STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1966-007

On semi-regular and minimal Hausdorff embeddings

by

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June 1966

The Mathematical Centre at Amsterdam, founded the 11th of February, 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications, and is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.) and the Central Organization for Applied Scientific Research in the Netherlands (T.N.O.), by the Municipality of Amsterdam and by several industries.

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§1. Introduction

It is well known that a space can be embedded in a compact Hausdorff space if and only if it is completely regular. Thus since every compact Hausdorff space is minimal Hausdorff, we have a minimal Hausdorff embedding of every completely regular space. This result has been extended by Banaschewski [1] who proved that every semi-regular Hausdorff space can be densely embedded in a minimal Hausdorff space. In [2], Berri asked for a determination of the class of spaces which can be embedded in minimal Hausdorff spaces. It is the purpose of this paper to show that this is precisely the class of all Hausdorff spaces. First, however, we obtain a general semi-regular embedding theorem which seems to be of interest in itself.

We wish to express our gratitude to Prof. J. de Groot for his substantial improvement of the proof.

§2. Definitions and Notation

(X,τ) will denote the topological space having underlying set X and topology τ. If $A \subset X$, $Cl_\tau A$ ($Int_\tau A$) will denote the closure (interior) of A with respect to τ.

A set, V, in a topological space will be called <u>regular open</u> provided that it is the same as the interior of its closure; i.e. $V = Int_\tau(Cl_\tau V)$. A topology will be called <u>semi-regular</u> provided that it has a base consisting of regular open sets.

A topological space (X,τ) will be called <u>minimal Hausdorff</u> provided that τ is Hausdorff and there exists no strictly coarser (weaker) Hausdorff topology on X.

A topological space will be called <u>Urysohn</u> provided that distinct points have disjoint <u>closed</u> neighbourhoods.

The <u>weight</u> of a topological space, $(X,\mathcal{T})$, is the least cardinal number, $\underline{m}$, such that there exists a base for $\mathcal{T}$ with cardinality $\underline{m}$.

§3. Main results

<u>Theorem 1.</u> [1] Any topological space, $(X,\mathcal{T})$, can be embedded as a nowhere dense closed subspace in a connected semi-regular space $(Y,\hat{\mathcal{T}})$. Furthermore, the embedding preserves the separation properties T_0, T_1, Hausdorff, and Urysohn; and the weight of $(Y,\hat{\mathcal{T}})$ will be equal to the weight of $(X,\mathcal{T})$ if the weight of $(X,\mathcal{T})$ is infinite.

<u>Proof.</u>
Let $\mathcal{B}$ be a base for $(X,\mathcal{T})$ such that the cardinality of $\mathcal{B}$ is equal to the weight, $\underline{m}$, of $(X,\mathcal{T})$. For each $\alpha \in \mathcal{B}$, let I_α be a copy of the real line open ray $(0,+\infty)$. If $\alpha \neq \beta$ we require that $I_\alpha \cap I_\beta = \emptyset$. (If $A \subset (0,+\infty)$ is considered to be a subset of I_α, we label it A_α.) Let $\hat{0}$ be an additional point and let $Z = \bigcup \{I_\alpha \mid \alpha \in B\} \cup \{\hat{0}\}$. We now proceed to define a topology on Z.
Let J be the set of positive integers and let Q be the set of positive rational numbers. For each $n \in J$ let

$$U_n = \{\hat{0}\} \cup \{(0, \tfrac{1}{n})_\alpha \mid \alpha \in \mathcal{B}\}.$$

Let

$$\mathcal{P} = \{U_n \mid n \in J\} \cup \{(a,b)_\alpha \mid a,b \in Q,\ \alpha \in \mathcal{B}\}.$$

Then $\mathcal{P}$ is a base for a connected topology, $\mathcal{T}'$ on Z and

$$\text{card } (\mathcal{P}) = \max (\aleph_0, \underline{m}).$$

Now we let $Y = X \cup Z$. For each $\alpha \in \mathcal{B}$ and each $n \in J$, let

$$\hat{\alpha}_n = \alpha \cup \{(n,+\infty)_\beta \mid \beta \in \mathcal{B} \text{ and } \beta \subset \alpha\}.$$

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[1] A slightly stronger version of the theorem which asserts that any space $(X,\mathcal{T})$ can be embedded in a semi-regular space composed entirely of disjoint copies of $(X,\mathcal{T})$ has also been established (cf. [4]).

Then

$$\hat{\mathfrak{B}} = \{\hat{\alpha}_n \mid \alpha \in \mathfrak{B},\ n \in J\} \cup \mathfrak{P}$$

satisfies the requirements of a base for a topology, $\hat{\tau}$, on Y. We now proceed to show that $(Y,\hat{\tau})$ has the properties asserted by the theorem. Clearly $\hat{\tau}$ restricted to Z is τ', $Z \in \hat{\tau}$, and if $U \in \hat{\mathfrak{B}}$, $U \cap Z \neq \emptyset$. Thus Z is a connected, dense, open subset of $\hat{\tau}$. Consequently, since $\hat{\tau}$ restricted to X is τ, we have that (X,τ) is embedded as a nowhere dense closed subspace of the connected space $(Y,\hat{\tau})$. It is clear that the weight of $(Y,\hat{\tau})$ is equal to the cardinality of $\mathfrak{P}$, which is equal to the weight of (X,τ) if the latter is infinite. Thus it remains to be shown that $(Y,\hat{\tau})$ is semi-regular and preserves the separation properties mentioned.

Proof of semi-regularity

We show that every number of $\hat{\mathfrak{B}}$ is regular open. If $V \in \mathfrak{P}$, then it is clear that there exists some integer, n, such that

$$V \subset \{\hat{0}\} \cup \{(0,n)_\alpha \mid \alpha \in \mathfrak{B}\} = W_n.$$

Thus if $x \in X$ and $x \in \beta$, then

$$\hat{\beta}_{n+1} \cap V \subset \hat{\beta}_{n+1} \cap W_n = \emptyset.$$

Consequently $Cl_{\hat{\tau}}\, V = Cl_{\tau'}\, V \subset Z$, which combined with the fact that $Z \in \hat{\mathfrak{T}}$, implies that

$$Int_{\hat{\tau}}\, Cl_{\hat{\tau}}\, V = Int_{\tau'}\, Cl_{\tau'}\, V = V.$$

Therefore V is $\hat{\tau}$-regular open. Now suppose that $\hat{\alpha}_n \in \hat{\mathfrak{B}} - \mathfrak{P}$, and $q \in Cl_{\hat{\tau}}\, \hat{\alpha}_n - \hat{\alpha}_n$. If $q \in Z$, then there exists some $\beta \subset \alpha$ such that $q = n \in I_\beta$. Thus $q \notin Int_{\hat{\tau}}\, Cl_{\hat{\tau}}\, \hat{\alpha}_n$. If $q \in X$ and $\hat{\delta}_m$ is any member of $\hat{\mathfrak{B}}$ which contains q, then $\delta \not\subset \alpha$; hence $\hat{\delta}_m \not\subset Cl_{\hat{\tau}}\, \hat{\alpha}_n$ since $\{m + 1\}_\delta \subset \hat{\delta}_m - Cl_{\hat{\tau}}\, \hat{\alpha}_n$. Consequently, $\hat{\alpha}_n$ is $\hat{\tau}$-regular open.

Proof of separation preservation

First note that distinct points of Z have disjoint closed $\hat{\tau}$-neighbourhoods. Now suppose that $x \in X$ and $z \in Z$. If $z \in V \in \mathfrak{P}$ and $x \in \alpha \in \mathfrak{B}$, then, as above, there exists an $n \in J$ such that $V \subset W_n$. Clearly $x \in \hat{\alpha}_{n+1}$ and

$$\mathrm{Cl}_{\hat{\tau}} V \cap \mathrm{Cl}_{\hat{\tau}} \hat{\alpha}_{n+1} = \emptyset.$$

Hence x and z have Urysohn separation. Furthermore, if $\alpha \in \mathfrak{B}$ and $x \notin \alpha$, then $x \notin \hat{\alpha}_n$ for all $n \in J$. Also if $\alpha, \beta \in \mathfrak{B}$ and $\alpha \cap \beta = \emptyset$, then $\hat{\alpha}_n \cap \hat{\beta}_n = \emptyset$ for all $n \in J$; and if $\mathrm{Cl}_{\tau}\, \alpha \cap \mathrm{Cl}_{\tau}\, \beta = \emptyset$, then $\mathrm{Cl}_{\hat{\tau}}\, \hat{\alpha}_n \cap \mathrm{Cl}_{\hat{\tau}}\, \hat{\beta}_n = \emptyset$ for all $n \in J$. Thus if (X,τ) is T_0, T_1, Hausdorff or Urysohn, $(Y,\hat{\tau})$ will be T_0, T_1, Hausdorff or Urysohn, respectively.

Theorem 2. Every Hausdorff space can be embedded in a minimal Hausdorff space.

Proof.

By theorem 1 any Hausdorff space can be embedded in a semi-regular Hausdorff space. By the result of Banaschewski cited above, this space can be embedded in a minimal Hausdorff space.

Remark.

Several generalizations of theorem 2 appear in [3]; e.g. an arbitrary Hausdorff space can be embedded as a closed subspace of a minimal Hausdorff space with the same weight and there exists a separation and weight preserving embedding of any space into a central-compact space (central-compactness being a generalization of the minimal Hausdorff properties to spaces with weaker separation).

References

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[4] G.E. Strecker and E. Wattel A coherent embedding of an arbitrary space in a semi-regular space, Math. Centrum Amsterdam, Afd. Zuivere Wisk., ZW 1966-006.