Centrum voor Wiskunde en Informatica

Centre for Mathematics and Computer Science

N.M. Temme

Laguerre polynomials: asymptotics for large degree

Department of Applied Mathematics Report AM-R8610 November

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

Laguerre Polynomials: Asymptotics for Large Degree

N.M. Temme
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are considered for large values of the degree n. The paper surveys results of ERDÉLYI (1960), that gives for fixed $\alpha \geqslant 0$ and for $x \in \mathbb{R}$ as uniformity parameter two asymptotic forms: the Bessel function case and the Airy function case. Next, more recent results of BAUMGARTNER (1980) and OLVER (1980) for Whittaker functions are interpreted for Laguerre polynomials; the parameter α can then be considered as a second uniformity parameter. A new method is given for obtaining similar asymptotic forms by using integral representations of $L_n^{(\alpha)}(x)$.

1980 Mathematics Subject Classifications: 41A60, 33A65, 34E05, 33A30
Keywords & Phrases: Laguerre polynomials, uniform asymptotic expansion, Whittaker functions, Bessel functions, saddle point method
Note: This paper is presented at the 2nd International Symposium on Orthogonal Polynomials and their Applications, Segovia (Spain), 22-27 September, 1986.

1. INTRODUCTION

We consider the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!} \tag{1.1}$$

for large values of the degree n. A number of writers have dealt with this problem. The earlier investigations are summarized in SZEGÖ (1958). Depending on the value of x, several asymptotic forms are given. For $x = 4n + \mathcal{O}(\sqrt{n})$ an Airy function is used, and it describes the transition of the oscillatory to the monotonic region in the x-interval. A more complete description is given by TRICOMI (summarized in [2]). Let

$$\kappa = n + (\alpha+1)/2. \tag{1.2}$$

Tricomi distinguished four cases:

(i)	x near 0,	Hilb's type formula
(ii)	$0 < x < 4\kappa$,	osillatory region,
(iii)	x near 4κ,	turning point region
(iv)	$x > 4\kappa$,	monotonic region.

In (i) a Bessel form is used and it describes the transition of $x<0$ (a monotonic region) to the oscillatory region. The early zeros of $L_n^{(\alpha)}(x)$ can be approximated in terms of zeros of the Bessel function $J_\alpha(z)$. The transition of (ii) to (iv) is described by an Airy asymptotic form. The regions of validity in Tricomi's results do not overlap. ERDÉLYI (1960) has given two essentially new asymptotic forms that cover the whole real x-axis. This seems to be the state of the art, in September 1986.

In this paper we summarize Erdélyi's results, and we interprete more recent results of BAUMGARTNER (1980) and OLVER (1980) for Whittaker functions to Laguerre polynomials. These results allow α to be a second uniformity parameter. Olver's asymptotic form is written in terms of an Hermite polynomial. We modify his result in order to obtain a larger uniformity domain for α.

The asymptotic forms of Erdélyi, Baumgartner and Olver are obtained from differential equations (Liouville-Green, WKB-method). We describe a method for obtaining similar expansions from

Report AM-R8610
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

integral representations of $L_n^{(\alpha)}(x)$. The same method can be used for the more general Laguerre functions, or Whittaker functions. OLVER (1975) mentioned this type of problem as an unsolved problem in the asymptotic estimation of special functions.

2. ERDÉLYI'S ASYMPTOTIC FORMS

Let α,a,b be fixed numbers, $\alpha\geqslant 0$, $0<a<b<1$, and let $\nu=4\kappa$, where κ is given in (1.2).

2.1. Bessel function as approximant

Let $x\leqslant b$. Then

$$L_n^{(\alpha)}(4\kappa x) = \frac{\Gamma(n+\alpha+1)}{n!}\, 2^{-\alpha-\frac{1}{2}}\kappa^{-\alpha}x^{-(\alpha+1)/2}e^{2\kappa x}(\psi/\psi')^{\frac{1}{2}}\{J_\alpha(4\kappa\psi)+\epsilon/\kappa\}, \tag{2.1}$$

as $n\to\infty$, where

$$\begin{aligned}\psi(x) &= \tfrac{1}{2}i[\sqrt{x^2-x}-\operatorname{arcsinh}\sqrt{-x}], \quad x\leqslant 0,\\ &= \tfrac{1}{2}[\sqrt{x-x^2}+\arcsin\sqrt{x}], \quad 0\leqslant x<1.\end{aligned}$$

The prime in ψ' denotes differentation with respect to x; ϵ in (2.1) is estimated by means of

$$\epsilon = \mathcal{O}\{\sqrt{|x|/(1-x)}\tilde{J}_\alpha(4\kappa\psi)\}, \quad \text{as} \quad \frac{1}{n}\sqrt{|x|/(1-x)}\to 0,$$

uniformly with respect to $x\in(-\infty,b]$. The function $\tilde{J}_\alpha(z)$ is defined by: let δ be a fixed positive number so that $J_\alpha(z)\neq 0$ when $0<|z|\leqslant\delta$ and $\alpha\geqslant 0$; then

$$\begin{aligned}\tilde{J}_\alpha(z) &= J_\alpha(z) \quad \text{if } z \text{ is imaginary or } 0\leqslant z\leqslant\delta,\\ &= \{|J_\alpha(z)|^2+|Y_\alpha(z)|^2\}^{\frac{1}{2}}, \quad \text{if} \quad z>\delta.\end{aligned}$$

2.2. Airy function as approximant

Let $x\geqslant a$, then

$$L_n^{(\alpha)}(4\kappa x) = \frac{(-1)^n}{n!}\sqrt{-\pi/\phi'}\,2^{5/6}\kappa^{k+1/6}(x\nu)^{-(\alpha+1)/2}e^{2\kappa x-\kappa}\{Ai(-\nu^{2/3}\phi)+\epsilon\}, \tag{2.2}$$

as $n\to\infty$, where

$$\begin{aligned}\phi(x) &= [3\beta(x)/2]^{2/3}, \quad 0<x\leqslant 1,\\ &= -[3\gamma(x)/2]^{2/3}, \quad x\geqslant 1,\\ \beta(x) &= \tfrac{1}{4}\pi-\psi(x)=\tfrac{1}{2}[\arccos\sqrt{x}-\sqrt{x-x^2}],\\ \gamma(x) &= \tfrac{1}{2}[\sqrt{x^2-x}-\operatorname{arccosh}\sqrt{x}].\end{aligned}$$

The error ϵ in (2.2) is estimated by means of

$$\epsilon = \mathcal{O}[\frac{1}{\nu x}\tilde{A}i(-\nu^{2/3}\phi)] \quad \text{as} \quad n\to\infty,$$

uniformly with respect to $x\in[a,\infty)$. The function $\tilde{A}i(z)$ is defined by

$$\begin{aligned}\tilde{A}i(z) &= Ai(z) \quad \text{if} \quad z\geqslant 0,\\ &= \{|Ai(z)|^2+|Bi(z)|^2\}^{\frac{1}{2}}, \quad \text{if} \quad z<0.\end{aligned}$$

3. Baumgartner's asymptotic form

Let

$$\kappa = n+(\alpha+1)/2, \quad \tau = \alpha/(2\kappa), \quad \nu = 4\kappa, \quad b = 2e^{-1/2}(1+\tau)^{(1+\tau)/4\tau}(1-\tau)^{(\tau-1)/4\tau}, \tag{3.1}$$

and let x_1, x_2 be the zeros of

$$R = |x^2-x+\tfrac{1}{4}\tau^2|^{\frac{1}{2}}, \tag{3.2}$$

that is

$$x_1 = \tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-\tau^2}, \quad x_2 = \tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-\tau^2}. \tag{3.3}$$

Then

$$L_n^{(\alpha)}(4\kappa x) = \frac{\Gamma(n+\alpha+1)}{n!}(b\kappa)^{-\alpha}x^{-(\alpha+1)/2}e^{2\kappa x}(\xi/\xi')^{\frac{1}{2}}\{J_\alpha(2\kappa\sqrt{\xi})+\epsilon\}, \tag{3.4}$$

as $n \to \infty$, uniformly with respect to $\tau \in [0,\tau_0]$ and $x \in [0, x_0 x_2]$, where τ_0, x_0 are fixed numbers in $(0,1)$. A bound for $|\epsilon|$ in (3.4) is available, and also error estimates in terms of $\mathcal{O}$-symbols. Our notation is slightly different from Baumgartner's notation. The relation between x and ξ is as follows.

If $0 \leq x \leq x_1$, $0<\tau<1$, we have $0 \leq \xi \leq \tau^2$ and

$$-2R+\ln\left\{\frac{1-2x-2R}{\sqrt{1-\tau^2}}\right\}+\tau\ln\left\{\frac{\tau R - x+\frac{1}{2}\tau^2}{\sqrt{1-\tau^2}}\right\}-\tau\ln x$$

$$=2\tau\ln\{\tau+\sqrt{\tau^2-\xi}\}-\tau\ln\xi-2\sqrt{\tau^2-\xi}. \tag{3.5}$$

If $x_1 \leq x < x_2$, $0<\tau<1$, we have $\xi \geq \tau^2$ and

$$2R-\arcsin\frac{1-2x}{\sqrt{1-\tau^2}}-\tau\arcsin\frac{x-\frac{1}{2}\tau^2}{x\sqrt{1-\tau^2}}+\frac{\pi}{2}(1-\tau)=$$

$$=2\sqrt{\xi-\tau^2}-2\tau\arctan\sqrt{\xi/\tau^2-1}. \tag{3.6}$$

At the turning point $x=x_1$ ($\xi=\tau^2$), the argument of the Bessel function in (3.4) equals $2\kappa\tau=\alpha$. When α is large the Bessel function then describes the transition of the monotonic region ($x<x_1$) to the oscillatory region ($x_1<x<x_2$). When in $J_\alpha(\alpha z)$ order and argument are nearly equal ($z \sim 1$), Airy functions can be used to describe the asymptotic behaviour for $\alpha \to \infty$.

Baumgartner has not interpreted his result for negative x-values. When we replace in (3.5) $\tau\ln x$, $\tau\ln\xi$ with $\tau\ln(-x)$, $\tau\ln(-\xi)$, respectively, we can use (3.5) also for $x<0$, where $\xi<0$. The Bessel function in (3.4) becomes a modified Bessel function if $\xi<0$.

The uniformity domain for α follows from $0 \leq \tau \leq \tau_0$, i.e.,

$$0 \leq \frac{\alpha}{2n+\alpha+1} \leq \tau_0, \quad \text{or} \quad 0 \leq \alpha \leq \frac{\tau_0}{1-\tau_0}(2n+1). \tag{3.7}$$

In other words: $0 \leq \alpha \leq \alpha_0 n$, where α_0 is any fixed positive number.

4. OLVER'S (MODIFIED) ASYMPTOTIC FORM

Since in OLVER (1980) the analogue of α plays the part of the large parameter, we modify Olver's result to obtain a more appropriate form for Laguerre polynomials. The relation between Laguerre polynomials and Whittaker functions is

$$L_n^{(\alpha)}(z) = \frac{(-1)^n}{n!} z^{-\frac{1}{2}(\alpha+1)} e^{\frac{1}{2}z} W_{\kappa,\mu}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} z^{-\frac{1}{2}(\alpha+1)} e^{\frac{1}{2}z} M_{\kappa,\mu}(z), \tag{4.1}$$

with κ as in (3.1) and $\mu=\alpha/2$.

The change of parameters in Olver's paper (§2) is $\zeta \to \zeta\sqrt{\tau}$, $x \to \tau x/4$. We repeat the basic steps in Olver's method in terms of our choice of parameters.

The functions $M_{\kappa,\mu}(z)$, $W_{\kappa,\mu}(z)$ are solutions of Whittaker's equation

$$\frac{d^2y}{dz^2} = \left\{ \frac{1}{4} - \frac{\kappa}{z} + \frac{\mu^2 - \frac{1}{4}}{z^2} \right\} y(z).$$

A first transformation $z=4\kappa x$ enables us to write

$$\frac{d^2w}{dx^2} = \left\{ \kappa^2 \frac{4(x-x_1)(x-x_2)}{x^2} - \frac{1}{4x^2} \right\} w(x), \tag{4.2}$$

where x_1, x_2 are given in (3.3), with solutions $M_{\kappa,\mu}(4\kappa x)$, $W_{\kappa,\mu}(4\kappa x)$.

We consider κ as the large parameter, x and $\tau=\mu/\kappa=\alpha/(2\kappa)$ as uniformity parameters, $x\in(0,\infty)$, $\tau\in(0,1]$. Baumgartner has used (4.2) to obtain the Bessel asymptotic form. We apply the Liouville-Green transformation to (4.2) by introducing $\eta=\eta(x)$ and $W(\eta)$ by writing

$$w(x) = \sqrt{\frac{dx}{d\eta}}\, W(\eta), \quad (\eta^2-\rho^2)\left[\frac{d\eta}{dx}\right]^2 = \frac{4(x-x_1)(x-x_2)}{x^2}, \tag{4.3}$$

where ρ is the non-negative number defined by

$$\int_{-\rho}^{\rho} \sqrt{\rho^2-\eta^2}\, d\eta = 2\int_{x_1}^{x_2} \sqrt{-x^2+x-\tfrac{1}{4}\tau^2}\, \frac{dx}{x}.$$

Evaluation of the integrals yields

$$\rho = \sqrt{2(1-\tau)}. \tag{4.4}$$

The relation between η and x is one-to-one, with

$$\eta(0)=-\infty, \quad \eta(x_1)=-\rho, \quad \eta(x_2)=\rho, \quad \eta(+\infty)=+\infty. \tag{4.5}$$

Solving the differential equation for η in (4.3) with the above boundary conditions, we obtain the following relations. Let $0<\tau<1$ and let R be given by (3.2).

(i) $\rho\leqslant\eta<\infty$, $x_2\leqslant x<\infty$:

$$\eta\sqrt{\eta^2-\rho^2} - \rho^2 \operatorname{arccosh}\frac{\eta}{\rho} = 4R - 2\tau\ln\left\{\frac{2x-\tau^2-2\tau R}{2x\sqrt{1-\tau^2}}\right\} - 2\ln\left\{\frac{2R+2x-1}{\sqrt{1-\tau^2}}\right\}; \tag{4.6}$$

(ii) $-\rho\leqslant\eta\leqslant\rho$, $x_1\leqslant x\leqslant x_2$:

$$\eta\sqrt{\rho^2-\eta^2} + \rho^2\arcsin\frac{\eta}{\rho} = 4R - 2\tau\arctan\frac{x-\frac{1}{2}\tau^2}{\tau R} - 2\arctan\frac{1-2x}{2R}; \tag{4.7}$$

(iii) $-\infty<\eta\leqslant-\rho$, $0<x\leqslant x_1$:

$$-\eta\sqrt{\rho^2-\eta^2}-\rho^2\operatorname{arccosh}\frac{-\eta}{\rho} = -4R+2\tau\ln\left\{\frac{\tau^2-2x+2\tau R}{2x\sqrt{1-\tau^2}}\right\}+2\ln\left\{\frac{1-2x-2R}{\sqrt{1-\tau^2}}\right\}. \quad (4.8)$$

If $\tau=1$ we have

$$\frac{1}{2}\eta^2 = 2x-\ln(2x)-1, \quad \operatorname{sign}(\eta) = \operatorname{sign}(x-\frac{1}{2}). \quad (4.9)$$

The differential equation (4.2) transforms into

$$\frac{d^2W}{d\eta^2} = \{\kappa^2(\eta^2-\rho^2)+\psi(\rho^2,\eta)\}W(\eta) \quad (4.10)$$

in which

$$\psi(\rho^2,\eta) = -\frac{\dot{x}^2}{4x^2}+\dot{x}^{\frac{1}{2}}\frac{d^2}{d\eta^2}(\dot{x}^{-\frac{1}{2}}), \quad \dot{x} = \frac{dx}{d\eta},$$

$$= \frac{3\eta^2+2\rho^2}{4(\eta^2-\rho^2)^2}-\{4x^3+(1-4\tau^2)x+\tau^2\}\frac{(\eta^2-\rho^2)x}{64(x^2-x+\frac{1}{4}\tau^2)^3}.$$

As in Olver's paper, it can be shown that

$$\psi(\rho^2,\eta) = \mathcal{O}\{1/(\eta^2+1)\}, \quad \eta\in\mathbb{R},$$

with $\tau\in(0,1]$.

The above results can be interpreted for $\tau>1$, but for the Laguerre polynomial these values do not make sense.

From Olver's theory it follows that the Whittaker function $W_{\kappa,\mu}(z)$ can be written as

$$W_{\kappa,\mu}(4\kappa x) = (2\kappa)^{\frac{1}{4}}x^{\frac{1}{2}}[2e^{-1}\kappa(1-\frac{1}{4}\rho^2)]^{\kappa(1-\frac{1}{4}\rho^2)}\cdot$$

$$\cdot\{\frac{\eta^2-\rho^2}{x^2-x+\frac{1}{4}\tau^2}\}^{1/4}\{U(-\frac{1}{2}\kappa\rho^2, \eta\sqrt{2\kappa})+\epsilon\}, \quad (4.11)$$

in which $U(a,z)$ is a parabolic cylinder function. The remainder ϵ is, in some sense, small when $\kappa\to\infty$, $x\in(0,\infty)$, $\tau\in(0,1]$. From Olver's theory an upper bound for $|\epsilon|$ can be constructed.

By using (4.1) it follows that for the Laguerre polynomials the quantity $-1/2\kappa\rho^2$ in the U-function in (4.11) can be written as (see (3.1), (4.4)) $-1/2\kappa\rho^2=-(\kappa-\mu)=-n-1/2$. Hence, the parabolic cylinder function reduces to a Hermite polynomial:

$$U(-n-\frac{1}{2},z) = D_n(z)=e^{-\frac{1}{2}n}e^{-\frac{1}{4}z^2}H_n(z/\sqrt{2}), \quad (4.12)$$

where $D_\nu(z)$ is another notation for parabolic cylinder functions.

By combining (4.1), (4.11), it follows that

$$L_n^{(\alpha)}(4\kappa x) = \frac{(-1)^n}{n!}2^{-\alpha-\frac{1}{2}n-\frac{3}{4}}\kappa^{-\frac{1}{2}\alpha-\frac{1}{4}}\left[\frac{n+\alpha+\frac{1}{2}}{e}\right]^{n+\alpha+\frac{1}{2}}x^{-\frac{1}{2}\alpha}\cdot$$

$$\cdot\{\frac{\eta^2-\rho^2}{x^2-x+\frac{1}{4}\tau^2}\}^{\frac{1}{4}}e^{2\kappa x-\frac{1}{2}\kappa\eta^2}\{H_n(\eta\sqrt{\kappa})+\tilde{\epsilon}\}, \quad (4.13)$$

where τ and κ are given in (3.1), ρ in (4.4) and $\mu=\alpha/2$. This form can be viewed as an 4asymptotic estimate with κ as the large parameter. It holds for $x\in(0,\infty)$, $\tau\in(0,1)$.

The latter gives for α the condition

$$0 < \frac{\alpha}{2n+1+\alpha} < 1,$$

which is indeed satisfied for all $\alpha \in (0, \infty)$.

We cannot claim that (4.13) holds uniformly in the (x, τ)-domain $[0, \infty) \times [0, 1]$, that is, inclusive the origin in both intervals. The reason is that for $\tau \to 0$ the mapping $x \to \eta(x)$ tends to a limit mapping in a non-uniform way. For instance, $\tau = 0$ gives in (3.3) $x_1 = 0$, and in (4.5) $\eta(0) = -\infty$, as well as $\eta(0) = -\sqrt{2}$. However, $\tau \to 0$ is allowed as long as x is bounded away from x_1, $x > x_1$.

A safe condition is to exclude in the (x, τ) domain a small rectangle $[0, x_0] \times [0, \tau_0]$, where x_0, τ_0 are arbitrarily small fixed positive numbers.

Recall that Baumgartner's result (3.4) is valid for x bounded away from x_2 $(x < x_2)$ and for α satisfying (3.7). It follows that (3.4) and (4.13) describe the asymptotic behaviour of $L_n^{(\alpha)}(4\kappa x)$ in overlapping domains of the (α, x) quarter plane $[0, \infty) \times [0, \infty)$.

Olver's asymptotic estimate of the Whittaker function $W_{\kappa,\mu}(z)$ also yields (4.13) after reparametrization. He shows that his result is valid for $\alpha \to \infty$, uniformly with respect to $x \in [0, \infty)$ and $n \in [0, n_0 \alpha]$, where n_0 is positive and fixed. It follows that in (4.13) these conditions can be used also.

The asymptotic estimate (4.13) has the Hermite polynomial as approximant. This polynomial has the same number of zeros as $L_n^{(\alpha)}(4\kappa x)$ itself. The zeros of $L_n^{(\alpha)}(4\kappa x)$ occur in the region $x_1 < x < x_2$. Let $l_{n,m}^{(\alpha)}, h_{n,m}$ be the m-th zeros of $L_n^{(\alpha)}(z), H_n(z)$, $m = 1, 2, \cdots, n$. For given α and n, we can compute

$$\eta_{n,m} = \frac{h_{n,m}}{\sqrt{\kappa}}, \quad m = 1, 2, \cdots, n \tag{4.14}$$

with κ defined in (3.1). Upon inverting (4.7) we can obtain $x_{n,m}$, giving the estimate

$$l_{n,m}^{(\alpha)} \sim 4\kappa x_{n,m}, \quad m = 1, 2, \cdots, n. \tag{4.15}$$

From asymptotic expansions of Hermite polynomials (see, for instance, SKOVGAARD (1959)) it follows that $h_{n,0}$, $h_{n,n}$ have the asymptotic estimate

$$-h_{n,0} = h_{n,n} = \sqrt{2n+1} - \epsilon(n)$$

where ϵ is a positive function of n, $\epsilon(n) = \mathcal{O}(n^{-1/6})$, as $n \to \infty$. It follows that the numbers $\eta_{n,m}$ of (4.14) belong to the interval $[-\rho, \rho]$, when n is large, $\alpha \geq 0$.

The estimate (4.15) is valid for $n \to \infty$, uniformly with respect to $m \in \{1, 2, \cdots, n\}$ and $\alpha \geq 0$.

m \ α	0	1	5	10	25	50	75	100
1	1.7	2.3	3.2	3.7	4.4	5.0	5.3	5.6
2	2.4	2.7	3.4	3.8	4.5	5.0	5.4	5.6
3	2.8	3.0	3.5	3.9	4.5	5.1	5.4	5.6
4	3.0	3.2	3.6	4.0	4.6	5.1	5.4	5.7
5	3.2	3.4	3.8	4.1	4.6	5.1	5.5	5.7
6	3.4	3.5	3.9	4.2	4.7	5.2	5.5	5.7
7	3.5	3.6	4.0	4.2	4.7	5.2	5.5	5.8
8	3.7	3.8	4.1	4.3	4.8	5.3	5.6	5.8
9	3.8	3.9	4.1	4.4	4.9	5.3	5.6	5.8
10	3.9	4.0	4.2	4.5	4.9	5.4	5.6	5.8

TABLE 4.1 Correct decimal digits in the approximations of zeros of $L_{10}^{(\alpha)}(x)$.

In Table 4.1 we show for $n = 10$ the "correct number of decimal digits" in the approximation (4.15). That is, we show

$$^{10}\log \left| \frac{l_{10,m}^{(\alpha)} - \tilde{l}_{10,m}^{(\alpha)}}{l_{10,m}^{(\alpha)}} \right|, \quad m = 1, \cdots, 10$$

where $\tilde{l}_{n,m}^{(\alpha)}$ is the approximation obtained by the procedure described in (4.14), (4.15). It follows that the large zeros are better approximated than the small zeros. Furthermore, large values of α give better approximations, and the approximations are quite uniform with respect to m.

The asymptotic representation (4.13) of $L_n^{(\alpha)}(z)$ in terms of the Hermite polynomial seems to be new. In [5, p. 251] the limit

$$\lim_{\alpha\to\infty}\{\alpha^{-\frac{1}{2}n}L_n^{(\alpha)}(\alpha+t\sqrt{\alpha})\} = \frac{(-1)^n}{n!}2^{-\frac{1}{2}n}H_n(t/\sqrt{2}) \tag{4.16}$$

is given, without reference to a source. We verify this relation by using special values of the parameters in (4.13). When α is large with respect to n, we have $\tau\to 1$. In the limit $\tau=1$, the relation between η and x is given by (4.9), and $\rho=0$. So we have if $\alpha>>n$

$$L_n^{(\alpha)}(4\kappa x) \sim \frac{(-1)^n}{n!}2^{-\alpha-\frac{1}{2}n-\frac{3}{4}}\kappa^{-\frac{1}{2}\alpha-\frac{1}{4}}\left[\frac{n+\alpha+\frac{1}{2}}{e}\right]^{(n+\alpha+\frac{1}{2})/2}$$

$$x^{-\frac{1}{2}}\sqrt{\frac{\eta}{x-\frac{1}{2}}}(2ex)^{\kappa}H_n(\eta\sqrt{\kappa}).$$

Writing $4\kappa x=\alpha+t\sqrt{\alpha}$, we observe that $x\to 1/2$. In this limit, η can be replaced with $x-1/2$. A few further calculations give indeed (4.16). It is valid for fixed values of t and n, although (4.16) can be replaced with an asymptotic relation in which $t=o(\sqrt{\alpha})$, $n=o(\alpha)$, as $\alpha\to\infty$.

5. Asymptotic methods based on integral representations

In this section we show how asymptotic forms like (3.4) and (4.13) can be obtained by using integral representations of $L_n^{(\alpha)}(x)$. This approach is not discussed earlier in the literature, and it deserves a more detailed analysis than we can give in the present paper. A completely different method now produces the relations between ξ and x (in § 3) or between η and x (in § 4).

The generating series

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(z)t^n = (1-t)^{-\alpha-1}e^{zt/(t-1)}, \quad |t|<1$$

yields the Cauchy type integral

$$L_n^{(\alpha)}(z) = \frac{1}{2\pi i}\int e^{zt/(t-1)}(1-t)^{-\alpha-1}t^{-n-1}dt, \tag{5.1}$$

where the contour is a small circle around $t=0$.

5.1. *Bessel asymptotic form*

The transformation $t=1-s$ gives for (5.1)

$$L_n^{(\alpha)}(4\kappa x) = \frac{e^{4\kappa x}}{2\pi i}\int_{-\infty}^{(0^+)}e^{\kappa\phi(s)}\frac{ds}{s\sqrt{1-s}}, \tag{5.2}$$

where

$$\phi(s) = \frac{-4x}{s}+\tau\ln\frac{1-s}{s^2}-\ln(1-s);$$

κ and τ are given in (3.1). Initially, the contour in (5.2) is a small circle around $s=1$. In (5.2) this circle is deformed and the new contour runs from $s=-\infty$ (phase $s=-\pi$) to $s=-\infty$ (phase $s=\pi$) and it encircles the origin counter-clockwise, cutting the real axis between 0 and 1. We introduce branch cuts from 1 to $+\infty$, and from 0 to $-\infty$, although the integrand in (5.2) has only a pole at

$s=0$.

The saddle points of $\phi(s)$ are defined by $\phi'(s)=0$, giving

$$s_1 = \frac{\tau+2x-2\sqrt{x^2-x+\frac{1}{4}\tau^2}}{\tau+1}, \quad s_2 = \frac{\tau+2x+2\sqrt{x^2-x+\frac{1}{4}\tau^2}}{\tau+1}. \tag{5.3}$$

The saddle points coincide when $x=x_1$ or $x=x_2$, where x_i are given in (3.3). As in § 3 we assume that $0\leqslant\tau<1$, $x<x_2$. When $x=x_1$ the saddle points are

$$s_1 = s_2 = 1-\sqrt{(1-\tau)/(1+\tau)}\,. \tag{5.4}$$

When τ is small this value is small. At $s=0$ the integrand of (5.2) has an essential singularity, combined with an algebraic singularity. Also, when $x\to 0$ and $\tau\to 0$ both s_1,s_2 approach $s=0$. In the same sense these phenomena occur in the integral representation of the Bessel function

$$J_\beta(2\kappa\sqrt{\xi}) = \frac{\xi^{\beta/2}}{2\pi i}\int_{-\infty}^{(0^+)} e^{\kappa\psi(t)}\frac{dt}{t}, \tag{5.5}$$

with

$$\psi(t) = t-\xi/t-2\sigma\ln t, \quad 2\sigma=\beta/\kappa. \tag{5.6}$$

Saddle points are

$$t_1 = \sigma-\sqrt{\sigma^2-\xi}, \quad t_2 = \sigma+\sqrt{\sigma^2-\xi}\,. \tag{5.7}$$

For $0<x<x_1$, the graphs of $\phi(s)$, $\psi(t)$ on $(0,1)$, $(0,\infty)$, respectively are shown in Fig. 5.1. If $x<x_1$ the saddle points s_2,t_2 are used for a saddle point contour; when $x_1\leqslant x<x_2$ both saddle points s_1,s_2 and t_1,t_2 are used.

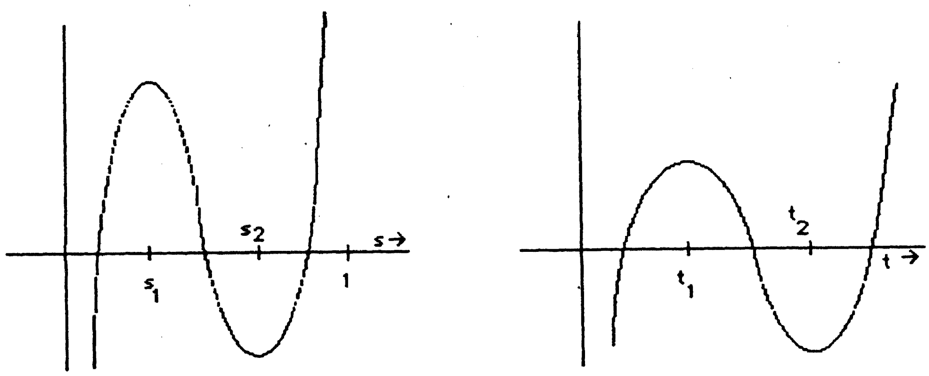

FIGURE 5.1. Graphs of $\phi(s)$, $\psi(t)$; $0<s<1$, $0<t<\infty$.

For $s\in(0,1)$, $t\in(0,\infty)$ we define the mapping $s\to t(s)$, by writing

$$\phi(s) = \psi(t)+A \tag{5.8}$$

where A and ξ, σ of (5.6) are to be determined. The mapping is one-to-one when we prescribe the corresponding points

$$s=0\leftrightarrow t=0, \quad s=s_i\leftrightarrow t=t_i \quad (i=1,2), \quad s=1\leftrightarrow t=+\infty.$$

In order to use this mapping for transforming (5.2) to (5.5) we need the analytic continuation of the mapping (5.8) to complex s and t-values. Also, to use values of x near x_1 and $x \to 0$, $\tau \to 0$, a more detailed analysis is needed, which will not be given here.

For a uniform representation of $L_n^{(\alpha)}(z)$ in terms of the Bessel function, we require that the mapping $s \to t(s)$ is analytic at $s=0$, $t=0$. The cancellation of singularities $-4x/s$ in $\phi(s)$ and $-\xi/t$ in $\psi(t)$ is possible if an only if $s \sim (4x/\xi)t$ as $t \to 0$. To make the mapping single valued at $t=0$, the log-terms $\ln s$ and $\ln t$ should have the same multiplying constants. This determines the value σ in (5.6): $\sigma = \tau$.

The value ξ in (5.6) follows from

$$\phi(s_1) - \phi(s_2) = \psi(t_1) - \psi(t_2),$$

giving (3,5), (3,6). Finally, the value A in (5.8) is given by $\psi(s_i) - \psi(t_i)$ ($i = 1$ or 2), giving

$$A = -\tau - 2x + \tfrac{1}{2}(1+\tau)\ln(1+\tau) - \tfrac{1}{2}(1-\tau)\ln(1-\tau) + \tau\ln(\xi/4x).$$

We proceed without a detailed discussion of the mapping (5.8). We obtain for (5.2)

$$L_n^{(\alpha)}(4\kappa x) = \frac{e^{4\kappa x + \kappa A}}{2\pi i} \int_{-\infty}^{(0^+)} e^{\kappa\psi(t)} f(t) \frac{dt}{t}, \tag{5.9}$$

where

$$f(t) = \frac{t}{s\sqrt{1-s}} \frac{ds}{dt} = \frac{s\sqrt{1-s}}{(\tau+1)t} \frac{(t-t_1)(t-t_2)}{(s-s_1)(s-s_2)}. \tag{5.10}$$

In (5.6), (5.7) the value σ equals $\tau = \alpha/(2\kappa)$. When $0 \leqslant x < x_1$, t_2 is the dominant saddle point in (5.9). A first approximation is obtained by replacing $f(t)$ with

$$f(t_2) = \tfrac{1}{2}\sqrt{2}\left\{\frac{\tau^2 - \xi}{x^2 - x + \frac{1}{4}\tau^2}\right\}^{\frac{1}{4}},$$

which value follows from (5.10) by using l'Hôpital's rule.

We obtain

$$\begin{aligned} L_n^{(\alpha)}(4\kappa x) &\sim e^{4\kappa x + \kappa A} \xi^{-\alpha/2} f(t_2) J_\alpha(2\kappa\sqrt{\xi}) \\ &= C \frac{\Gamma(n+\alpha+1)}{n!} (b\kappa)^{-\alpha} x^{-(\alpha+1)/2} e^{2\kappa x} (\xi/\xi')^{\frac{1}{2}} J_\alpha(2\kappa\sqrt{\xi}) \end{aligned}$$

(confer (3.4)), with

$$C = (\tfrac{1}{2}\kappa b^2)^\alpha \frac{n!}{\Gamma(n+\alpha+1)} = 1 + \mathcal{O}(\kappa^{-1}), \tag{5.11}$$

as $\kappa \to \infty$, uniformly with respect to $\tau \in [0, \tau_0]$, τ_0 a fixed number in (0,1).

More terms in the asymptotic estimate of $L_n^{(\alpha)}(z)$ can be obtained by using an integration by parts procedure. For a similar approach, see [10, § 6]. Write

$$f(t) = c_0 + \tfrac{1}{2}d_0(t - \xi/t) + \frac{(t-t_1)(t-t_2)}{t} g(t)$$

with

$$c_0 = \tfrac{1}{2}\{f(t_1) + f(t_2)\}, \quad d_0 = \tfrac{1}{2}\{f(t_1) - f(t_2)\}/(t_1 - t_2). \tag{5.12}$$

Then (5.9) can be written as

$$L_n^{(\alpha)}(4\kappa x) = e^{4\kappa x + \kappa A} \xi^{-\alpha/2} \{c_0 J_\alpha(2\kappa\sqrt{\xi}) + \sqrt{\xi} d_0 J'_\alpha(2\kappa\sqrt{\xi}) + \delta\} \tag{5.13}$$

where

$$\delta = \frac{\xi^{\alpha/2}}{2\pi i}\int e^{\kappa\psi(t)}\frac{(t-t_1)(t-t_2)}{t^2}g(t)dt = \frac{\xi^{\alpha/2}}{2\pi i\kappa}\int g(t)de^{\kappa\psi(t)} = \frac{\xi^{\alpha/2}}{2\pi i\kappa}\int f_1(t)e^{\kappa\psi(t)}\frac{dt}{t}$$

with $f_1(t) = -tg'(t)$. Repeating this procedure we obtain the formal expansion

$$L_n^{(\alpha)}(4\kappa x) \sim e^{4\kappa x+\kappa A}\xi^{-\alpha/2}\{J_\alpha(2\kappa\sqrt{\xi})\sum_{s=0}^{\infty}c_s\kappa^{-s} + \sqrt{\xi}J_\alpha{}'(2\kappa\sqrt{\xi})\sum_{s=0}^{\infty}d_s\kappa^{-s}\}. \tag{5.14}$$

The coefficients c_s, d_s follow from (5.12) and

$$f_s(t) = -tg'_{s-1}(t) = c_s + \tfrac{1}{2}d_s(t-\xi/t) + \frac{(t-t_1)(t-t_2)}{t}g_s(t), \quad s \geqslant 1,$$

where $f_0 = f$, $g_0 = g$. Observe that in (5.13) δ is a remainder. Each step in the integration by parts procedure gives a new remainder δ_s, say. Since $f(t_1) = f(t_2)$, $d_0 = 0$ and δ in (5.13) satisfies $Cc_0\delta = \epsilon$, where ϵ is the remainder in (3.4); C, c_0 are given in (5.11), (5.12).

5.2. Hermite polynomial as approximant

The transformation $t = (s-1)/(s+1)$ gives for (5.1)

$$L_n^{(\alpha)}(4\kappa x) = \frac{(-1)^{n+1}2^{-\alpha}e^{2\kappa x}}{2\pi i}\int_{+\infty}^{(1^+)} e^{\kappa\Phi(s)}\frac{ds}{\sqrt{1-s^2}}, \tag{5.15}$$

where

$$\Phi(s) = -2xs + \tau\ln(1-s^2) + \ln\frac{1+s}{1-s}; \tag{5.16}$$

κ and τ are given in (3.1). We define a branch cut from 1 to $+\infty$ for the multi-valued functions in $\Phi(s)$, although the point $s=1$ is only a pole of the integrand in (5.15). The saddle points of $\Phi(s)$ are

$$s_1 = \frac{\frac{1}{2}\tau - \sqrt{x^2 - x + \frac{1}{4}\tau^2}}{x}, \quad s_2 = \frac{\frac{1}{2}\tau + \sqrt{x^2 - x + \frac{1}{4}\tau^2}}{x}. \tag{5.17}$$

When $\tau \in (0,1)$ and x is large, we have

$$s_1 = -1 + \frac{\tau+1}{2x} + \mathcal{O}(x^{-2}), \quad s_2 = 1 - \frac{1-\tau}{2x} + \mathcal{O}(x^{-2}).$$

When x crosses the turning point x_2 from above, the saddle points collide and become complex at

$$s_1 = s_2 = \tau/(1+\sqrt{1-\tau^2}).$$

When x decreases further the saddle points turn around $s=1$, they collide (when $x = x_1$) at

$$s_1 = s_2 = (1+\sqrt{1-\tau^2})/\tau,$$

and they become real with values in $(1,\infty)$. When τ is close to unity ($\tau<1$), the collisions occur near $s=1$, a singular point of $\Phi(s)$, but in fact a pole of order $n+1$ of the integrand of (5.15).

We give a representation of the parabolic cylinder function showing similar asymptotic phenomena. We have (see [1, p. 687])

$$U(-\tfrac{1}{2}\kappa\rho^2, \eta\sqrt{2\kappa}) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\kappa\rho^2)e^{-\frac{1}{2}\kappa\eta^2}(2\kappa)^{\frac{1}{4}-\frac{1}{4}\kappa\rho^2}}{2\pi i}\int_{-\infty}^{(0^+)} e^{\kappa\Psi(t)}\frac{dt}{\sqrt{t}} \tag{5.18}$$

where

$$\Psi(t) = -t^2 + 2\eta t - \tfrac{1}{2}\rho^2\ln t. \tag{5.19}$$

The saddle points are

$$t_1 = \frac{1}{2}\eta - \frac{1}{2}\sqrt{\eta^2-\rho^2}, \quad t_2 = \frac{1}{2}\eta + \frac{1}{2}\sqrt{\eta^2-\rho^2}. \tag{5.20}$$

It is easily verified that the behaviour of t_1, t_2 is quite similar to that of s_1, s_2. When η crosses the turning points $\pm\rho$ and when ρ is small, the interesting area in the t-plane is the neighborhood of $t=0$. In Fig. 5.2 we sketch the paths of steepest descent for the three different situations. In the

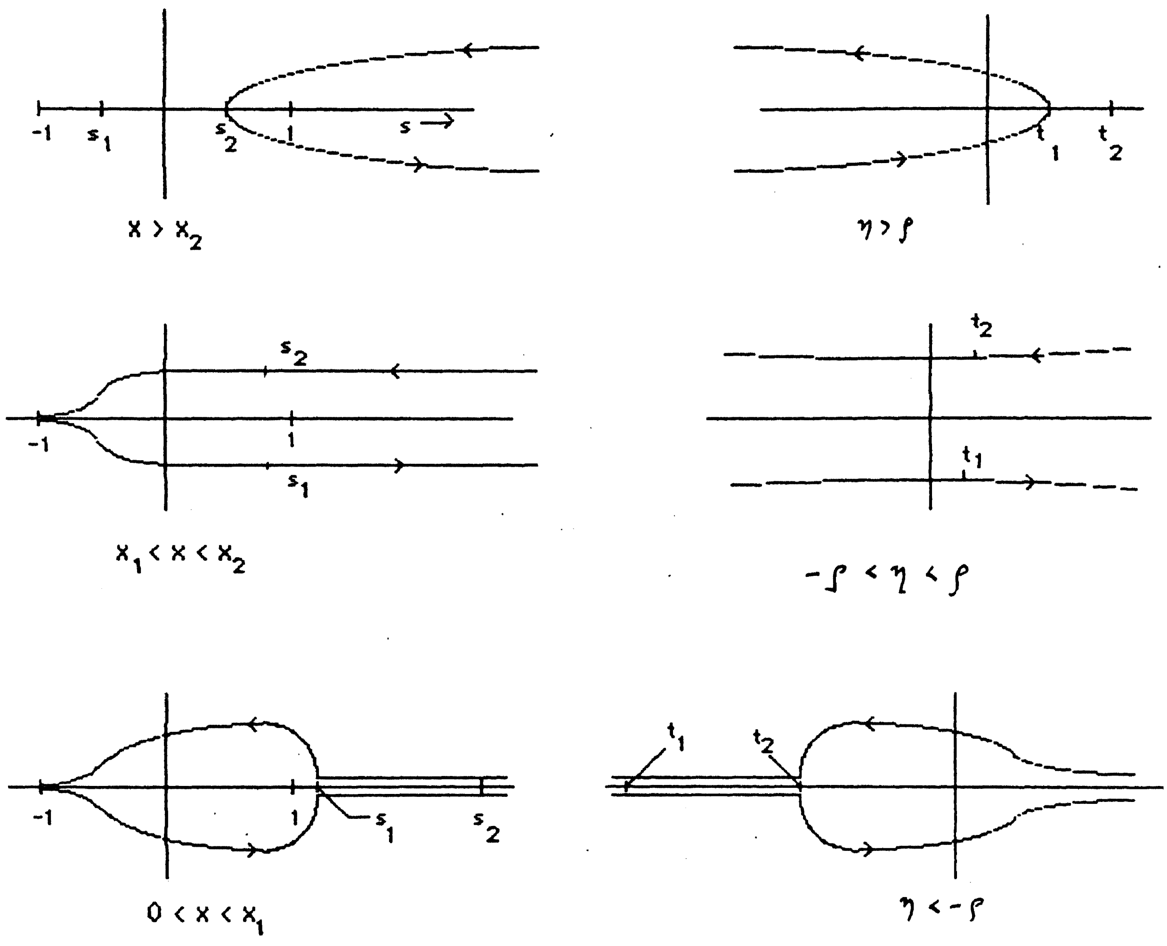

FIGURE 5.2. Steepest descent curves in s and t plane

third situation ($0<x<x_1$) the contributions from s_1 to $+\infty$ (on both sides of the cut) cancel, since for Laguerre polynomials $s=1$ is a pole. The same conclusion holds for the t-plane when $(1+\kappa\rho^2)/2=n$, a positive integer. If $s=1$ ($t=0$) is indeed an algebraic singularity, the saddle point

$s=s_2$ $(t=t_1)$ gives an important contribution. This change in behaviour is typical for $U(-a-1/2,z)$ when a crosses non-negative integer values and $-z$ is a large positive number.

Now we define the mapping $s \to t(s)$ by writing

$$\Phi(s) = \Psi(t)+B \tag{5.21}$$

with corresponding points $t(-1)=+\infty$, $t(s_1)=t_2$, $t(s_2)=t_1$, $t(1)=0$. To obtain a mapping that is analytic at $s=1$, we must have $1/2\rho^2=1-\tau$, see (4.4). The value η in (5.19) is computed by eliminating B in (5.21): $\Phi(s_1)-\Phi(s_2)=\Psi(t_2)-\Psi(t_1)$. Finally, B follows from $B=\Phi(s_1)-\Psi(t_2)$, giving $B=-1/2(\tau+1)-\tau\ln x+1/2(\tau+1)\ln(\tau+1)-1/2\eta^2-1/2(1-\tau)\ln 2$.

Again, a more detailed discussion of the mapping (5.21) is needed to show that the transformation of (5.15) to an integral of type (5.18) is well defined. We only give the result

$$L_n^{(\alpha)}(4\kappa x) = \frac{(-1)^{n+1}2^{-\alpha}e^{2\kappa x+\kappa B}}{2\pi i} \int_{-\infty}^{(0^+)} e^{\kappa\Psi(t)}h(t)\frac{dt}{\sqrt{t}}, \tag{5.22}$$

where

$$h(t) = \frac{\sqrt{t}}{\sqrt{1-s^2}}\frac{ds}{dt} = \frac{-\sqrt{1-s^2}}{x\sqrt{t}}\frac{(t-t_1)(t-t_2)}{(s-s_1)(s-s_2)},$$

with s_i,t_i given in (5.17), (5.20). At the saddle point t_1 the value of h is

$$h(t_1) = -\tfrac{1}{2}\sqrt{2}\left\{\frac{\eta^2-\rho^2}{x^2-x+\frac{1}{4}\tau^2}\right\}^{\frac{1}{4}}.$$

A first approximation now reads

$$L_n^{(\alpha)}(4\kappa x) \sim \frac{(-1)^{n+1}2^{-\alpha}\kappa^{n/2}e^{2\kappa x+\kappa B}}{n!} h(t_1)H_n(\eta\sqrt{\kappa}),$$

which is exactly the estimate in (4.13). More terms follows from an integration by parts procedure.

Write

$$h(t) = \gamma_0+\delta_0 t+(t-t_1)(t-t_2)g(t),$$

with

$$\gamma_0 = \frac{t_1h(t_2)-t_2h(t_1)}{t_1-t_2}, \quad \delta_0 = \frac{h(t_1)-h(t_2)}{t_1-t_2}. \tag{5.23}$$

It follows that

$$L_n^{(\alpha)}(4\kappa x) = \frac{(-1)^{n+1}2^{-\alpha}e^{2\kappa x+\kappa B}\kappa^{n/2}}{n!}[\gamma_0H_n(\eta\sqrt{\kappa})+\frac{\delta_0}{2\sqrt{\kappa}}H_n'(\eta\sqrt{\kappa})+\tilde{\delta}],$$

$$\tilde{\delta} = \frac{n!\kappa^{-n/2}}{2\pi\kappa i}\int e^{\kappa\Psi(t)}h_1(t)\frac{dt}{\sqrt{t}}, \quad h_1(t) = \tfrac{1}{2}\sqrt{t}\frac{d}{dt}[\sqrt{t}g(t)].$$

Since $h(t_1)=h(t_2)$, $\delta_0=0$, and $\tilde{\delta}$ is the same as $\tilde{\epsilon}$ in (4.13). Continuing the above procedure, we obtain the formal expansion

$$L_n^{(\alpha)}(4\kappa x) \sim \frac{(-1)^{n+1}2^{-\alpha}\kappa^{n/2}e^{2\kappa x+\kappa B}}{n!}\{H_n(\eta\sqrt{\kappa})\sum_{s=0}^{\infty}\gamma_s\kappa^{-s}+\frac{1}{2\sqrt{\kappa}}H_n'(\eta\sqrt{\kappa})\sum_{s=0}^{\infty}\delta_s\kappa^{-s}\}.$$

The coefficients follow from (5.23) and

$$h_s(t) = \tfrac{1}{2}\sqrt{t}\frac{d}{dt}[\sqrt{t}g_{s-1}(t)]=\gamma_s+\delta_s t+(t-t_1)(t-t_2)g_s(t), \quad s\geqslant 1,$$

where $h_0=h$, $g_0=g$.

REFERENCES

1. M.A. ABRAMOWITZ & I.A. STEGUN (1964). *Handbook of mathematical functions,* Nat. Bur. Stand. Appl. Math. Ser. 55, Washington D.C.
2. G.B. BAUMGARTNER, JR. (1980). *Uniform asymptotic approximations for the Whittaker function* $M_{k,m}(z)$, PhD-Thesis, Illinois Institute of Technology, Chicago.
3. A. ERDÉLYI, ed. (1953). *Bateman Manuscript project,* Higher transcendental functions, Vol. II, McGraw-Hill, New York.
4. A. ERDÉLYI, (1960). *Asymptotic forms for Laguerre polynomials,* J. Indian Math. Soc. *24*, 235-250.
5. W. MAGNUS, F. OBERHETTINGER, R.P. SONI, (1966). Formulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin.
6. F.W.J. OLVER, (1975). *Unsolved problems in the asymptotic estimation of special functions.* In: *Theory and application of special functions,* (R.A. Askey, ed.) pp. 99-142, Academic Press, 1975.
7. F.W.J. OLVER, (1980). *Whittaker functions with both parameters large: uniform approximations in terms of parabolic cylinder functions,* Proc. Royal Soc. Edinb. *84A*, 213-234.
8. H. SKOVGAARD, (1959). *Asymptotic forms of Hermite polynomials,* Technical Report 18, Department of Mathematics, Cal. Inst. Technology, Pasadena.
9. G. SZEGÖ, (1958). *Orthogonal polynomials,* Amer. Math. Soc. Colloq. Publ., Vol. 23, New York.
10. N.M. TEMME, (1985). *Special functions as approximants in uniform asymptotic expansions of integrals; a survey.* Rend. Sem. Mat. Univers. Politecn. Torino (Special issue on Special Functions: Theory and Computation) 289-317.