43	27.26	29.79	.981	.987	.903	.940	.885	.756
		1	.5	23.24	22.82	24.16	27.60	1.019	.962	.842	.981	.834	.631
		1	1	19.35	18.48	20.57	23.69	1.047	.941	.817	1	.792	.5
.01	.5	.1	0	172.21	174.71	170.09	160.88	.986	1.012	1.070	.976	1	1
		.1	.1	88.58	87.75	89.64	110.30	1.010	.988	.803	1	.956	.5
		.25	0	76.03	78.83	74.11	71.27	.965	1.026	1.067	.944	1	1
		.25	.1	54.88	54.63	55.60	64.23	1.005	.987	.854	.987	.942	.701
		.25	.25	40.61	39.68	42.00	50.99	1.023	.967	.796	1	.909	.5
		.5	0	44.03	47.28	42.41	41.49	.931	1.038	1.061	.899	1	1
		.5	.1	36.72	37.45	36.95	39.97	.981	.994	.919	.952	.939	.810
		.5	.25	30.62	29.77	31.65	36.79	1.020	.959	.825	.987	.890	.645
		.5	.5	24.79	23.80	26.49	31.29	1.042	.936	.792	1	.855	.5
		1	0	28.06	31.99	26.85	26.69	.877	1.045	1.051	.830	1	1
		1	.1	25.39	27.20	25.34	26.33	.933	1.002	.964	.888	.941	.879
		1	.25	22.66	22.88	23.58	25.64	.990	.961	.884	.940	.885	.756
		1	.5	19.90	19.16	21.49	24.24	1.038	.926	.821	.981	.834	.631
		1	1	17.11	16.06	18.97	21.46	1.065	.902	.797	1	.792	.5

TABLE 5 (continued)

Efficiencies of tests for the bivariate linear hypothesis

$n_1 = 7$ (i.e. 8 levels in one-way MANOVA)

α	β	θ_1	θ_2	N_{LR}	N_P	N_H	N_R	$\frac{N_{LR}}{N_P}$	$\frac{N_{LR}}{N_H}$	$\frac{N_{LR}}{N_R}$	$e^B_{P,LR}$	$e^B_{H,LR}$	$e^B_{R,LR}$
.01	.9	.1	0	312.72	315.15	310.47	290.32	.992	1.007	1.077	.976	1	1
		.1	.1	158.33	157.54	159.23	196.88	1.005	.994	.804	1	.956	.5
		.25	0	132.61	135.32	130.45	123.05	.980	1.017	1.078	.944	1	1
		.25	.1	94.82	94.64	95.26	110.74	1.002	.995	.856	.987	.942	.701
		.25	.25	68.38	67.53	69.47	85.67	1.013	.984	.798	1	.909	.5
		.5	0	72.70	75.80	70.64	67.44	.959	1.029	1.078	.899	1	1
		.5	.1	60.27	61.16	60.06	64.99	.985	1.003	.927	.952	.939	.810
		.5	.25	48.91	48.48	49.81	59.10	1.009	.982	.828	.987	.890	.645
		.5	.5	38.57	37.66	39.90	48.73	1.024	.967	.791	1	.855	.5
		1	0	42.86	46.74	41.00	39.81	.917	1.045	1.077	.830	1	1
		1	.1	38.64	40.87	37.96	39.27	.945	1.018	.984	.888	.941	.879
		1	.25	34.06	34.78	34.47	38.12	.979	.988	.893	.940	.885	.756
		1	.5	29.15	28.69	30.36	35.57	1.016	.960	.819	.981	.834	.631
		1	1	23.91	22.92	25.53	30.39	1.043	.936	.787	1	.792	.5
.001	.5	.1	0	245.91	249.61	242.45	225.87	.985	1.014	1.089	.976	1	1
		.1	.1	125.35	124.14	126.88	162.75	1.010	.988	.770	1	.956	.5
		.25	0	107.39	111.58	104.38	98.96	.962	1.029	1.085	.944	1	1
		.25	.1	76.86	76.54	77.76	91.81	1.004	.988	.837	.987	.942	.701
		.25	.25	56.21	54.99	58.14	73.58	1.022	.967	.764	1	.909	.5
		.5	0	61.19	65.94	58.58	56.60	.928	1.045	1.081	.899	1	1
		.5	.1	50.62	51.78	50.75	55.21	.978	.997	.917	.952	.939	.810
		.5	.25	41.37	40.70	43.06	51.59	1.017	.961	.802	.987	.890	.645
		.5	.5	33.27	32.07	35.53	43.80	1.037	.936	.760	1	.855	.5
		1	0	37.99	43.65	35.95	35.40	.870	1.057	1.073	.830	1	1
		1	.1	34.11	36.82	33.77	35.08	.926	1.010	.972	.888	.941	.879
		1	.25	30.12	30.67	31.20	34.42	.982	.965	.875	.940	.885	.756
		1	.5	26.04	25.31	28.15	32.78	1.029	.925	.794	.981	.834	.631
		1	1	21.89	20.75	24.42	28.80	1.055	.896	.760	1	.792	.5
	.9	.1	0	403.56	407.16	400.28	371.34	.991	1.008	1.087	.976	1	1
		.1	.1	203.85	202.62	205.12	262.84	1.006	.994	.776	1	.956	.5
		.25	0	170.90	174.80	167.77	157.26	.978	1.019	1.087	.944	1	1
		.25	.1	121.81	121.56	122.36	145.28	1.002	.995	.838	.987	.942	.701
		.25	.25	87.57	86.41	89.08	113.78	1.013	.983	.770	1	.909	.5
		.5	0	93.39	97.78	90.47	85.89	.995	1.032	1.087	.899	1	1
		.5	.1	77.16	78.39	76.82	83.76	.984	1.004	.921	.952	.939	.810
		.5	.25	62.35	61.82	63.61	77.43	1.009	.980	.805	.987	.890	.645
		.5	.5	48.91	47.76	50.73	64.07	1.024	.964	.763	1	.855	.5
		1	0	54.67	59.97	52.04	50.33	.912	1.051	1.086	.830	1	1
		1	.1	49.09	52.16	48.11	49.87	.941	1.020	.984	.888	.941	.879
		1	.25	43.07	44.09	43.59	48.81	.977	.988	.882	.940	.885	.756
		1	.5	36.65	36.11	38.26	45.98	1.015	.958	.797	.981	.834	.631

TABLE 6

Deficiencies of tests for the bivariate linear hypothesis

$n_1 = 3,\ \beta = .5$

α	θ_1	θ_2	$N_P - N_{LR}$	$N_H - N_{LR}$	$N_R - N_{LR}$	$N_H - N_R$	$d^B_{P,LR}$	$d^B_{H,LR}$	$d^B_{R,LR}$	$d^B_{H,R}$
.05	.1	0	.57	-.47	-1.56	1.09		-92.10	-92.10	65.72
	.1	.1	-.63	.66	7.75	-7.11	-39.14			
	.5	0	.89	-.32	-.30	-.02		-14.98	-14.98	2.13
	.5	.1	-.10	.39	1.71	-1.32				
	.5	.25	-.77	.83	2.67	-1.84				
	.5	.5	-1.09	1.01	2.62	-1.62	-6.14			
	1	0	1.27	-.20	-.12	-.07		-7.52	-7.52	.56
	1	.1	.42	.27	.75	-.47				
	1	.25	-.43	.68	1.48	-.80				
	1	1	-1.19	1.07	1.87	-.80	-3.10			
.01	.1	0	1.05	-.88	-4.44	3.56		-101.32	-101.32	65.72
	.1	.1	-1.06	1.13	16.14	-15.00	-43.63			
	.5	0	1.50	-.68	-1.00	.33		-16.97	-16.97	2.13
	.5	.1	-.09	.54	2.91	-2.37				
	.5	.25	-1.04	1.25	4.95	-3.70				
	.5	.5	-1.38	1.52	4.89	-3.37	-7.00			
	1	0	2.01	-.51	-.55	.05		-8.57	-8.57	.56
	1	.1	.75	.28	1.06	-.78				
	1	.25	-.39	.95	2.46	-1.52				
	1	1	-1.37	1.61	3.35	-1.74	-3.51			
.001	.1	0	1.83	-1.53	-8.34	6.81		-109.50	-109.50	65.72
	.1	.1	-1.69	1.85	30.18	-28.32	-47.63			
	.5	0	2.42	-1.16	-1.90	.75		-18.78	-18.78	2.13
	.5	.1	-.07	.76	4.73	-3.97				
	.5	.25	-1.41	1.86	8.53	-6.67				
	.5	.5	-1.77	2.26	8.51	-6.25	-7.84			
	1	0	3.10	-.89	-1.08	.19		-9.57	-9.57	.56
	1	.1	1.20	.32	1.53	-1.21				
	1	.25	-.37	1.34	3.92	-2.58				
	1	1	-1.59	2.36	5.63	-3.27	-3.94			

APPENDIX

We prove propositions 1 and 2 and make some remarks on the computation of the power functions of the multivariate tests in the final example.

PROOF OF PROPOSITION 1.

Fix an alternative $\theta \in \mathbb{R}^p$. Under this alternative we can write

$$\sum_{j=1}^{p} \bar{Y}_j^2 = \sum_{j=1}^{p} \theta_j^2 + 2\sum_{j=1}^{p} \theta_j(\bar{Y}_j-\theta_j) + \sum_{j=1}^{p} (\bar{Y}_j-\theta_j)^2$$

$$= \sum_{j=1}^{p} \theta_j^2 + 2n^{-\frac{1}{2}} \sum_{j=1}^{p} \theta_j U_j + n^{-1} \sum_{j=1}^{p} U_j^2,$$

where U_j has a standard normal distribution $(j=1,\ldots,p)$. The characteristic function of $n^{\frac{1}{2}}(\bar{Y}_1^2-\theta_1^2)$ evaluated at t is

$$E\exp\,(2it\theta_1U_1+itn^{-\frac{1}{2}}U_1^2)$$

$$= E\exp(2it\theta_1U_1).\{1+itn^{-\frac{1}{2}}U_1^2+\tfrac{1}{2}(it)^2n^{-1}U_1^4+\tfrac{1}{6}(it)^3n^{-\frac{3}{2}}U_1^6\} + R_{1,n}(t)$$

$$= \exp(-2\theta_1^2t^2).\{1+itn^{-\frac{1}{2}}+4\theta_1^2(it)^3n^{-\frac{1}{2}}+\tfrac{3}{2}(it)^2n^{-1}+12\theta_1^2(it)^4n^{-1}+$$

$$+8\theta_1^4(it)^6n^{-1}+\tfrac{5}{2}(it)^3n^{-3/2}+30\theta_1^2(it)^5n^{-3/2}+40\theta_1^4(it)^7n^{-3/2}+$$

$$+\tfrac{32}{3}\theta_1^6(it)^9n^{-3/2}\} + R_{1,n}(t),$$

and hence, putting $\tau = \|\theta\|$, the characteristic function f_n of $\frac{1}{2}\tau^{-1}n^{\frac{1}{2}}\sum_{j=1}^{p}(\bar{Y}_j^2-\theta_j^2)$ is equal to

$$f_n(t) = \exp(-\tfrac{1}{2}t^2).\{1+\tfrac{1}{2}\tau^{-1}n^{-\frac{1}{2}}(pit+(it)^3)+\tfrac{1}{8}\tau^{-2}n^{-1}(p(p+2)(it)^2+$$

$$+2(p+2)(it)^4+(it)^6) + \tfrac{1}{48}\tau^{-3}n^{-3/2}(p(p+2)(p+4)(it)^3+$$

$$+3(p+2)(p+4)(it)^5+3(p+4)(it)^7+(it)^9)\} + R_n(t).$$

The remainder terms $R_{1,n}(t)$ and $R_n(t)$ in these expansions are of order $\mathcal{O}(n^{-2})$, as $n \to \infty$, for each fixed t.

Neglecting $R_n(t)$, inversion of f_n yields an expansion of the cdf F_n of $\frac{1}{2}\tau^{-1}n^{\frac{1}{2}} \sum_{j=1}^{p} (\overline{Y}_j^2-\theta_j^2)$. Application of Esséen's smoothing lemma (cf. FELLER (1971) Lemma 14.3.2 with $T = n^2$) and a more careful analysis of $R_n(t)$ shows that inclusion of $R_n(t)$ would only add a term of order $\mathcal{O}(n^{-2})$ to the expansion of F_n. Hence, as $n \to \infty$,

$$F_n(x) = \Phi(x)-\tfrac{1}{2}\tau^{-1}n^{-\frac{1}{2}}\{p\Phi^{(1)}(x)+\Phi^{(3)}(x)\}$$

$$+\tfrac{1}{8}\tau^{-2}n^{-1}\{p(p+2)\Phi^{(2)}(x)+2(p+2)\Phi^{(4)}(x)+\Phi^{(6)}(x)\}$$

$$-\tfrac{1}{48}\tau^{-3}n^{-3/2}\{p(p+2)(p+4)\Phi^{(3)}(x)+3(p+2)(p+4)\Phi^{(5)}(x)$$

$$+3(p+4)\Phi^{(7)}(x)+\Phi^{(9)}(x)\}+\mathcal{O}(n^{-2}).$$

Defining $v_{\beta,n} = F_n^{-1}(1-\beta)$, it follows after a little algebra that

$$v_{\beta,n} = u_\beta+\tfrac{1}{2}\tau^{-1}(u_\beta^2+p-1)n^{-\frac{1}{2}}+\tfrac{1}{4}(p-1)\tau^{-2}u_\beta n^{-1}-\tfrac{1}{12}(p-1)\tau^{-3}(u_\beta^2-1)n^{-3/2}$$

$$+\mathcal{O}(n^{-2}).$$

Let c_n be the critical value of the χ^2-test based on $n\sum_{j=1}^{p}\overline{Y}_j^2$ with power β at θ. Then

$$c_n = \tau^2 n + 2\tau u_\beta n^{\frac{1}{2}} + u_\beta^2 + p - 1 + \tfrac{1}{2}(p-1)\tau^{-1}u_\beta n^{-\frac{1}{2}}$$

$$-\tfrac{1}{6}(p-1)\tau^{-2}(u_\beta^2-1)n^{-1} + \mathcal{O}(n^{-3/2}),$$

and therefore under the null hypothesis

(A1) $$\log\alpha = \log P_0\{n\sum_{j=1}^{p}\overline{Y}_j^2 \geq c_n\} = \log \Pr\{\chi_p^2 \geq c_n\}$$

$$= -\tfrac{1}{2}c_n+(\tfrac{1}{2}p-1)(\log(\tfrac{1}{2}c_n)-\log\Gamma(\tfrac{1}{2}p)+(p-2)c_n^{-1}+\mathcal{O}(c_n^{-2}) =$$

$$= -\tfrac{1}{2}\tau^2 n - \tau u_\beta n^{\frac{1}{2}} + (\tfrac{1}{2}p-1)\log n - \tfrac{1}{2}(u_\beta^2+p-1)$$

$$+ (\tfrac{1}{2}p-1)\log(\tfrac{1}{2}\tau^2) - \log\Gamma(\tfrac{1}{2}p) + \frac{1}{4}(3p-7)\tau^{-1}u_\beta n^{-\frac{1}{2}}$$

$$+ \frac{1}{12}\tau^{-2}(11u_\beta^2 - 5pu_\beta^2 + 6p^2 - 7p - 11)n^{-1} + \mathcal{O}(n^{-3/2}).$$

Considering the linear MP test, let c_n' be the critical value of the test based on $n^{\frac{1}{2}}\tau^{-1}\sum_{j=1}^{p}\theta_j\overline{Y}_j$ with power β at θ. Since $c_n' = n^{\frac{1}{2}}\tau + u_\beta$, it follows that under the null hypothesis

$$\text{(A2)}\qquad \log\alpha = \log P_0\{n^{\frac{1}{2}}\tau^{-1}\sum_{j=1}^{p}\theta_j\overline{Y}_j \geq c_n'\} = \log(1-\Phi(c_n'))$$

$$= -\tfrac{1}{2}(c_n')^2 - \tfrac{1}{2}\log(2\pi) - \log c_n' - (c_n')^2 + \mathcal{O}((c_n')^{-3})$$

$$= -\tfrac{1}{2}\tau^2 n - \tau u_\beta n^{\frac{1}{2}} - \tfrac{1}{2}\log n - \tfrac{1}{2}u_\beta^2 - \tfrac{1}{2}\log(2\pi\tau^2) - \tau^{-1}u_\beta n^{-\frac{1}{2}}$$

$$+ \tfrac{1}{2}\tau^{-2}u_\beta^2 n^{-1} - \tau^{-2}n^{-1} + \mathcal{O}(n^{-3/2}).$$

Equating (A1) and (A2) after substitution of $n = N_{\chi^2}$ in (A1) and $n = N_{MP}$ in (A2) yields the desired result. □

<u>REMARK 1</u>. The randomization procedure underlying the above proof is as follows. If for sample sizes n and n+1 a test with exact power β at θ has sizes α_n and α_{n+1} respectively ($\alpha_{n+1} < \alpha \leq \alpha_n$), $N = n+q_n$ where q_n satisfies $\log\alpha = q_n \log\alpha_{n+1} + (1-q_n)\log\alpha_n$. Note that this procedure differs from the randomization described in section 2.

<u>REMARK 2</u>. The expansions given in the proof are expansions in powers of $\|\theta\| n^{\frac{1}{2}}$. Hence the convergence of the Bahadur deficiency expansion is rather slow if $\|\theta\| N_{MP}^{\frac{1}{2}}$ is small. Since $\|\theta\| N_{MP}^{\frac{1}{2}} \sim -2\log\alpha$, unsatisfactory approximations to the actual deficiencies by the leading terms of the expansion are not surprising for the traditional values of α (.05 or .01, say).

<u>PROOF OF PROPOSITION 2.</u>

Let $M_N = X_{[\frac{1}{2}N]+1:N}$ be the sample median in a sample of size N from the double exponential distribution with cdf $F_\theta(x) = 1-\frac{1}{2}e^{-(x-\theta)}$ if $x \geq \theta$, $F_\theta(x) = \frac{1}{2}e^{x-\theta}$ if $x < \theta$. Let $\pi_M^N(\theta)$ denote the power at θ of the size-α test rejecting the hypothesis $H_0 : \theta = 0$ for large values of M_N (the M-test). The critical value of the test is denoted by c_N.

Since α is fixed $c_N > 0$ is of order $\mathcal{O}(N^{-\frac{1}{2}})$ as $N \to \infty$. Note that $P_\theta\{M_N > c_N\} = \Pr\{Y_N \leq [\frac{1}{2}N] \mid p_\theta\}$ where Y_N has a binomial distribution with parameters N and $p_\theta = F_\theta(c_N)$. We consider contiguous alternatives $\{\theta_N\}$ such that the power at θ_N is fixed at β $(0 < \beta < 1)$ for all N. It follows that $p_\theta = F_\theta(c_N) = \frac{1}{2} + \mathcal{O}(N^{-\frac{1}{2}})$ as $N \to \infty$, both for $\theta = \theta_N$ and $\theta = 0$. By normal approximation (see Theorem 3.1 in MOLENAAR (1970) for the remainder term)

$$\Pr\{Y_N \leq k_N \mid p\} = \Phi(2(k_N+\tfrac{1}{2}-Np)N^{-\frac{1}{2}})+\mathcal{O}(N^{-1})$$

as $N \to \infty$, if $p = \frac{1}{2} + \mathcal{O}(N^{-1})$ and $|k_N-Np| = \mathcal{O}(N^{\frac{1}{2}})$. Hence, putting $F_0(c_N) = \frac{1}{2}+\delta_N$,

$$\alpha = P_0\{M_N > c_N\} = \Phi(2([\tfrac{1}{2}N]+\tfrac{1}{2}-\tfrac{1}{2}N-N\delta_N)N^{-\frac{1}{2}})+\mathcal{O}(N^{-1})$$

implying

$$\delta_N = \tfrac{1}{2}u_\alpha N^{-\frac{1}{2}}+([\tfrac{1}{2}N]+\tfrac{1}{2}-\tfrac{1}{2}N)N^{-1}+\mathcal{O}(N^{-3/2}).$$

First assume $\beta > \frac{1}{2}$, i.e. $\theta_N > c_N$ for large N. In that case

$$\begin{aligned} p_{\theta_N} &= F_{\theta_N}(c_N) = \tfrac{1}{4}e^{-\theta_N}(1-F_0(c_N))^{-1} = \tfrac{1}{2}e^{-\theta_N}(1-2\delta_N)^{-1} \\ &= \tfrac{1}{2}-\tfrac{1}{2}\theta_N+\delta_N+\tfrac{1}{4}\theta_N^2-\delta_N\theta_N+2\delta_N^2+\mathcal{O}(N^{-3/2}) \\ &= \tfrac{1}{2}-\tfrac{1}{2}\theta_N+\tfrac{1}{2}u_\alpha N^{-\frac{1}{2}}+\tfrac{1}{4}\theta_N^2+([\tfrac{1}{2}N]+\tfrac{1}{2}-\tfrac{1}{2}N)N^{-1}-\tfrac{1}{2}u_\alpha\theta_N N^{-\frac{1}{2}}+\tfrac{1}{2}u_\alpha^2 N^{-1}+\mathcal{O}(N^{-3/2}). \end{aligned}$$

Hence, if $\beta > \frac{1}{2}$,

$$\text{(A3)} \qquad \pi_M^N(\theta_N) = P_{\theta_N}(M_N > c_N) = \Phi(2([\tfrac{1}{2}N]+\tfrac{1}{2}-Np_{\theta_N})N^{-\frac{1}{2}})+\mathcal{O}(N^{-1}) =$$

$$= \Phi(\theta N^{\frac{1}{2}}-u_\alpha+u_\alpha\theta_N-u_\alpha^2 N^{-\frac{1}{2}}-\tfrac{1}{2}\theta_N^2 N^{\frac{1}{2}})+\mathcal{O}(N^{-1})$$

as $N \to \infty$. Similarly, if $\beta < \frac{1}{2}$,

$$p_{\theta_N} = 1-e^{\theta_N}(1-F_0(c_N)) = 1-e^{\theta_N}(\tfrac{1}{2}-\delta_N)$$

$$= \tfrac{1}{2}-\tfrac{1}{2}\theta_N+\tfrac{1}{2}u_\alpha N^{-\frac{1}{2}}-\tfrac{1}{4}\theta_N^2+([\tfrac{1}{2}N]+\tfrac{1}{2}-\tfrac{1}{2}N)N^{-1}+\tfrac{1}{2}u_\alpha\theta_N N^{-\frac{1}{2}}+\mathcal{O}(N^{-3/2})$$

implying

$$\text{(A4)} \qquad \pi_M^N(\theta_N) = \Phi(\theta_N N^{\frac{1}{2}}-u_\alpha-u_\alpha\theta_N+\tfrac{1}{2}\theta_N^2 N^{\frac{1}{2}})+\mathcal{O}(N^{-1}).$$

Now let $\pi_S^{N_1}(\theta_N)$ denote the power at θ_N of the size-α sign test based on a sample of size N_1. By Lemma 3.5.1 in ALBERS (1974)

$$\text{(A5)} \qquad \pi_S^{N_1}(\theta_N) = \Phi(-u_\alpha+N_1^{\frac{1}{2}}\{2F_0(\theta_N)-1\})+\mathcal{O}(N_1^{-1})$$

$$= \Phi(-u_\alpha+\theta_N N_1^{\frac{1}{2}}-\tfrac{1}{2}\theta_N^2 N_1^{\frac{1}{2}})+\mathcal{O}(N_1^{-1})$$

as $N_1 \to \infty$ if N and N_1 are of the same order of magnitude.

Putting $\pi_M^N(\theta_N) = \pi_S^{N_1}(\theta_N)$ and using (A3), (A4) and (A5) yields $N_1 = N+\varepsilon_N N^{\frac{1}{2}}+\mathcal{O}(1)$ with $\varepsilon_N = 2u_\alpha(\theta_N N^{\frac{1}{2}})^{-1}(\theta_N N^{\frac{1}{2}}-u_\alpha)$ if $\beta > \frac{1}{2}$ and $\varepsilon_N = 2(\theta_N N^{\frac{1}{2}}-u_\alpha)$ if $\beta < \frac{1}{2}$. Substitution of $\theta_N N^{\frac{1}{2}}-u_\alpha = u_{1-\beta}+\mathcal{O}(N^{-\frac{1}{2}})$ (from (A3) and (A4)) proves the proposition for $\beta \neq \frac{1}{2}$. The case $\beta = \frac{1}{2}$ may be dealt with along the same lines using more detailed power expansions. We omit the details. □

Finally we briefly consider the power functions of the LR test, Pillai's test, Hotelling's test and Roy's test for the multivariate linear hypothesis. They can be obtained by integration of the joint density of the eigenvalues $\ell_1 \geq \ldots \geq \ell_p$ of $S_{h,m}(S_{h,m}+S_{e,m})^{-1}$, cf. section 2. For $p = 2$ this density has the form

$$\text{(A6)} \qquad f(\ell_1,\ell_2;n,n_1,\theta_1,\theta_2) = c\,\exp(\tfrac{1}{2}-(\theta_1+\theta_2)).$$

$$\cdot\ {}_1F_1(\tfrac{1}{2}(n+n_1);\tfrac{1}{2}n_1;\tfrac{1}{2}m\Theta,L)(\ell_1-\ell_2)\prod_{i=1}^{2}\ell_i^{\frac{1}{2}(n_1-3)}(1-\ell_i)^{\frac{1}{2}(n-3)},$$

where

$$c = \pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}(n+n_1))\Gamma(\tfrac{1}{2}(n+n_1)-\tfrac{1}{2})/\{\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}n-\tfrac{1}{2})\Gamma(\tfrac{1}{2}n_1)\Gamma(\tfrac{1}{2}n_1-\tfrac{1}{2})\},$$

$L = \mathrm{diag}(\ell_1,\ell_2)$ and ${}_1F_1$ is a hypergeometric function of two matrices (see CONSTANTINE (1963) p. 1279 or MUIRHEAD (1978) p. 25). If p = 2 the function ${}_1F_1$ can also be written as

$$\sum_{k=0}^{\infty} \frac{(\tfrac{1}{2}(n+n_1))_k(-\tfrac{1}{2}n)_k}{(\tfrac{1}{2}n_1-\tfrac{1}{2})_k(\tfrac{1}{2}n_1)_k} \frac{(-\tfrac{1}{4}m^2\theta_1\theta_2\ell_1\ell_2)^k}{k!} \cdot$$

$$\cdot \sum_{j=0}^{\infty} \frac{(\tfrac{1}{2}(n+n_1)+k)_j}{(\tfrac{1}{2}n_1+2k)_j} \frac{(\tfrac{1}{4}m(\theta_1+\theta_2)(\ell_1+\ell_2))^j}{j!} \; {}_2F_1(-\tfrac{1}{2}j;-\tfrac{1}{2}j+\tfrac{1}{2},1,x^2)$$

with $x = (\theta_1-\theta_2)(\ell_1-\ell_2)/\{(\theta_1+\theta_2)(\ell_1+\ell_2)\}$ and $(a)_0 = 1$, $(a)_k = a(a+1)\ldots(a+k-1)$, $k \geq 1$ (cf. MUIRHEAD (1975) expression (1.6), but note that a factor k! is missing in the denominator).

The cdf's of the four statistics T_m^{LR}, T_m^P, T_m^H and T_m^R, evaluated at t, can now be obtained by numerical integration of (A6) over the regions

$$R_{LR} = \{(\ell_1,\ell_2) : 0 \leq \ell_1 \leq \ell_2 \leq 1\} \setminus\{(\ell_1,\ell_2) : 0 \leq \ell_1 \leq 1-t,\ 0 \leq \ell_2 \leq \min(\ell_1,(1-t-\ell_1)/(1-\ell_1)\},$$

$$R_P = \{(\ell_1,\ell_2) : 0 \leq \ell_1 \leq \min(1,t),\ 0 \leq \ell_2 \leq \min(\ell_1,t-\ell_1)\},$$

$$R_H = \{(\ell_1,\ell_2) : 0 \leq \ell_1 \leq t/(1+t),\ 0 \leq \ell_2 \leq \min(\ell_1,t-\ell_1(1+t)/(1+t-\ell_1(2+t)))\},$$

$$R_R = \{(\ell_1,\ell_2) : 0 \leq \ell_1 \leq t,\ 0 \leq \ell_2 \leq \ell_1\},$$

respectively.

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