Queueing Syst
DOI 10.1007/s11134-017-9560-z

Heavy traffic limit for a tandem queue with identical service times

H. Christian Gromoll[1] · Bryce Terwilliger[1] · Bert Zwart[2]

Received: 6 December 2016 / Revised: 15 September 2017

Abstract We consider a two-node tandem queueing network in which the upstream queue is $M/G/1$ and each job reuses its upstream service requirement when moving to the downstream queue. Both servers employ the first-in-first-out policy. We investigate the amount of work in the second queue at certain embedded arrival time points, namely when the upstream queue has just emptied. We focus on the case of infinite-variance service times and obtain a heavy traffic process limit for the embedded Markov chain.

Keywords Tandem queue · Infinite variance · Feller process · Process limit

Mathematics Subject Classification 60K25 · 90B22

1 Introduction

One of the most remarkable queueing models in the literature is the tandem network consisting of two first-in-first-out (FIFO) queues, where the first queue is $M/G/1$ with arrival rate λ, and jobs reuse their original service requirement when moving to the second queue. This latter feature introduces dependence between the second queue's arrival and service processes, resulting in unusual behavior in the second queue. In

✉ Bert Zwart
Bert.Zwart@cwi.nl

H. Christian Gromoll
gromoll@virginia.edu

Bryce Terwilliger
bat5ct@virginia.edu

1 Department of Mathematics, University of Virginia, Charlottesville, VA 22904, USA

2 Centrum Wiskunde & Informatica, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Published online: 16 November 2017

his PhD thesis [1], Boxma derived explicit expressions for a variety of steady-state functionals for the second queue, in particular the steady-state waiting time of a job; see also [2].

Apart from being a rare example of a nonproduct-form tandem queueing network for which an explicit analysis of the downstream queue is possible, this model also shows unusual behavior in heavy traffic. In particular, a variety of results have been derived in the case where the variance of service requirements is finite; see [3,4] and references therein for an overview. These results imply that the amount of work in the second queue is of smaller order than the amount of work in the first queue as the system load ρ (which is identical for both queues) increases to 1. For service times with bounded support, it is even shown in [5] that the expected value of the waiting time in the second queue is finite for $\rho = 1$.

The intuition behind these results is that the amount of work in the first queue is driven by sums, but in the second queue is driven by maxima. More precisely, letting M_k be the largest service time in the kth busy period of an $M/G/1$ queue, and letting I_k be an exponential random variable with rate λ (which can be interpreted as the duration of the kth idle period), the workload R_k in the second queue at the end of the kth busy period of the first queue satisfies the recursion

$$R_{k+1} = \max\{R_k - I_k, M_k\}, \quad k \geq 1; \tag{1}$$

cf. [2].

The goal of this paper is to analyze the Markov chain R_k in detail, in the regime where $\rho \to 1$ and in the situation where normalized sums and normalized maxima are comparable, i.e., the case where service times have a regularly varying tail of index in the range $(1, 2)$. This is the range not covered in [3,4]. We not only focus on the invariant distribution of this Markov chain, but also on its behavior at the process level.

A key ingredient of our analysis is a limit theorem for the distribution tail of M_k in heavy traffic. It turns out that it is not possible to use the tail behavior of M_k for fixed ρ, as suggested in [6]. Rather, we prove a new lower bound for the tail of M_k that is in the same spirit of an upper bound derived in [5]. A rescaled version of the distribution of M_k is then shown to converge to a limit that is expressed through a certain function $\kappa(y)$, shown to be the unique solution of a particular equation. Once this result for the limiting distribution of M_k is established, it is possible to utilize techniques from [7] to determine a Markov (in particular a Feller) process that is the limit of an appropriately scaled and normalized version of the Markov chain (1).

A model related to (1) is treated in [8], which investigates the extreme-value behavior of a Markov chain modeling the evolution of world records in improving populations. Though the models are different, one could connect them by interpreting $R_k - I_k$ as a discounted world record. A main difference is that, in [8], the random variables M_k have a fixed distribution, while we need to consider how M_k behaves in heavy traffic, which represents a substantial part of our effort.

Though the Markov chain (1) is of intrinsic interest, it gives a somewhat coarse description of the workload evolution in the second queue. It is also of interest to consider the evolution of the workload in the second queue during busy periods of the first queue, to consider joint convergence of both queues in heavy traffic, and to drop the assumption that interarrival times are exponential. These questions are beyond the scope and techniques of this paper, and will be pursued elsewhere.

The paper is organized as follows. Section 2 provides a detailed model description and presents our main results. Section 3 focuses on the behavior of M_k in heavy traffic. The process limit of (1) is investigated in Sect. 4 (dealing with convergence of one-dimensional distributions) and Sect. 5 (focusing on convergence of the entire process).

1.1 Notation

The following notation will be used throughout. Let $\mathbb{N} = \{1, 2, \ldots\}$ and let $\mathbb{R}$ denote the real numbers. Let $\mathbb{R}_+ = [0, \infty)$. For $a, b \in \mathbb{R}$, write $a \vee b$ for the maximum, $a \wedge b$ for the minimum, $[a]^+ = 0 \vee a$, $[a]^- = 0 \vee -a$, and $[a]$ for the integer part of a. A sum over an empty set of indices is defined to be zero.

We say a nonnegative function f is regularly varying with parameter ν if

$$\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\nu$$

for each $\lambda > 0$, and we say it is regularly varying at zero if this holds for $x \to 0$ instead. A random variable V is regularly varying with parameter ν if $x \mapsto \mathbb{P}\{V > x\}$ is regularly varying with parameter $-\nu$. Note that if a nonnegative random variable V is regularly varying with parameter ν, then $\mathbb{E}[|V|^\gamma] < \infty$ if and only if $\gamma < \nu$.

For a function F of bounded variation, we denote the Lebesgue–Stieltjes measure associated with F by $dF(x)$ or $F(dx)$. In particular, when $F(x) = (x/\kappa)^{-\nu}$, we denote this signed measure $(dx/\kappa)^{-\nu}$.

For a distribution function $F(x) = \mathbb{P}\{V \leq x\}$ we write $\bar{F}(x) = 1 - F(x)$.

Let $\mathbb{D} = \mathbb{D}([0, \infty), \mathbb{R})$ be the space of real-valued, right-continuous functions on $[0, \infty)$ with finite left limits. We endow $\mathbb{D}$ with the Skorohod J_1-topology, which makes $\mathbb{D}$ a Polish space [9]. If X and Y have the same distribution, we write $X \sim Y$. We write $X_n \Rightarrow X$ if X_n converges in distribution to X.

2 Model description and main results

In this section, we give a precise description of the tandem queue, specify our assumptions, and state our main result.

2.1 Definition of the model

We formulate a model equivalent to the one in Boxma [2]. The tandem queueing system consists of two queues Q1 and Q2 in series; both Q1 and Q2 are single-server queues employing the FIFO policy, with an unlimited buffer. Jobs enter the tandem system at Q1. After completion of service at Q1, a job immediately enters Q2, and when service at Q2, which has exactly the same length as previously experienced in Q1, is completed, it leaves the tandem system. We assume the system is empty at time zero.

Arrivals to Q1 are given by the *exogenous arrival process* $E(\cdot)$, a Poisson process with parameter λ. The *service times* of these arriving jobs are given by an i.i.d. sequence $\{V_i, i \in \mathbb{N}\}$ with distribution function F. That is, V_i is the amount of service required from each server by the ith arrival. We assume throughout that $1 - F$ is regularly varying with parameter $-\nu$, $1 < \nu < 2$, so that $\mathbb{E}[V_1] < \infty$ and $\mathbf{Var}(V_1) = \infty$.

Assume the traffic intensity $\rho = \lambda\mathbb{E}[V_1] \leq 1$ so that the number of jobs in a typical busy period of Q1 is a proper random variable, and when $\rho < 1$ the expected number of jobs in a busy period is $1/(1-\rho)$. Let M_i denote the service time of the largest job in the ith busy period of Q1, and denote the distribution function of M_i by m. The distribution function m does not depend on i because the busy periods correspond to independent and identically distributed cycles. For $w > 0$, Boxma [5] shows that $m(w)$ is the unique solution to

$$m(w) = \int_0^w e^{-\lambda t \bar{m}(w)} \mathrm{d}F(t). \tag{2}$$

Jobs departing Q1 immediately enter Q2. Jobs only arrive to Q2 from Q1, so the arrival process at Q2 is the departure process from Q1. At Q2, the service requirement of the ith job is V_i, equal to its service requirement at Q1, so no additional randomness is introduced in the second queue.

For $t \geq 0$, let

$$I(t) = \sup_{s \leq t} \left[\sum_{i=1}^{E(s)} V_i - s \right]^{-}. \tag{3}$$

We interpret $I(t)$ as the cumulative amount of idle time experienced by the first server up to time t.

Let $W_i(t)$ denote the (immediate) workload at time t at Qi, $i = 1, 2$, which is the total amount of time that the server must work in order to satisfy the remaining service requirement of each job present at the queue at time t, ignoring future arrivals. These processes are defined in the usual way: for $t \geq 0$,

$$W_1(t) = \sum_{i=1}^{E(t)} V_i - t + I(t).$$

The departure process from Q1 may be written $D(t) = \max\{k \geq 0 : \sum_{i=1}^{k} V_i \leq t - I(t)\}$. Then, $W_2(t)$ is defined analogously to $W_1(t)$ using $D(t)$ in place of $E(t)$ and the Q2 idleness process in place of $I(t)$; this latter process is defined as in (3) with $D(\cdot)$ in place of $E(\cdot)$.

This paper concerns the workload in the second queue at particular points in time. Let t_i be the arrival time to Q1 of the last job in the ith busy period at Q1. Let $\tilde{t}_i$ be the time this job arrives to Q2. For $i \in \mathbb{N}$,

$$\tilde{t}_i = t_i + W_1(t_i).$$

Let R_n be the workload in the second queue at the time of the arrival to Q2 of the last job in the nth busy period of Q1. For $n \in \mathbb{N}$,

$$R_n = W_2(\tilde{t}_n).$$

The random variable R_n is the largest sojourn time in Q2 experienced by any job in the nth busy period of Q1. The reason for this is that, as long as Q1 is not idling, the next interarrival time to Q2 is identical to the next service requirement, or amount of work to be added to Q2. If this service requirement is less than the current Q2 workload, the workload will simply decrease and then increase by the same amount, returning to its previous level. If this service requirement is greater than the current workload, the workload will decrease to zero and then jump to a level equal to the incoming service requirement, higher than the previous level.

In this way the Q2 workload performs a series of returns to a given level until a job arrives that is larger than all previous jobs in the busy period, causing the level to be set higher. Although the last job of a Q1 busy period may not be the largest, it will by definition return the Q2 workload to the highest level it attains for the busy period (or set it to a new highest level if this job happens to be the largest in the busy period). Thus, the Q2 workload R_n at time $\tilde{t}_n$ is equal to the highest workload and thus largest sojourn time encountered upon arrival by any job in the nth Q1 busy period.

The above description is only valid during busy periods of the first queue. Idleness in the first queue complicates the dynamics substantially. Nevertheless, Boxma [2] Theorem 6.1 describes the steady-state distribution of R_n when $\rho < 1$:

$$\lim_{n\to\infty} \mathbb{P}\{R_n \leq w\} = m(w) \exp\left(-\lambda \int_w^\infty \bar{m}(y)\,\mathrm{d}y\right). \tag{4}$$

In this paper, we establish a limit theorem for the whole chain R_n as the traffic intensity $\rho \to 1$.

2.2 Heavy traffic limit theorems

Now we consider a sequence of tandem queueing systems indexed by $n \in \mathbb{N}$. Each model in the sequence is defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each $n \in \mathbb{N}$, the arrival process $E^{(n)}$ is a Poisson process with parameter $\lambda^{(n)}$, and the service times are given by the same sequence $\{V_i\}_{i=1}^\infty$ of i.i.d. regularly varying random variables with parameter $\nu \in (1, 2)$. Assume that $\mathbb{E}[V_1] > 0$ and that $\{V_i\}_{i=1}^\infty$ is independent of each $E^{(n)}$. When necessary, we will apply a superscript (n) to indicate the nth model.

Asymptotic assumptions We make the following asymptotic assumptions about our sequence of models as $n \to \infty$. We want the traffic intensity $\rho^{(n)}$ increasing to 1 with fixed service times $\{V_i\}$, so let $\lambda = 1/\mathbb{E}[V_1]$ and assume $\lambda^{(n)} \uparrow \lambda$ so that $\rho^{(n)} = \lambda^{(n)}\mathbb{E}[V_1] \uparrow 1$. Additionally, we assume this occurs at an appropriate rate, namely

$$\left(\frac{1-\rho^{(n)}}{n\bar{F}(n)}\right) \to \gamma \geq 0. \tag{5}$$

We are now ready to state the first main result of our study. Let T_ν be a Pareto(ν) random variable and let Γ denote the gamma function.

Theorem 2.1 *Under the above assumptions, for $y > 0$,*

$$\lim_{n\to\infty} n\bar{m}^{(n)}(ny) = \kappa(y)/y, \tag{6}$$

where $\kappa = \kappa(y)$ satisfies the equation

$$\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right] - \kappa\gamma y^{\nu-1}\left(\frac{-1}{\Gamma(1-\nu)}\right) = (\lambda\kappa)^\nu, \tag{7}$$

$\kappa(y)$ is constant when $\gamma = 0$, and is regularly varying of index $1-\nu$ when $\gamma > 0$.

To give an idea of the proof, observe that Boxma's equation (2) for the distribution function m of the largest job in a busy period is nearly the Laplace transform of V evaluated at $\lambda\bar{m}(w)$. Since (2) holds for each model, we scale time and space by n as in the law of large numbers, then apply an Abelian theorem to show that $n\bar{m}^{(n)}(n\cdot)$, the sequence of rescaled distribution functions, converges. We then find appropriate asymptotic bounds, establishing subsequential limits. These limits can all be characterized as the solution to an equation which is shown to be unique, implying convergence. A detailed proof of this result is provided in Sect. 3.

We now turn to our results pertaining to the behavior of (1) in heavy traffic. For each $n \in \mathbb{N}$, let $\{Y_n(k), k = 0, 1, 2, \ldots\}$ be a Markov chain in $[0, \infty)$ with transition function $\mu_n(x, B) = \mathbb{P}\left\{\max(x - I^{(n)}/n, M^{(n)}/n) \in B\right\}$, where $I^{(n)}$ is an exponential random variable with parameter $\lambda^{(n)}$ independent of the random variable $M^{(n)}$ which is the largest job in a busy period. Observe that, using (1) (which is the recursion corresponding to Proposition 4.1), $Y_n(k) \sim \frac{1}{n}R_k^{(n)}$. Let $X_n(t) = Y_n([nt])$. Our next result describes convergence of the one-dimensional distributions of (1).

Theorem 2.2 *For each $t \geq 0$, $X_n(t) \Rightarrow Z_t$ with*

$$\mathbb{P}\{Z_t \leq x\} = \exp\left(-\lambda\int_x^{x+t/\lambda} \kappa(y)/y\,\mathrm{d}y\right).$$

In particular, when $\gamma > 0$,

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}\{Z_t \leq x\} &= \exp\left(-\lambda\int_x^\infty \kappa(y)/y\mathrm{d}y\right)\\ &= \lim_{n\to\infty} m^{(n)}(nx)\exp\left(-\lambda^{(n)}\int_x^\infty n\bar{m}^{(n)}(nt)\mathrm{d}t\right).\end{aligned} \tag{8}$$

The second statement (8) follows from the first together with Theorem (2.1): when $\rho < 1$, we can rescale space by n in the steady-state distribution for (1) given by (4), which becomes $m^{(n)}(nx)\exp\left\{-\lambda^{(n)}\int_x^{\infty} n\bar{m}^{(n)}(nt)\mathrm{d}t\right\}$. Note that the limit of the steady-state distributions agrees with the limit of the one-dimensional distributions, showing that the limits $t \to \infty$ and $n \to \infty$ can be interchanged. The proof of the first statement is more involved and described in Sect. 4.

We conclude this section with the following theorem for the scaled process.

Theorem 2.3 *Suppose $\{X_n(0)\}$ has limiting distribution ν. There is a Markov process X corresponding to a Feller semigroup $\{T(t)\}$ with initial distribution ν and sample paths in $D_{\mathbb{R}}[0, \infty)$ such that $X_n \Rightarrow X$.*

The generator of X can informally be written as

$$\hat{A}f(x) = \frac{-f'(x)}{\lambda} + \int_x^{\infty} f'(y)\kappa(y)y^{-1}\,\mathrm{d}y. \tag{9}$$

One can interpret the above as follows. The process drifts at rate $-1/\lambda$ and jumps come from the maximum process that is the limit of $n\bar{m}^{(n)}(n\mathrm{d}u)$, which also depends on the drift γ coming from the traffic intensity. A formal proof of Theorem 2.3 is given in Sect. 5.

3 The maximum service time in heavy traffic

The purpose of this section is to derive the asymptotic behavior of the distribution of M_k in heavy traffic. The section begins with two technical lemmas, for which the proofs can be skipped at first reading. After that, we derive asymptotic lower and upper bounds, which are sharp up to a constant, and provide an important stepping stone toward the derivation of the limit.

3.1 Some preliminary lemmas

The following lemma is intuitive because the supremum over a larger set of similar objects must also be larger. Recall that $\lambda^{(n)} \uparrow \lambda$ and let $m^{(\infty)}$ be the distribution function of the largest job in a busy period in a system where the arrival process is Poisson and $\rho = 1$.

Lemma 3.1 *As $n \to \infty$,*

$$\bar{m}^{(n)}(x) \uparrow \bar{m}^{(\infty)}(x), \quad x \geq 0.$$

Proof Apply (2) to a convergent subsequence of $\bar{m}^{(n)}(x)$ and pass to the limit via dominated convergence. Since (2) has unique solutions, the limit must equal $\bar{m}^{(\infty)}(x)$. For monotonicity, observe that differentiating (2) for fixed x with respect to λ yields

$$\frac{\mathrm{d}m}{\mathrm{d}\lambda} = \frac{-\int_0^x t(1-m)e^{-\lambda t(1-m)}\,\mathrm{d}F(t)}{1-\lambda\int_0^x te^{-\lambda t(1-m)}\mathrm{d}F(t)},$$

which is negative because $\lambda \int_0^x t e^{-\lambda t(1-m)} \mathrm{d}F(t) \le \lambda \int_0^x t \,\mathrm{d}F(t) \le \rho$ and implies $\frac{d\bar{m}}{d\lambda}$ is positive for λ less than the critical value. □

The next lemma uses an Abelian theorem. Recall that $\rho^{(n)} = \lambda^{(n)} \mathbb{E}[V_1]$. Our assumption that the $\{V_i\}$ are regularly varying with parameter $1 < \nu < 2$ implies that we can write $1 - F(t) = \left(\frac{-1}{\Gamma(1-\nu)}\right) t^{-\nu} l(t)$ for a slowly varying function l.

Lemma 3.2 *Fix $y > 0$. Then,*

$$\lim_{n\to\infty} \frac{\left(\frac{-1}{\Gamma(1-\nu)}\right) \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V} \middle| V > ny\right] - \frac{\bar{m}^{(n)}(ny)(1-\rho^{(n)})(ny)^{\nu}}{l(ny)}}{\left(\lambda^{(n)} ny \bar{m}^{(n)}(ny)\right)^{\nu} \left(\frac{l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)}{l(ny)}\right)} = 1.$$

Proof Since arrivals are Poisson and $\rho^{(n)} \le 1$, we have $m^{(n)}(ny) = \int_0^{ny} e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t)$ by (2). So,

$$\bar{m}^{(n)}(ny) = 1 - \int_0^{\infty} e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t) + \int_{ny}^{\infty} e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t). \qquad (10)$$

For fixed $y > 0$, write

$$\begin{aligned}
\int_{ny}^{\infty} e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t) &= \int_0^{\infty} e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} 1_{(ny,\infty)}(t) \mathrm{d}F(t) \\
&= \mathbb{E}\left[e^{-\lambda^{(n)} V \bar{m}^{(n)}(ny)} 1_{(ny,\infty)}(V)\right] \\
&= \mathbb{P}\{V > ny\} \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V} \middle| V > ny\right] \\
&= \left(\frac{-1}{\Gamma(1-\nu)}\right) (ny)^{-\nu} l(ny) \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V} \middle| V > ny\right].
\end{aligned}$$

Substituting this into Eq. (10),

$$\begin{aligned}
\bar{m}^{(n)}(ny) = 1 &- \int_0^{\infty} e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t) \\
&+ \left(\frac{-1}{\Gamma(1-\nu)}\right) (ny)^{-\nu} l(ny) \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V} \middle| V > ny\right].
\end{aligned}$$

Rearranging, and using $\lambda^{(n)}\mathbb{E}[V] = \rho^{(n)}$, we have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t) - 1 + \lambda^{(n)}\bar{m}^{(n)}(ny)\mathbb{E}[V] \\
&\quad = \left(\frac{-1}{\Gamma(1-\nu)}\right)(ny)^{-\nu} l(ny)\mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V}\,\middle|\, V > ny\right] \\
&\qquad - \bar{m}^{(n)}(ny)(1-\rho^{(n)}).
\end{aligned} \tag{11}$$

Next, dividing by $\left(\lambda^{(n)}\bar{m}^{(n)}(ny)\right)^\nu l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)$ and multiplying the right-hand side by $(ny)^\nu/(ny)^\nu$ we have

$$\begin{aligned}
&\frac{\int_0^\infty e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t) - 1 + \lambda^{(n)}\bar{m}^{(n)}(ny)\mathbb{E}[V]}{\left(\lambda^{(n)}\bar{m}^{(n)}(ny)\right)^\nu l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)} \\
&= \frac{\left(\frac{-1}{\Gamma(1-\nu)}\right) l(ny)\mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V}\,\middle|\, V > ny\right] - \bar{m}^{(n)}(ny)(1-\rho^{(n)})(ny)^\nu}{\left(\lambda^{(n)} ny \bar{m}^{(n)}(ny)\right)^\nu l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)}.
\end{aligned} \tag{12}$$

The limit as $n \to \infty$ on the left-hand side is 1 by [10] Theorem 8.1.6. To justify the use of Theorem 8.1.6, we note the left-hand side of Eq. (12) is, in the notation used in Theorem 8.1.6, $(\hat{F}(s) - 1 + s\mathbb{E}[V])/(s^\nu l(1/s))$. So, $\bar{F}(x) = -1/\Gamma(1-\nu)x^{-\nu}l(x)$ is equivalent to $(\hat{F}(s) - 1 + s\mathbb{E}[V])/(s^\nu l(1/s)) \to 1$, where $1, < \nu < 2$ and $s = s(n) = \lambda^{(n)}\bar{m}^{(n)}(ny)$. Since $\lambda^{(n)} \uparrow \lambda < \infty$ and $\bar{m}^{(n)}(\cdot)$ is increasing in n by Lemma 3.1, $m^{(\infty)}$ is a proper probability distribution yields $s(n) \le \lambda\bar{m}^{(\infty)}(ny) \downarrow 0$ as $n \to \infty$. □

3.2 Asymptotic lower and upper bounds

We are now ready to derive lower and upper bounds for $ny\bar{m}^{(n)}(ny)$ that are shown to converge in $(0, \infty)$ for each $y > 0$.

Lemma 3.3 *For all $y \ge 0$,*

$$\limsup_{n\to\infty} ny\bar{m}^{(n)}(ny) \le \max\left[2^{2/\nu}\mathbb{E}[V], 1\right].$$

Proof If $\lambda^{(n)}(ny)\bar{m}^{(n)}(ny) \ge 1$, we take $A = 2$ and $\delta = \nu/2$ in Potter's Theorem [10] 1.5.6 so that, for n sufficiently large,

$$(1/2)\left(\lambda^{(n)} ny\bar{m}^{(n)}(ny)\right)^{-\nu/2} \le \left(\frac{l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)}{l(ny)}\right).$$

The terms $\bar{m}^{(n)}(ny)(1-\rho^{(n)})(ny)^{\nu}$ and $\frac{-1}{\Gamma(1-\nu)}$ are nonnegative and $l(ny)$ is eventually positive. So, for n sufficiently large,

$$\begin{aligned}
&\frac{\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V}\middle|V>ny\right]-\frac{\bar{m}^{(n)}(ny)(1-\rho^{(n)})(ny)^{\nu}}{l(ny)}}{\left(\lambda^{(n)}ny\bar{m}^{(n)}(ny)\right)^{\nu}\left(\frac{l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)}{l(ny)}\right)}\\
&\leq\frac{1}{\left(\lambda^{(n)}ny\bar{m}^{(n)}(ny)\right)^{\nu}\left(\frac{l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)}{l(ny)}\right)}\\
&\leq\frac{1}{\left(\lambda^{(n)}ny\bar{m}^{(n)}(ny)\right)^{\nu}\left((1/2)\left(\lambda^{(n)}ny\bar{m}^{(n)}(ny)\right)^{-\nu/2}\right)}\\
&=\frac{2}{\left(\lambda^{(n)}ny\bar{m}^{(n)}(ny)\right)^{\nu/2}}.
\end{aligned}$$

Lemma 3.2 gives

$$\liminf_{n\to\infty}\frac{2}{\left(\lambda^{(n)}ny\bar{m}^{(n)}(ny)\right)^{\nu/2}}\geq 1,$$

when $\limsup_{n\to\infty}\lambda^{(n)}ny\bar{m}(ny)\geq 1$. Since $\lambda^{(n)}\to 1/\mathbb{E}[V]$, we have

$$\limsup_{n\to\infty} ny\bar{m}^{(n)}(ny)\leq\max\left[2^{2/\nu}\mathbb{E}[V],1\right]. \qquad \square$$

The following inequality holds even in the case $\rho^{(n)}=1$ for each n.

Lemma 3.4 *For each compact set in $\mathcal{K}\subset\mathbb{R}_+$, there exists a constant $L>0$ such that, for all $y\in\mathcal{K}$,*

$$\liminf_{n\to\infty} ny\bar{m}^{(n)}(ny)\geq L.$$

Proof Fix $y\in\mathcal{K}$ and let $K=\sup_n ny\bar{m}^{(n)}(ny)$, which is finite by Lemma 3.3. Note that, for all $t\leq K$, there exists, under our assumptions, a constant C_K independent of n such that $e^{-\lambda^{(n)},t}\leq 1-\lambda^{(n)}t+C_K t^2$ for each n and each $t\in[0,K]$. Inserting this inequality into Boxma's equation (2), we obtain

$$\begin{aligned}
m^{(n)}(ny) &= \int_0^{ny} e^{-\lambda^{(n)} t \bar{m}^{(n)}(ny)} \mathrm{d}F(t) \\
&\le \int_0^{ny} \left(1 - \lambda^{(n)} t \bar{m}^{(n)}(ny) + C_K t^2 (\bar{m}^{(n)}(ny))^2\right) \mathrm{d}F(t) \\
&= F(ny) - \lambda^{(n)} \bar{m}^{(n)}(ny) \int_0^{ny} t \mathrm{d}F(t) + C_K \left(\bar{m}^{(n)}(ny)\right)^2 \int_0^{ny} t^2 \mathrm{d}F(t).
\end{aligned}$$

Consequently,

$$\bar{m}^{(n)}(ny) \ge \bar{F}(ny) + \lambda^{(n)} \bar{m}^{(n)}(ny) \int_0^{ny} t \mathrm{d}F(t) - C_K \left(\bar{m}^{(n)}(ny)\right)^2 \int_0^{ny} t^2 \mathrm{d}F(t).$$

This implies

$$\bar{m}^{(n)}(ny) \left(1 - \lambda^{(n)} \int_0^{ny} t \mathrm{d}F(t) + \bar{m}^{(n)}(ny) C_K \int_0^{ny} t^2 \mathrm{d}F(t)\right) \ge \bar{F}(ny).$$

Since the second factor on the left-hand side is positive, we see that

$$\bar{m}^{(n)}(ny) \ge \bar{F}(ny) \left(1 - \lambda^{(n)} \int_0^{ny} t \mathrm{d}F(t) + \bar{m}^{(n)}(ny) C_K \int_0^{ny} t^2 \mathrm{d}F(t)\right)^{-1}.$$

So we see that

$$\frac{1}{ny\bar{m}^{(n)}(ny)} \le \frac{1 - \lambda^{(n)} \int_0^{ny} t \mathrm{d}F(t)}{ny\bar{F}(ny)} + ny\bar{m}^{(n)}(ny) C_K \frac{\int_0^{ny} t^2 \mathrm{d}F(t)}{(ny)^2 \bar{F}(ny)}. \tag{13}$$

To derive our desired result, we need to show that the limit superior on the right-hand side of this equation is finite. Since we already know from Lemma 3.3 that $\limsup_{n\to\infty} ny\bar{m}^{(n)}(ny) \le K$, it suffices to investigate both fractions.

Both will be dealt with using Karamata's theorem (Theorems 1.6.4 and 1.6.5 in [10]). Set $w = ny$. An application of these results in our setting yields

$$\lim_{w\to\infty} \frac{\int_0^w t^2 \mathrm{d}F(t)}{w^2 \bar{F}(w)} = \frac{\nu}{2-\nu} \quad \text{and} \quad \lim_{w\to\infty} \frac{\int_w^\infty t \mathrm{d}F(t)}{w\bar{F}(w)} = \frac{\nu}{\nu-1}. \tag{14}$$

For the first fraction, write

$$\frac{1 - \lambda^{(n)} \int_0^w t \mathrm{d}F(t)}{w\bar{F}(w)} = \frac{1-\rho^{(n)}}{w\bar{F}(w)} + \frac{\lambda^{(n)} \int_w^\infty t \mathrm{d}F(t)}{w\bar{F}(w)}. \tag{15}$$

The first term on the right-hand side of (15) converges to $\gamma y^{\nu-1}$ due to our heavy traffic assumption (5) and since $\bar{F}$ is regularly varying. The second term converges to

$\lambda\nu/(\nu-1)$ by the second equality in (14). Applying the first equality in (14) to the second fraction in (13) yields a limiting upper bound of $\gamma y^{\nu-1}+\frac{\lambda\nu}{\nu-1}+\frac{KC_K\nu}{2-\nu}$. So

$$\liminf_{n\to\infty} ny\bar{m}^{(n)}(ny) \geq \left(\gamma y^{\nu-1}+\frac{\lambda\nu}{\nu-1}+\frac{KC_K\nu}{2-\nu}\right)^{-1},$$

which is bounded below by some $L>0$ for all $y\in\mathcal{K}$. □

3.3 Properties of κ

In this section, we show that $ny\bar{m}^{(n)}(ny)$ converges to $\kappa(y)$ and we describe several properties of $\kappa(y)$ for fixed $1<\nu<2$, $\lambda>0$, and $\gamma\geq 0$. We begin with several technical lemmas.

Lemma 3.5 *If* $\lim_{n\to\infty}\bar{m}^{(n)}(ny)ny = \kappa$ *for finite* κ, *and* $n^{\nu-1}\left(\frac{1-\rho^{(n)}}{l(n)}\right)\to\gamma\left(\frac{-1}{\Gamma(1-\nu)}\right)$, *we have*

$$\lim_{n\to\infty}\frac{\bar{m}^{(n)}(ny)(1-\rho^{(n)})(ny)^{\nu}}{l(ny)} = \kappa\gamma y^{\nu-1}\left(\frac{-1}{\Gamma(1-\nu)}\right).$$

Proof

$$\begin{aligned}\frac{\bar{m}^{(n)}(ny)(1-\rho^{(n)})(ny)^{\nu}}{l(ny)} &= \left(\bar{m}^{(n)}(ny)ny\right)\left(\frac{n^{\nu-1}(1-\rho^{(n)})}{l(n)}\right)\left(\frac{l(n)}{l(ny)}\right)\left(y^{\nu-1}\right)\\ &\to \kappa\gamma\left(\frac{-1}{\Gamma(1-\nu)}\right)y^{\nu-1}.\end{aligned}$$

□

We will need the following simple fact.

Lemma 3.6 *Let* $f,g:\mathbb{R}_+\to\mathbb{R}_+$ *with* $f(x)\to\infty$ *and* $g(x)\to\infty$ *as* $x\to\infty$, *and* $f(x)/g(x)\to c>0$ *as* $x\to\infty$. *Let* L *be slowly varying. Then,*

$$\lim_{x\to\infty}\frac{L(f(x))}{L(g(x))}=1.$$

Proof By Karamata's representation theorem, we have

$$\frac{L(f(x))}{L(g(x))} = \frac{\exp\left(\eta(f(x))+\int_B^{f(x)}\frac{\epsilon(t)}{t}\,\mathrm{d}t\right)}{\exp\left(\eta(g(x))+\int_B^{g(x)}\frac{\epsilon(t)}{t}\,\mathrm{d}t\right)}.$$

Taking the natural log of each side, it suffices to show

$$\eta(f(x)) - \eta(g(x)) + \int_B^{f(x)} \frac{\epsilon(t)}{t}\,\mathrm{d}t - \int_B^{g(x)} \frac{\epsilon(t)}{t}\,\mathrm{d}t \to 0,$$

as x goes to infinity. Since η is convergent and f, g go to infinity, we need only show the signed integral

$$\int_{g(x)}^{f(x)} \frac{\epsilon(t)}{t}\,\mathrm{d}t$$

converges to zero. $\epsilon(t)$ is a bounded positive function, so integrating yields

$$\left| \int_{g(x)}^{f(x)} \frac{\epsilon(t)}{t}\,\mathrm{d}t \right| \le \sup_{t\in[g(x)\wedge f(x),\, g(x)\vee f(x)]} \epsilon(t) |\ln(f(x)/g(x))|.$$

Since $g(x) \wedge f(x)$ goes to infinity as $x \to \infty$, $\epsilon(t) \to 0$ as $t \to \infty$, and $\ln(f(x)/g(x)) \to \ln(c)$ as $x \to \infty$, we have $\ln\left(\frac{L(f(x))}{L(g(x))}\right) \to 0$ as $x \to \infty$. □

Corollary 3.7 *Fix $y > 0$. If $\lim_{n\to\infty} \lambda^{(n)} = \lambda$ and $\lim_{n\to\infty} \bar{m}^{(n)}(ny)ny = \kappa$ for $0 < \kappa < \infty$, we have*

$$\lim_{n\to\infty} \left(\frac{l\left(\frac{1}{\lambda^{(n)}\bar{m}^{(n)}(ny)}\right)}{l(ny)} \right) = 1.$$

Recall that T_ν is a Pareto ν random variable if

$$\mathbb{P}\{T_\nu > x\} = \begin{cases} x^{-\nu}, & \text{if } x \ge 1, \\ 1, & \text{if } x < 1. \end{cases}$$

Clearly T_ν is regularly varying with parameter ν.

Proposition 3.8 *Fix $y > 0$. Then, if $\lambda^{(n)} \to \lambda$, $ny\bar{m}^{(n)}(ny) \to \kappa > 0$, and V is regularly varying with parameter ν, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V} \,\middle|\, V > ny \right] = \mathbb{E}\left[e^{-\lambda\kappa T_\nu} \right].$$

Proof Observe that

$$\mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V} \,\middle|\, V > ny \right] = \int_0^\infty e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)t} 1_{(ny,\infty)}(t) \frac{\mathrm{d}F(t)}{1 - F(ny)}.$$

Now substitute $u = \bar{m}^{(n)}(ny)t$ to obtain,

$$\mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V} \,\middle|\, V > ny \right] = \int_0^\infty e^{-\lambda^{(n)}u} 1_{(ny\bar{m}^{(n)}(ny),\infty)}(u) \frac{F(\mathrm{d}u/\bar{m}^{(n)}(ny))}{1 - F(ny)}.$$

Thus, we have

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V}\,\middle|\, V > ny\right] = \lim_{n\to\infty}\int_0^\infty e^{-\lambda^{(n)}u} 1_{(\kappa,\infty)}(u)\, \frac{F(\mathrm{d}u/\bar{m}^{(n)}(ny))}{1-F(ny)},$$

because $e^{-\lambda^{(n)}u} \le 1$, and

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^\infty \left(1_{(\kappa,\infty)}(u) - 1_{(ny\bar{m}^{(n)}(ny),\infty)}(u)\right) \frac{F\left(\frac{\mathrm{d}u}{\bar{m}^{(n)}(ny)}\right)}{1-F(ny)}\\
&= \lim_{n\to\infty}\left(\frac{1-F\left(\frac{\kappa}{\bar{m}^{(n)}(ny)}\right)}{1-F(ny)} - \frac{1-F\left(\frac{ny\bar{m}^{(n)}(ny)}{\bar{m}^{(n)}(ny)}\right)}{1-F(ny)}\right)\\
&= \lim_{n\to\infty}\left(\frac{1-F\left(\frac{\kappa ny}{ny\bar{m}^{(n)}(ny)}\right)}{1-F(ny)} - \frac{1-F\,(ny)}{1-F(ny)}\right)\\
&= 0,
\end{aligned}$$

by Lemma 3.6 since $1-F$ is regularly varying with parameter $-\nu$.

The measure $\frac{F(\mathrm{d}u/\bar{m}^{(n)}(ny))}{1-F(ny)}$ converges weakly to the measure $(\mathrm{d}u/\kappa)^{-\nu}$ as $n\to\infty$, since, for all $0 \le a < b$, as in the previous display,

$$\begin{aligned}
&\lim_{n\to\infty}\int 1_{(a,b]}\frac{F(\mathrm{d}u/\bar{m}^{(n)}(ny))}{1-F(ny)}\\
&= \lim_{n\to\infty}\frac{F(b/\bar{m}^{(n)}(ny))}{1-F(ny)} - \frac{F(a/\bar{m}^{(n)}(ny))}{1-F(ny)}\\
&= \left(\frac{a}{\kappa}\right)^{-\nu} - \left(\frac{b}{\kappa}\right)^{-\nu}.
\end{aligned}$$

For all $\epsilon > 0$, there exists N such that $n > N$ implies $|e^{-\lambda^{(n)}u} - e^{-\lambda u}| < \epsilon$, uniformly in u. Combining with the above weak convergence,

$$\lim_{n\to\infty}\int_0^\infty \left|e^{-\lambda^{(n)}u} - e^{-\lambda u}\right| 1_{(\kappa,\infty)}(u)\,\frac{F(\mathrm{d}u/\bar{m}^{(n)}(ny))}{1-F(ny)} = 0.$$

So, we have

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V}\,\middle|\, V > ny\right] = \lim_{n\to\infty}\int_0^\infty e^{-\lambda u} 1_{(\kappa,\infty)}(u)\, \frac{F(\mathrm{d}u/\bar{m}^{(n)}(ny))}{1-F(ny)}.$$

Using weak convergence again and the fact that the limit measure has no atoms gives

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\lambda^{(n)}\bar{m}^{(n)}(ny)V}\,\middle|\, V > ny\right] = \kappa^\nu \int_\kappa^\infty e^{-\lambda t}\,(\mathrm{d}t)^{-\nu}.$$

Finally, substitute $t = x\kappa$ to get

$$\kappa^{\nu}\int_{\kappa}^{\infty} e^{-\lambda t}\,(\mathrm{d}t)^{-\nu} = \int_{1}^{\infty} e^{-\lambda\kappa x}\,(\mathrm{d}x)^{-\nu} = \mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right]. \qquad \square$$

The equation that describes $\kappa(y)$ is (7), as shown in the following lemma.

Lemma 3.9 *Let T_ν be Pareto ν, $1 < \nu < 2$, $\gamma \geq 0$ and $\lambda > 0$. The equation in the variable $\kappa > 0$*

$$\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right] - \kappa\gamma y^{\nu-1}\left(\frac{-1}{\Gamma(1-\nu)}\right) = (\lambda\kappa)^{\nu}$$

has exactly one solution for all $y > 0$.

Proof The left-hand side is a strictly decreasing continuous function in κ and the right-hand side is a strictly increasing continuous function in κ. When $\kappa = 0$ the left-hand side is $\frac{-1}{\Gamma(1-\nu)} > 0$ and the right-hand side is 0. As $\kappa \to \infty$, the left-hand side goes to 0 if $\gamma = 0$ and $-\infty$ if $\gamma > 0$; the right-hand side goes to infinity. Thus, (7) has exactly one solution. $\square$

We are finally in a position to prove Theorem 2.1.

Proof of Theorem 2.1 Let $\tilde{\kappa}$ be a limit point of $ny\bar{m}^{(n)}(ny)$. Then, $0 < \tilde{\kappa} < \infty$ by Lemmas 3.4 and 3.3. Let n_r be a subsequence such that $\lim_{r\to\infty} n_r y\bar{m}^{(n_r)}(n_r y) = \tilde{\kappa}$. By Lemma 3.2 we have

$$\lim_{r\to\infty}\frac{\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda^{(n_r)}\bar{m}^{(n_r)}(n_r y)V}\,\middle|\,V > n_r y\right] - \frac{\bar{m}^{(n_r)}(n_r y)(1-\rho^{(n_r)})(n_r y)^{\nu}}{l(n_r y)}}{\left(\lambda^{(n_r)} n_r y\bar{m}^{(n_r)}(n_r y)\right)^{\nu}\left(\frac{l\left(\frac{1}{\lambda^{(n_r)}\bar{m}^{(n_r)}(n_r y)}\right)}{l(n_r y)}\right)} = 1. \tag{16}$$

Lemmas 3.8, 3.5, and 3.7 reduce Eq. (16) to

$$\frac{\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\tilde{\kappa}T_\nu}\right] - \tilde{\kappa}\gamma y^{\nu-1}\left(\frac{-1}{\Gamma(1-\nu)}\right)}{(\lambda\tilde{\kappa})^{\nu}} = 1.$$

Thus, any limit point of $ny\bar{m}^{(n)}(ny)$ satisfies Eq. (7), of which the solution is called $\kappa(y)$, so Lemma 3.9 implies the limit point is unique, so $\lim_{n\to\infty} ny\bar{m}^{(n)}(ny) = \kappa$. $\square$

The properties of $\kappa(y)$ as a function of y are established below.

3.4 Properties of $\kappa(y)$

In this section, we describe several properties of the function κ. In particular, $\kappa(y)$ is uniformly bounded above and regularly varying with parameter $1-\nu$. First we need asymptotic properties of an inverse function.

Lemma 3.10 *Suppose* $G : (0,\infty) \to (0,\infty)$ *is nonincreasing, invertible,* $\lim_{t\downarrow 0} G(t) = \infty$*, and G is regularly varying at zero with parameter* $-\alpha$ *for* $0 \le \alpha \le \infty$*. Then,* G^{-1} *is regularly varying at infinity with parameter* $-1/\alpha$*.*

Proof Define $h : (0,\infty) \to (0,\infty)$ by $h(t) = 1/t$. We have that $G \circ h$ is regularly varying at infinity with parameter α, $\lim_{t\to\infty} G \circ h(t) = \infty$, and $G \circ h$ is nondecreasing. Thus, Proposition 0.8 in [11] gives that $(G \circ h)^{-1} = h \circ G^{-1}$ is regularly varying at infinity with parameter $1/\alpha$. Since the parameter of a composition of regularly varying functions at infinity is the product of the parameters, and h is regularly varying at infinity with parameter -1, we have that $h \circ h \circ G^{-1} = G^{-1}$ is regularly varying with parameter $-1/\alpha$. □

Lemma 3.11 *For fixed* (λ, γ, ν)*,* $\kappa(y)$ *defined implicitly by Eq.* (7) *is continuous and regularly varying with parameter* $1-\nu$ *if* $\gamma > 0$ *and* $\kappa(y)$ *is constant if* $\gamma = 0$*. Moreover,* $\kappa(y) \le \frac{1}{\lambda}\left(\frac{-1}{\Gamma(1-\nu)}\right)^{1/\nu}$*.*

Proof If $\gamma = 0$, then κ satisfies $\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right] = (\lambda\kappa)^\nu$, so κ is constant. If $\gamma > 0$, then κ satisfies

$$\left(\frac{\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right] - (\lambda\kappa)^\nu}{\kappa\gamma\left(\frac{-1}{\Gamma(1-\nu)}\right)}\right)^{1/(\nu-1)} = y.$$

Since

$$\begin{aligned} &\kappa \mapsto \left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right] \text{ is strictly decreasing,} \\ &\kappa \mapsto -(\lambda\kappa)^\nu \text{ is strictly decreasing and,} \\ &\kappa \mapsto \kappa\gamma\left(\frac{-1}{\Gamma(1-\nu)}\right) \text{ is strictly increasing,} \end{aligned}$$

and each of these functions is continuous, we see that the inverse function $\kappa \mapsto y(\kappa)$ is strictly decreasing and continuous. So, $y \mapsto \kappa(y)$ is continuous.

The moment generating function of T_ν is continuous at zero, so

$$\kappa \mapsto \left(\frac{\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right] - (\lambda\kappa)^\nu}{\gamma\left(\frac{-1}{\Gamma(1-\nu)}\right)}\right)^{1/(\nu-1)}$$

is a slowly varying function at zero. Thus, $y(\kappa)$ is regularly varying at zero with parameter $-1/(\nu-1)$. So, by Lemma 3.10 we have that $\kappa(y)$ is regularly varying at infinity with parameter $1-\nu$.

From Eq. (7) we have

$$\begin{aligned}\kappa &= \frac{1}{\lambda}\left(\left(\frac{-1}{\Gamma(1-\nu)}\right)\mathbb{E}\left[e^{-\lambda\kappa T_\nu}\right]-\kappa\gamma y^{\nu-1}\left(\frac{-1}{\Gamma(1-\nu)}\right)\right)^{1/\nu}\\ &\le \frac{1}{\lambda}\left(\frac{-1}{\Gamma(1-\nu)}\right)^{1/\nu}.\end{aligned}$$ □

The following corollary follows from the monotonicity of $\bar{m}$ and replacing ny in the proof of Theorem 2.1 with $ny+b$. Uniform convergence follows from pointwise convergence and, for each n, $n\bar{m}^{(n)}(ny+b)$ is nonincreasing and converging to zero as y goes to infinity, while $\frac{\kappa(y)}{y}$ is continuous and converges to zero. This property also follows from the fact that $\kappa(y)$ is regularly varying, and is necessary in the next section.

Corollary 3.12 *Under the assumptions of Lemma 3.2, let b be a real number. Then,*

$$\lim_{n\to\infty} n\bar{m}^{(n)}(ny+b)=\frac{\kappa(y)}{y}.$$

Moreover, the convergence is uniform on intervals bounded away from zero.

4 Convergence of the one-dimensional distributions

In this section, we first write the waiting time in the second queue in terms of independent random variables. Here we are using the fact that for the *M*/*G*/1 queue the length of an idle period is independent of the service times in the preceding busy period.

Recall that M_k is the largest service time the in the kth busy period in the first queue and I_k is the duration of the idle period in the first queue between the kth and $(k+1)$st busy period.

Proposition 4.1 *For each $n\ge 1$,*

$$R_n=\max_{k=1}^{n}\left(M_k-\sum_{j=k}^{n-1} I_j\right),$$

Proof This follows by induction in (1). □

We now turn to the distribution of $R^{(n)}_{[nt]}$. We first investigate what happens if we replace idle periods by their mean, and then show that we can indeed make such a simplification.

The proof of the following preliminary result is a standard application of weak convergence by considering the sequence of measures $\phi_n=\sum_{k=1}^{[nt]}\frac{1}{n}\delta_{k/n}$.

Proposition 4.2 *If* $nf(n, ny) \to g(y)$ *uniformly on* $[0, t]$, *and* g *is continuous on* $[0, t]$, *then*

$$\lim_{n\to\infty} \sum_{k=1}^{[nt]} f(n, k) = \int_0^t g(y)\, \mathrm{d}y.$$

Proposition 4.3 *Under the assumptions of Lemma 3.2, for fixed* $t \geq 0$ *and* $x > 0$ *we have*

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{n} \max_{k=1}^{[nt]} \left(M_k^{(n)} - \frac{k-1}{\lambda}\right) \leq x\right\}$$
$$= \begin{cases} \left(1 + \frac{t}{x\lambda}\right)^{-\lambda\kappa}, & \text{if } \gamma = 0, \\ \exp\left\{-\lambda \int_x^{x+t/\lambda} \kappa(y)/y \,\mathrm{d}y\right\}, & \text{if } \gamma > 0. \end{cases}$$

Proof Since $\{M_k\}$ are independent random variables,

$$\begin{aligned}
\mathbb{P}\left\{\frac{1}{n} \max_{k=1}^{[nt]} \left(M_k^{(n)} - \frac{k-1}{\lambda}\right) \leq x\right\} &= \mathbb{P}\left\{\max_{k=1}^{[nt]} \left(M_k^{(n)} - \frac{k-1}{\lambda}\right) \leq nx\right\} \\
&= \prod_{k=1}^{[nt]} \mathbb{P}\left\{M_k^{(n)} - \frac{k-1}{\lambda} \leq nx\right\} \\
&= \prod_{k=1}^{[nt]} \mathbb{P}\left\{M_k^{(n)} \leq nx + \frac{k-1}{\lambda}\right\} \\
&= \prod_{k=1}^{[nt]} m^{(n)}\left(nx + \frac{k-1}{\lambda}\right) \\
&= \exp\left\{\sum_{k=1}^{[nt]} \ln\left(m^{(n)}\left(nx + \frac{k-1}{\lambda}\right)\right)\right\}.
\end{aligned}$$

Let $f(n, k) = \ln\left(m^{(n)}\left(nx + \frac{k-1}{\lambda}\right)\right)$. Then, for fixed $y > 0$,

$$\begin{aligned}
nf(n, ny) &= n \ln\left(m^{(n)}\left(nx + \frac{ny-1}{\lambda}\right)\right) \\
&= \ln\left(\left(1 - \frac{n\bar{m}^{(n)}\left(nx + ny/\lambda - \frac{1}{\lambda}\right)}{n}\right)^n\right).
\end{aligned}$$

The function $\ln((1 - z/n)^n) \to -z$ uniformly on compact intervals as $n \to \infty$ and $n\bar{m}^{(n)}\left(n(x + y/\lambda) - \frac{1}{\lambda}\right) \to \kappa(x+y/\lambda)/(x+y/\lambda)$ uniformly on $y \in [0, t]$ as $n \to \infty$, by Corollary 3.12, because $x > 0$. Thus, $nf(n, ny) \to -\kappa(x + y/\lambda)/(x + y/\lambda)$, uniformly for $y \in [0, t]$ since, for each n, $m^{(n)}$ is nondecreasing and the limit is continuous. Now, continuity of the exponential function and Proposition 4.2 gives

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{n}\max_{k=1}^{[nt]}\left(M_k^{(n)} - \frac{k-1}{\lambda}\right) \le x\right\} = \exp\left\{-\int_0^t \frac{\kappa(x+y/\lambda)}{x+y/\lambda}\mathrm{d}y\right\}$$
$$= \exp\left\{-\lambda\int_x^{x+t/\lambda} \kappa(y)/y\,\mathrm{d}y\right\}.$$

Note that the above proof holds for all $\gamma \ge 0$; the case $\gamma = 0$ is just a rewriting of the previous expression since κ is constant. □

The sequence of idle periods is i.i.d. exponential $\lambda^{(n)}$ in the nth system. Since the largest job in a busy period is independent of the idle period that follows, it is convenient to reindex the sequence of idle periods. This is why we write $\sum_{i=1}^{k-1} I_i^{(n)}$ instead of $\sum_{i=k}^{n-1} I_i^{(n)}$ in the following proposition. This proposition shows that, due to monotonicity of the maximum and Kolmogorov's theorem, we can replace $\sum_{i=1}^{k-1} I_i^{(n)}$ by the limit of its mean.

Proposition 4.4 *For all $x > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{n}\max_{k=1}^{n}\left(M_k^{(n)} - \sum_{i=1}^{k-1} I_i^{(n)}\right) \le x\right\}$$
$$= \lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{n}\max_{k=1}^{n}\left(M_k^{(n)} - \frac{k-1}{\lambda}\right) \le x\right\}. \tag{17}$$

Proof The right-hand side of (17) converges by Proposition 4.3. Using the inequality $-\max(|b_k|) \le \max(a_k+b_k) - \max(a_k) \le \max(b_k) \le \max(|b_k|)$, which implies that $|\max(a_k+b_k) - \max(a_k)| \le \max(|b_k|)$ for real numbers a_k and b_k, we have

$$\left|\frac{1}{n}\max_{k=1}^{n}\left(M_k^{(n)} - \sum_{i=1}^{k-1} I_i^{(n)}\right) - \frac{1}{n}\max_{k=1}^{n}\left(M_k^{(n)} - \frac{k-1}{\lambda}\right)\right|$$
$$= \left|\frac{1}{n}\max_{k=1}^{n}\left(M_k^{(n)} - \frac{k-1}{\lambda} + \frac{k-1}{\lambda} - \frac{k-1}{\lambda^{(n)}} + \frac{k-1}{\lambda^{(n)}} - \sum_{i=1}^{k-1} I_i^{(n)}\right)\right.$$
$$\left.-\frac{1}{n}\max_{k=1}^{n}\left(M_k^{(n)} - \frac{k-1}{\lambda}\right)\right|$$
$$\le \frac{1}{n}\max_{k=1}^{n}\left|\frac{k-1}{\lambda} - \frac{k-1}{\lambda^{(n)}} + \frac{k-1}{\lambda^{(n)}} - \sum_{i=1}^{k-1} I_i^{(n)}\right|$$
$$\le \frac{(n-1)|\lambda - \lambda^{(n)}|}{n\lambda\lambda^{(n)}} + \frac{1}{n}\max_{k=1}^{n}\left(\left|\frac{k-1}{\lambda^{(n)}} - \sum_{i=1}^{k-1} I_i^{(n)}\right|\right).$$

so, it suffices to show

$$\frac{1}{n}\max_{k=1}^{n}\left(\left|\sum_{i=1}^{k-1}\left(I_i^{(n)}-\frac{1}{\lambda^{(n)}}\right)\right|\right)\to 0$$

in probability as $n\to\infty$. This follows from Kolmogorov's maximal inequality. For each $\epsilon>0$,

$$\begin{aligned}
&\mathbb{P}\left\{\frac{1}{n}\max_{k=1}^{n}\left(\left|\sum_{i=1}^{k-1}\left(I_i^{(n)}-\frac{1}{\lambda^{(n)}}\right)\right|\right)\geq\epsilon\right\}\\
&=\mathbb{P}\left\{\frac{1}{n}\max_{k=1}^{n-1}\left(\left|\sum_{i=1}^{k}\left(I_i^{(n)}-\frac{1}{\lambda^{(n)}}\right)\right|\right)\geq\epsilon\right\}\\
&=\mathbb{P}\left\{\max_{k=1}^{n-1}\left(\left|\sum_{i=1}^{k}\left(I_i^{(n)}-\frac{1}{\lambda^{(n)}}\right)\right|\right)\geq n\epsilon\right\}\\
&\leq\frac{1}{(n\epsilon)^2}\frac{n-1}{(\lambda^{(n)})^2}\to 0
\end{aligned}$$

as $n\to\infty$, because $\lambda^{(n)}\to\lambda>0$. □

Finally, we prove the main result for this section, which implies Theorem 2.2. Recall that we have assumed $\rho^{(n)}=\lambda^{(n)}\mathbb{E}[V]$ and $\frac{1-\rho^{(n)}}{n(1-F(n))}\to\gamma\geq 0$ as $n\to\infty$, and that $1-F(t)=\left(\frac{-1}{\Gamma(1-\nu)}\right)t^{-\nu}l(t)$ for $1<\nu<2$ and l a slowly varying function. Recall that $\lambda=\mathbb{E}[V]^{-1}$ and let $\kappa(y)$ be such that the parameters $(\kappa,\lambda,\nu,\gamma,y)$ satisfy (7).

Theorem 4.5 *For fixed $t\geq 0$ and $x>0$ we have*

$$\lim_{n\to\infty}\mathbb{P}\left\{\frac{1}{n}R_{[nt]}^{(n)}\leq x\right\}=\begin{cases}\left(1+\frac{t}{x\lambda}\right)^{-\lambda\kappa}, & \text{if } \gamma=0,\\ \exp\left\{-\lambda\int_x^{x+t/\lambda}\kappa(y)/y\,dy\right\}, & \text{if } \gamma>0.\end{cases}$$

Proof By Proposition 4.1

$$\mathbb{P}\left\{\frac{1}{n}R_{[nt]}^{(n)}\leq x\right\}=\mathbb{P}\left\{\frac{1}{n}\max_{k=1}^{[nt]}\left(M_k^{(n)}-\sum_{j=k}^{[nt]-1}I_j^{(n)}\right)\leq x\right\}.$$

For each n, the i.i.d. collections $\{I_k^{(n)}\}$ and $\{M_k^{(n)}\}$ are independent, so

$$\max_{k=1}^{[nt]}\left(M_k^{(n)}-\sum_{j=k}^{[nt]-1}I_j^{(n)}\right)\sim\max_{k=1}^{[nt]}\left(M_k^{(n)}-\sum_{j=1}^{k-1}I_j^{(n)}\right).$$

By Proposition 4.4,

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{n} R^{(n)}_{[nt]} \leq x\right\} = \lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{n} \max_{k=1}^{[nt]} \left(M^{(n)}_k - (k-1)/\lambda\right) \leq x\right\},$$

and so by Proposition 4.3,

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{n} R^{(n)}_{[nt]} \leq x\right\} = \begin{cases} \left(1 + \frac{t}{x\lambda}\right)^{-\lambda\kappa}, & \text{if } \gamma = 0, \\ \exp\left\{-\lambda \int_x^{x+t/\lambda} \kappa(y)/y \, \mathrm{d}y\right\}, & \text{if } \gamma > 0. \end{cases} \tag{18}$$

□

Definition 4.6 Let $\Phi(t, x)$ be the right-hand side of (18) for $t \geq 0$ and $x > 0$, $\Phi(t, x) = 0$ for $t \geq 0$ and $x < 0$, $\Phi(t, 0) = 0$ for $t > 0$, and $\Phi(0, 0) = 1$.

Note that $x \mapsto \Phi(t, x)$ is a distribution function for each $t \geq 0$. To see this, recall that $\kappa(y)$ is constant when $\gamma = 0$ so we may write $\Phi(t, x) = \exp\left\{-\lambda \int_x^{x+t/\lambda} \kappa(y)/y \, \mathrm{d}y\right\}$ for all $\gamma \geq 0$. By the proof of Lemma 3.4 we have

$$\kappa(y) \geq \left(\gamma + \frac{\lambda\nu}{\nu - 1} + \frac{K C_K \nu}{2 - \nu}\right)^{-1}$$

for $0 \leq y \leq 1$, so $\int_x^{x+t/\lambda} \kappa(y)/y \, \mathrm{d}y$ goes to ∞ as x goes to zero. Thus, for $t > 0$ we have $\Phi(t, x) \downarrow 0$ as $x \downarrow 0$. Since κ is bounded, for each $t > 0$ we have $\Phi(t, x) \to 1$ as $x \to \infty$. For $t > 0$, $\Phi(t, \cdot)$ is strictly increasing because $\kappa(y)/y$ is strictly decreasing.

5 Process level convergence

To prove Theorem 2.3, we begin with a description of the limit process.

Definition 5.1 Let Z_t be the random variable with distribution function $\Phi(t, x)$, and observe that $Z_0 = 0$.

Then, for fixed t, $\frac{1}{n} R^{(n)}_{[nt]}$ converges in distribution to Z_t. Note that we have $Z_t \Rightarrow 0$ as $t \downarrow 0$ since $Z_t \geq 0$ a.s. and, for any $x > 0$, $\lim_{t\to 0} \mathbb{P}\{Z_t > x\} = 0$, for all $\gamma \geq 0$, so $Z_t \to 0$ in probability.

Now we observe a property of Z_t analogous to $N(0, s) + N(0, t) \sim N(0, s + t)$ for $N(0, s)$ and $N(0, t)$ independent normal random variables with mean zero and variances s and t.

Lemma 5.2 *Let Z_t, Z_s, and Z_{s+t} be independent random variables with distribution given by Eq.* (18). *Then*

$$\mathbb{P}\{\max[Z_t - s/\lambda, Z_s] \leq x\} = \mathbb{P}\{Z_{s+t} \leq x\}. \tag{19}$$

Proof Compute

$$\begin{aligned}
&\mathbb{P}\{\max[Z_t - s/\lambda, Z_s] \le x\} \\
&\quad = \mathbb{P}\{Z_t \le x + s/\lambda\}\,\mathbb{P}\{Z_s \le x\} \\
&\quad = \exp\left\{-\lambda \int_{x+s/\lambda}^{x+s/\lambda+t/\lambda} \kappa(y)/y\,\mathrm{d}y\right\} \exp\left\{-\lambda \int_{x}^{x+s/\lambda} \kappa(y)/y\,\mathrm{d}y\right\} \\
&\quad = \exp\left\{-\lambda \int_{x}^{x+(s+t)/\lambda} \kappa(y)/y\,\mathrm{d}y\right\} \\
&\quad = \mathbb{P}\{Z_{s+t} \le x\}\,.
\end{aligned}$$

□

We are now ready to describe the generator of our limit process.

Definition 5.3 Let $\hat{C}([0,\infty))$ be the space of continuous functionals on $[0,\infty)$ that converge to zero at infinity. We give this space the topology of uniform convergence, induced by the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$ (see [7] p. 164). The subspace $\hat{C}_c^1([0,\infty))$ contains the continuously differentiable functionals with compact support, and we define

$$D = \{f \in \hat{C}_c^1([0,\infty)) : \text{for some } a > 0,\ |f'(x)| \le ax \text{ for all } x\}.$$

For each $t \ge 0$ and $f \in \hat{C}([0,\infty))$, define the operator

$$T(t)f(x) = \mathbb{E}\left[f(\max(x - t/\lambda, Z_t))\right]. \tag{20}$$

Note that $T(0) = I$.

A strongly continuous, positive, contraction semigroup on $\hat{C}([0,\infty))$ whose generator is conservative is called a *Feller semigroup*.

Lemma 5.4 *$\{T(t)\}$ defines a Feller semigroup on $\hat{C}([0,\infty))$ with generator A, which is an extension to $\hat{C}([0,\infty))$ of $\hat{A}$ given by*

$$\hat{A}f(x) = \frac{-f'(x)}{\lambda} + \int_x^\infty f'(y)\frac{\kappa(y)}{y}\,\mathrm{d}y, \quad f \in D. \tag{21}$$

Moreover, D is a core for A.

Proof For each $t \ge 0$ and $f \in \hat{C}([0,\infty))$, $T(t)f$ is continuous by bounded convergence. Since, for each $\epsilon > 0$, there exists N large enough for $x \ge N$ to imply $|f(x)| < \epsilon$, $x \ge N + t/\lambda$ implies $\max(x - t/\lambda, Z_t) \ge x - t/\lambda$ and therefore $|T(t)f(x)| < \epsilon$. Thus, $T(t) : \hat{C}([0,\infty)) \to \hat{C}([0,\infty))$. Clearly $T(t)$ is also positive, linear, and contractive on $\hat{C}([0,\infty))$.

For the semigroup property, let Z_s, Z_t, and Z_{s+t} be independent, with distributions given by (18). Then, by Fubini's theorem and Lemma 5.2,

$$\begin{aligned} T(s)T(t)f(x) &= \mathbb{E}\left[T(t)f(\max(x - s/\lambda, Z_s))\right] \\ &= \mathbb{E}\left[f(\max(\max(x - t/\lambda, Z_t) - s/\lambda, Z_s))\right] \\ &= \mathbb{E}\left[f(\max(x - (s+t)/\lambda, Z_t - s/\lambda, Z_s))\right] \\ &= \mathbb{E}\left[f(\max(x - (s+t)/\lambda, Z_{s+t}))\right] \\ &= T(s+t)f(x) \end{aligned} \tag{22}$$

for all $f \in \hat{C}([0,\infty))$, $x \in [0,\infty)$, and $s, t \geq 0$. Since $T(0) = I$, this implies $\{T(t)\}$ is a semigroup.

Next we show the semigroup is strongly continuous. Note that $f \in \hat{C}([0,\infty))$ is uniformly continuous. Write

$$\begin{aligned} \|T(t)f - f\| &= \sup_{x\in[0,\infty)} \left|\mathbb{E}\left[f(\max(x - t/\lambda, Z_t)) - f(x)\right]\right| \\ &\leq \sup_{x\in[0,\infty)} \mathbb{P}\{Z_t \leq x - t/\lambda\} \left|f(x - t/\lambda) - f(x)\right| \\ &\quad + \sup_{x\in[0,\infty)} \int_{[x-t/\lambda]^+}^{\infty} |f(y) - f(x)|\, \Phi(t, \mathrm{d}y). \end{aligned} \tag{23}$$

As $t \downarrow 0$ the first term goes to zero because it is bounded by

$$\sup_{x\in[t/\lambda,\infty)} |f(x - t/\lambda) - f(x)| \to 0$$

by uniform continuity of f. For the second term, let $\epsilon > 0$ be and $\eta > 0$ be such $\sup_{x,y\in[0,\eta)} |f(x) - f(y)| < \epsilon$. Then,

$$\begin{aligned} &\sup_{x\in[0,\infty)} \int_{[x-t/\lambda]^+}^{\infty} |f(y) - f(x)|\, \Phi(t, \mathrm{d}y) \leq \sup_{x\in[\eta,\infty)} \mathbb{P}\{Z_t \geq \eta - t/\lambda\}\, 2\, \|f\| \\ &\vee \left(\sup_{x\in[0,\eta)} \sup_{y\in[\eta,\infty)} \mathbb{P}\{Z_t \geq \eta\}\, |f(y) - f(x)| \right. \\ &\left. + \sup_{x,y\in[0,\eta)} \mathbb{P}\{Z_t < \eta\}\, |f(y) - f(x)| \right). \end{aligned} \tag{24}$$

We have $\sup_{x\in[\eta,\infty)} \mathbb{P}\{Z_t \geq \eta - t/\lambda\} \|f\| \to 0$ since f is bounded, $Z_t \Rightarrow 0$, and $\eta > 0$. Similarly,

$$\sup_{x\in[0,\eta)} \sup_{y\in[\eta,\infty)} \mathbb{P}\{Z_t \geq \eta\}\, |f(y) - f(x)| \leq \mathbb{P}\{Z_t \geq \eta\}\, 2\|f\| \to 0$$

since f is bounded and $Z_t \Rightarrow 0$. Lastly, $\sup_{x,y\in[0,\eta)} \mathbb{P}\{Z_t < \eta\}\,|f(y) - f(x)| < \epsilon$ and since ϵ is arbitrary, we conclude that $\|T(t)f - f\| \to 0$ as $t \downarrow 0$. Thus, $\{T(t)\}$ is a strongly continuous semigroup. A is conservative because $(f \equiv 1, f \equiv 0)$ is in the bounded-pointwise closure of A.

It remains to show that an extension A of $\hat{A}$ is the generator of $\{T(t)\}$ and that D is a core for A. Note that by l'Hôpital's rule, for $w > 0$,

$$\begin{aligned}
\lim_{t\to 0} \frac{1 - \Phi(t, w)}{t} &= \lim_{t\to 0} -\frac{\partial}{\partial t}\Phi(t, w) \\
&= \lim_{t\to 0} \frac{\kappa(w + t/\lambda)}{w + t/\lambda}\Phi(t, w) \\
&= \kappa(w)/w,
\end{aligned}$$

since κ is continuous by Lemma 3.11. Now let $f \in D$. Then,

$$\begin{aligned}
\frac{1}{t}\left(T(t)f(x) - f(x)\right) &= \mathbb{E}\left[\frac{1}{t}\left(f(\max(x - t/\lambda, Z_t)) - f(x)\right)\right] \\
&= \mathbb{P}\{Z_t \le x - t/\lambda\}\left(\frac{f([x - t/\lambda]^+) - f(x)}{t}\right) \\
&\quad + \frac{1}{t}\int_{[x-t/\lambda]^+}^{\infty} (f(y) - f(x))\ \Phi(t, \mathrm{d}y).
\end{aligned} \tag{25}$$

As $t \to 0$, the first term converges to $-f'(x)/\lambda$ if $x > 0$ and zero if $x = 0$, since f is differentiable and $Z_t \Rightarrow 0$.

For the second term, we note the integral is finite since f is bounded and $\Phi(t, \mathrm{d}y)$ is a probability measure, so Fubini's theorem yields

$$\begin{aligned}
&\frac{1}{t}\int_{[x-t/\lambda]^+}^{\infty} (f(y) - f(x))\,\Phi(t, \mathrm{d}y) \\
&= \frac{1}{t}\int_{[x-t/\lambda]^+}^{\infty}\int_x^y f'(w)\,\mathrm{d}w\,\Phi(t, \mathrm{d}y) \\
&= \frac{1}{t}\int_{[x-t/\lambda]^+}^{x}\int_{[x-t/\lambda]^+}^{w} -f'(w)\,\Phi(t, \mathrm{d}y)\mathrm{d}w \\
&\quad + \frac{1}{t}\int_x^{\infty}\int_w^{\infty} f'(w)\,\Phi(t, \mathrm{d}y)\mathrm{d}w \\
&= \frac{1}{t}\int_{[x-t/\lambda]^+}^{x} -f'(w)\left(\Phi(t, w) - \Phi(t, [x - t/\lambda]^+)\right)\mathrm{d}w \\
&\quad + \int_x^{\infty} f'(w)\left(\frac{1 - \Phi(t, w)}{t}\right)\mathrm{d}w.
\end{aligned} \tag{26}$$

The first integral converges to zero because f' is bounded and Φ is continuous; the region shrinks to zero at a rate proportional to t. For $x > 0$, the second integral converges to $\int_x^\infty f'(w)\kappa(w)/w\,\mathrm{d}w$ because $\left(\frac{1-\Phi(t,w)}{t}\right) \to \kappa(w)/w$ uniformly, since $\Phi(t,\cdot)$ is increasing and continuous for each t. If $x = 0$, then for any $\delta > 0$ we split the integral into the part over $[\delta,\infty]$, on which we may use the above uniform convergence, and must then show that

$$\int_0^\delta f'(w)\frac{1-\Phi(t,w)}{t}\mathrm{d}w \to 0, \quad \text{as} \quad \delta \to 0, \tag{27}$$

uniformly in t. To that end, we use the bound $1-\exp\{-\lambda z\} \le \lambda z$ to write

$$\begin{aligned} 1-\Phi(t,w) &= 1-\exp\left(-\lambda\int_w^{w+t/\lambda}\frac{\kappa(y)}{y}\mathrm{d}y\right)\\ &\le \lambda\frac{\|\kappa\|_\infty t}{\lambda w}. \end{aligned}$$

Combining with the bound $|f'(w)| \le aw$, the integral in (27) is bounded above by $\delta a\|\kappa\|_\infty$, achieving the desired result.

We have shown the generator of $\{T(t)\}$ extends $\hat{A}$. To show D is a core, note first that, for all $t \ge 0$ and $x > 0$,

$$\frac{\partial}{\partial x}\Phi(t,x) = -\lambda\left(\frac{\kappa(x+t/\lambda)}{x+t/\lambda} - \frac{\kappa(x)}{x}\right)\Phi(t,x), \tag{28}$$

so Z_t is a continuous random variable. Let $f \in D$. Clearly $T(t)f$ has compact support since $x - t/\lambda \vee Z_t \ge x - t/\lambda$. Since

$$\left|\frac{f(\max(x+h-t/\lambda,y)) - f(\max(x-t/\lambda,y))}{h}\right| \le \|f'\|_\infty,$$

and

$$\begin{aligned} &\frac{f(\max(x+h-t/\lambda,y)) - f(\max(x-t/\lambda,y))}{h}\\ &\to \begin{cases} f'(x-t/\lambda), & \text{if } y \le x-t/\lambda,\\ 0, & \text{if } y > x-t/\lambda, \end{cases} \end{aligned}$$

as $h \to 0$, and since $\Phi(t,\cdot)$ is a continuous distribution function, the dominated convergence theorem gives that $\frac{\partial}{\partial x}T(t)f(x)$ exists and is continuous. In particular, $\frac{\partial}{\partial x}T(t)f(x) \le \int_0^{[x-t/\lambda]^+}\|f'\|_\infty\|\frac{\partial}{\partial x}\Phi(t,y)\|_\infty\,\mathrm{d}y$, and so the derivative of $T(t)f$ is bounded by a linear function. Thus, $T(t)f \in D$ and [7] Proposition 1.3.3 implies that D is a core for A. □

We are ready to prove Theorem 2.3. Recall that, for each $n \ge 1$, $\{Y_n(k), k = 0,1,2,\ldots\}$ is the Markov chain in $[0,\infty)$ with transition function $\mu_n(x,\Gamma) = \mathbb{P}\left\{\max(x - I^{(n)}/n, M^{(n)}/n) \in \Gamma\right\}$, where $I^{(n)}$ is an exponential random variable with parameter $\lambda^{(n)}$ independent of the random variable $M^{(n)}$, which itself is the largest job in a busy period with service times equal in distribution to V and interarrival times equal in distribution to $I^{(n)}$. Recall that $Y_n(k)$ is equal in distribution to $\frac{1}{n}R_k^{(n)}$, and that we define $X_n(t) = Y_n([nt])$.

Let $T_n f(x) = \int f(y)\mu_n(x, \mathrm{d}y)$ and let $A_n = n(T_n - I)$. The proof will use the following formula for iterates of T_n.

Lemma 5.5 *For all $n \geq 1$, $t \geq 0$ and $x \geq 0$,*

$$T_n^{[nt]} f(x) = \mathbb{E}\left[f\left(\max\left(x - \frac{1}{n}\sum_{j=1}^{[nt]} I_j^{(n)}, \frac{1}{n} \max_{k=1}^{[nt]} \left(M_k^{(n)} - \sum_{j=k}^{[nt]-1} I_j^{(n)} \right)\right)\right)\right].$$

Proof If $[nt] = 2$, then

$$\begin{aligned}
T_n^{[nt]} f(x) &= T_n(T_n f)(x) \\
&= \int T_n f(y)\, \mu_n(x, \mathrm{d}y) \\
&= \int\int f(z)\, \mu_n(y, \mathrm{d}z)\, \mu_n(x, \mathrm{d}y) \\
&= \mathbb{E}\left[\int f(z)\, \mu_n(\max(x - I_2^{(n)}/n, M_2^{(n)}/n), \mathrm{d}z)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[f\left(\max\left(\max\left(x - I_2^{(n)}/n, M_2^{(n)}/n\right) - I_1^{(n)}/n, M_1^{(n)}\right)\right)\right]\right] \\
&= \mathbb{E}\left[f\left(\max\left(\max\left(x - I_2^{(n)}/n, M_2^{(n)}/n\right) - I_1^{(n)}/n, M_1^{(n)}\right)\right)\right],
\end{aligned}$$

where the iterated integral becomes the expectation of independent random variables $I_1^{(n)}, I_2^{(n)}, M_1^{(n)}, M_2^{(n)}$. Then,

$$\begin{aligned}
T_n^{[nt]} f(x) &= \mathbb{E}\left[f\left(\max\left(x - I_1^{(n)}/n - I_2^{(n)}/n, M_1^{(n)}/n - I_1^{(n)}/n, M_2^{(n)}/n\right)\right)\right] \\
&= \mathbb{E}\left[f\left(\max\left(x - \frac{1}{n}\sum_{j=1}^{2} I_j^{(n)}, \frac{1}{n} \max_{k=1}^{2} \left(M_k^{(n)} - \sum_{j=k}^{1} I_j^{(n)} \right)\right)\right)\right].
\end{aligned}$$

The general case follows by induction. □

Proof of Theorem 2.3 In order to conclude that $X_n \Rightarrow X$ we use [7], Chapter 4, Theorem 2.6. We have already shown that $T_n : \hat{C}([0, \infty)) \to \hat{C}([0, \infty))$ has the correct form and, by Lemma 5.4, $\{T(t)\}$ is a Feller semigroup with the stated generator. So by Theorem 2.6 in Ch. 4 of [7], it remains to show that for each fixed $f \in \hat{C}([0, \infty))$ and $t \geq 0$, we have the convergence $\lim_{n\to\infty} T_n^{[nt]} f = T(t) f$ in the space $\hat{C}([0, \infty))$, which is topologized by the uniform norm. That is, we must show uniform convergence on $[0, \infty)$ of $T_n^{[nt]} f$ to $T(t) f$.

To show this, we show $A_n f \to Af$ for each $f \in D$, which, by [7] Theorem 1.6.5, gives $T_n^{[nt]} f \to T(t) f$ uniformly on compact sets for each $f \in \hat{C}([0, \infty))$. Then, to upgrade to uniform convergence we use the fact that $T_n^{[nt]}f$ (and $T(t)f$) are uniformly small for large x; in particular, by Lemma 5.5,

$$\begin{aligned}
T_n^{[nt]}f(x) &= \mathbb{E}\left[f\left(\max\left(x-\frac{1}{n}\sum_{j=1}^{[nt]}I_j^{(n)}, \frac{1}{n}\max_{k=1}^{[nt]}\left(M_k^{(n)}-\sum_{j=k}^{[nt]-1}I_j^{(n)}\right)\right)\right)\right]\\
&= \mathbb{E}\left[f\left(\max\left(x-\frac{1}{n}\sum_{j=1}^{[nt]}I_j^{(n)}, \frac{1}{n}R_{[nt]}^{(n)}\right)\right)\right]\\
&\le \sup_{z\in[[x-y]^+,\infty)}|f(z)| + \|f\|_\infty \mathbb{P}\left\{\frac{1}{n}\sum_{j=1}^{[nt]}I_j^{(n)} > y\right\},
\end{aligned}$$

where the last inequality is true for any $y > 0$. By setting $y > \lambda t$, the limiting mean of the idle periods, and then choosing x sufficiently larger than y, both of the above terms can be made uniformly small in n and x.

To show $A_n f \to Af$, fix $f \in D$ so the derivative of f is bounded and let $a > 0$ be such that $|f'(x)| \le ax$. Write

$$\begin{aligned}
A_n f(x) &= \mathbb{E}\left[n(f(\max(x - I^{(n)}/n, M^{(n)}/n)) - f(x))\right]\\
&= \int_0^\infty \int_0^\infty n(f(\max(x-y/n, z/n)) - f(x))\lambda^{(n)}e^{-\lambda^{(n)}y}m^{(n)}(\mathrm{d}z)\,\mathrm{d}y\\
&= \int_0^\infty \int^{[nx-y]^+} n(f([x-y/n]^+) - f(x))\lambda^{(n)}e^{-\lambda^{(n)}y}m^{(n)}(\mathrm{d}z)\,\mathrm{d}y\\
&\quad + \int_0^\infty \int_{[nx-y]^+}^\infty n(f(z/n) - f(x))\lambda^{(n)}e^{-\lambda^{(n)}y}m^{(n)}(\mathrm{d}z)\,\mathrm{d}y\\
&= \int_0^\infty m^{(n)}([nx-y]^+)n(f([x-y/n]^+) - f(x))\lambda^{(n)}e^{-\lambda^{(n)}y}\,\mathrm{d}y\\
&\quad + \int_0^\infty \int_{[x-y/n]^+}^\infty (f(u) - f(x))nm^{(n)}(n\mathrm{d}u)\lambda^{(n)}e^{-\lambda^{(n)}y}\,\mathrm{d}y.
\end{aligned} \tag{29}$$

In the case $x = 0$, the first integral is zero. For $x > 0$ we have $m^{(n)}([nx-y]^+) \uparrow 1$ for each y as $n \to \infty$, since $m^{(n)} \le m^{(n+1)}$ by Lemma 3.1 and $m^{(\infty)}$ is proper. Also, as $n \to \infty$,

$$n(f([x-y/n]^+) - f(x)) = -n\int_{[x-y/n]^+}^x f'(z)\,\mathrm{d}z \to -yf'(x), \quad y \ge 0, \tag{30}$$

and $|n(f([x-y/n]^+) - f(x))| \le |y|\|f'\|_\infty$. So,

$$|m^{(n)}([nx-y]^+)n(f([x-y/n]^+) - f(x))\lambda^{(n)}e^{-\lambda^{(n)}y}| \le |y|\|f'\|_\infty \lambda e^{-\lambda^{(1)}y}$$

with $\lambda^{(n)} \uparrow \lambda$ positive and finite. Thus, dominated convergence gives

$$\int_0^\infty m^{(n)}([nx-y]^+)n(f([x-y/n]^+)-f(x))\lambda^{(n)}e^{-\lambda^{(n)}y}\,\mathrm{d}y \to -\frac{f'(x)}{\lambda} \quad (31)$$

for $x \geq 0$, since $f'(0)=0$.

Now we may write the last line of (29) as

$$\begin{aligned}&\int_0^\infty \int_{[x-y/n]^+}^x (f(u)-f(x))nm^{(n)}(n\mathrm{d}u)\lambda^{(n)}e^{-\lambda^{(n)}y}\,\mathrm{d}y \\ &\quad + \int_0^\infty \int_x^\infty (f(u)-f(x))nm^{(n)}(n\mathrm{d}u)\lambda^{(n)}e^{-\lambda^{(n)}y}\,\mathrm{d}y.\end{aligned} \quad (32)$$

If $x=0$, the first integral is zero. If $x>0$, then it goes to zero since

$$\begin{aligned}&\int_{[x-y/n]^+}^x (f(u)-f(x))nm^{(n)}(n\mathrm{d}u) \\ &\quad \leq y\|f'\|\left(m^{(n)}(nx)-m^{(n)}(n[x-y/n]^+)\right) \to 0\end{aligned}$$

for each y, since $\|n(f(u)-f(x))\| \leq y\|f'\|$ for $u \in [[x-y/n]^+, x]$. The integral $\int_0^\infty \int_{[x-y/n]^+}^x (f(u)-f(x))nm^{(n)}(n\mathrm{d}u)\lambda^{(n)}e^{-\lambda^{(n)}y}\,\mathrm{d}y \to 0$ since the inner integral is bounded by $y\|f'\|$. We may evaluate the outer integral of the second term in (32) immediately:

$$\begin{aligned}&\int_0^\infty \int_x^\infty (f(u)-f(x))nm^{(n)}(n\mathrm{d}u)\lambda^{(n)}e^{-\lambda^{(n)}y}\,\mathrm{d}y \\ &\quad = \int_x^\infty (f(u)-f(x))nm^{(n)}(n\mathrm{d}u) \\ &\quad = \int_x^\infty \int_x^u f'(z)\,\mathrm{d}z\,nm^{(n)}(n\mathrm{d}u) \\ &\quad = \int_x^\infty \int_z^\infty f'(z)nm^{(n)}(n\mathrm{d}u)\,\mathrm{d}z \\ &\quad = \int_x^\infty f'(z)n\bar{m}^{(n)}(nz)\,\mathrm{d}z.\end{aligned}$$

Letting K be the support of $f'(z)$, the integrand

$$|f'(z)n\bar{m}^{(n)}(nz)| \leq azn\bar{m}^{(n)}(nz)1_K(z) \to a\kappa(z)1_K(z)$$

uniformly, because $n\bar{m}^{(n)}(nz) \to \kappa(z)/z$ uniformly by monotonicity of $\bar{m}^{(n)}(nz)$ and because $z \in K$ is bounded. So, bounded convergence gives

$$\int_x^\infty f'(z) n\bar{m}^{(n)}(nz)\,\mathrm{d}z \to \int_x^\infty f'(z)\frac{\kappa(z)}{z}\,\mathrm{d}z. \qquad \square$$

Acknowledgements Funding was provided by NWO grant number 639.033.413.

References

1. Boxma, O.: Analysis of models for tandem queues. Ph.D. Thesis, University of Utrecht, Utrecht (1977)
2. Boxma, O.: On a tandem queueing model with identical service times at both counters, I. Adv. Appl. Probab **11**, 616–643 (1979)
3. Karpelevich, F.I., Kreĭnin, A.Y.: Heavy Traffic Limits for Multiphase Queues, vol. 137 of Translations of Mathematical Monographs. American Mathematical Society, Providence (1994). (Translated from the Russian manuscript by Kreĭnin and A. Vainstein)
4. Karpelevitch, F.I., Kreinin, A.Y.: Asymptotic analysis of queueing systems with identical service. J. Appl. Probab. **33**(1), 267–281 (1996)
5. Boxma, O.: On the longest service time in a busy period of the M/G/1 queue. Stoch. Process. Appl. **8**(1), 93–100 (1978)
6. Boxma, O.J., Deng, Q.: Asymptotic behaviour of the tandem queueing system with identical service times at both queues. Math. Methods Oper. Res. **52**(2), 307–323 (2000)
7. Ethier, S.N., Kurtz, T.G.: Markov Processes: Characterization and Convergence, vol. 282. Wiley, Hoboken (2009)
8. Ballerini, R., Resnick, S.I.: Records in the presence of a linear trend. Adv. Appl. Probab. **19**(4), 801–828 (1987)
9. Billingsley, P.: Convergence of Probability Measures. Tracts on Probability and Statistics. Wiley Series in Probability and Mathematical Statistics. Wiley, Hoboken (1968)
10. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Cambridge University Press, Cambridge (1987)
11. Resnick, S.I.: Heavy-tail Phenomena. Springer Series in Operations Research and Financial Engineering. Springer, New York (2007). (Probabilistic and statistical modeling)