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Edge-Disjoint Homotopic Paths in a Planar Graph with One Hole

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We prove the following theorem, conjectured by K. Mehlhorn: Let $G = (V,E)$ be a planar graph, embedded in $\mathbb{R}^2$. Let O denote the interior of the unbounded face. Let I be the interior of some fixed bounded face. Let $C_1, \ldots, C_k$ be curves in $\mathbb{R}^2 \setminus (I \cup O)$, with end points in $V \cap \delta(I \cup O)$, so that for each vertex v of G the degree of v in G has the same parity (mod 2) as the number of curves C_i beginning or ending in v (counting a curve beginning and ending in v for two). Then there exist pairwise edge-disjoint paths $P_1, \ldots, P_k$ in G so that P_i is homotopic to C_i in the space $\mathbb{R}^2 \setminus (I \cup O)$ for $i=1,\ldots,k$, if and only if for each dual path Q from $I \cup O$ to $I \cup O$ the number of edges in Q is not smaller than the number of times Q necessarily intersects the curves C_i.

The theorem generalizes a theorem of Okamura and Seymour. Its proof yields a polynomial-time algorithm finding the paths as required.

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1. THE THEOREM.

We prove the following theorem, conjectured by K. Mehlhorn in relation to the automatic design of integrated circuits.

THEOREM. *Let* $G=(V,E)$ *be a planar graph, embedded in the plane* $\mathbb{R}^2$. *Let* O *denote the interior of the unbounded face. Let* I *be the interior of some fixed bounded face. Let* $C_1,\dots,C_k$ *be curves in* $\mathbb{R}^2\setminus(I\cup O)$, *with end points in* $V\cap\delta(I\cup O)$, *so that for each vertex* v *of* G *the number*

$$(1)\qquad \deg_G(v)+\deg_{C_1,\dots,C_k}(v)$$

is even. Then there exist pairwise edge-disjoint paths $P_1,\dots,P_k$ *in* G *so that* $P_i\simeq C_i$ *in* $\mathbb{R}^2\setminus(I\cup O)$ $(i=1,\dots,k)$ *if and only if for each dual path* Q *from* $\delta(I\cup O)$ *to* $\delta(I\cup O)$ *we have:*

$$(2)\qquad e(Q)\geq\sum_{i=1}^{k}cr(Q,C_i).$$

We here use the following terminology and conventions. A graph may have multiple edges. $\delta(F)$ denotes the boundary of F. $\deg_G(v)$ is the degree of v in G. $\deg_{C_1,\dots,C_k}(v)=\sum_{i=1}^{k}\rho_i$ where ρ_i is the number of end points of C_i equal to v (so $\rho_i\in\{0,1,2\}$). By a *path* we mean a path not containing the same edge twice (it may contain vertices more than once). Each of the curves C_i is allowed to have self-intersections. $P\simeq C$ *in* $\mathbb{R}^2\setminus(I\cup O)$, or just $P\simeq C$, means that P and C are *homotopic* in the space $\mathbb{R}^2\setminus(I\cup O)$ (i.e., there exists a continuous function $F:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2\setminus(I\cup O)$ so that: $F(0,\cdot)$ follows P, $F(1,\cdot)$ follows C, $F(\cdot,0)$ is constant, and $F(\cdot,1)$ is constant - it follows that P and C have the same beginning points and have the same end points). A *dual path* in this paper always means a path in the dual graph

$$(3)\qquad Q=(F_0,e_1,F_1,e_2,F_2,\dots,F_{t-1},e_t,F_t),$$

where $F_0,\dots,F_t$ are faces, e_j is the edge separating F_{j-1} and F_j $(j=1,\dots,t)$, and where $F_j\in\{I,O\}$ if and only if $j\in\{0,t\}$. We denote by $e(Q)$ the number of edges in Q (so $e(Q)=t$ in (3)). Moreover,

$$(4)\qquad cr(Q,C):=\min\left\{\,|Q'\cap C'|\;\middle|\;Q'\simeq Q,\ C'\simeq C\right\}$$

(here we identify a dual path in the obvious way with a curve in $\mathbb{R}^2 \setminus (I \cup O)$, unique up to homotopy and the choice of the beginning and end point on the first and last edge of Q).

Note that I and O play a symmetric role: by turning the configuration inside out, I and O can be exchanged.

2. PROOF OF THE THEOREM.

Since necessity of the condition (2) is trivial, we prove sufficiency. Clearly, we may assume that G is connected. We apply induction on

$$(5) \qquad \sum_{v \in V} (2\deg_G(v) - \deg_{C_1,\ldots,C_k}(v))$$

(which number is nonnegative by (2)).

We may assume that no edge separates faces F and G so that $F,G \in \{I,O\}$ (possibly F=G=I or F=G=O). Suppose e separates such F and G. Contract e, yielding the graph G/e. Clearly, in the reduced configuration, (2) again holds, for each Q. Since (5) is decreased, there exist edge-disjoint paths $P_1 \simeq C_1, \ldots, P_k \simeq C_k$ in G/e. We may assume that no P_i contains a homotopic trivial circuit. Since for $Q := (F,e,G)$ we have $\sum_{i=1}^{k} cr(Q,C_i) \leq e(Q) = 1$, there exists at most one path which breaks down into two paths if we go back to the original graph G. To this path we add e in the appropriate position, and we obtain edge-disjoint paths $P_1 \simeq C_1, \ldots, P_k \simeq C_k$ in G.

We may assume also that for no $i \in \{1,\ldots,k\}$ there exists and edge e in G so that $e \simeq C_i$: deleting e and C_i makes (5) smaller, but will not violate condition (2).

If there exists no dual path Q having equality in (2), we can delete the edges on the boundary of O: after that the condition (2) is still satisfied. This follows from the fact that if Q connects O and O, then

$$(6) \qquad e(Q) > \sum_{i=1}^{k} cr(Q,C_i) \quad \text{and} \quad e(Q) \equiv \sum_{i=1}^{k} cr(Q,C_i) \pmod 2$$

(this last follows from (1)); so (2) also holds after deletion of the edges on $\delta(O)$. Each other dual path Q contains at most one edge on $\delta(O)$, so that (2) is also satisfied after deletion of the edges on $\delta(O)$.

If we delete the edges on $\delta(O)$, (5) is decreased, so by induction there exist paths as required in the smaller graph, and hence also in the original graph.

So we may assume we have equality in (2) for some dual path Q.

We will use the following terminology. Let $\delta(O)$ have n vertices and n edges. Number them in clockwise order by $0,1,2,\ldots,2n-2,2n-1$, where vertices get an odd number and edges an even number. Define

(7) $w_j :=$ vertex numbered $j-2nt$, with $t:=\lfloor j/2n \rfloor$, for $j\in\mathbb{Z}$, j odd;
$w_j :=$ edge numbered $j-2nt$, with $t:=\lfloor j/2n \rfloor$, for $j\in\mathbb{Z}$, j even.

Similarly, let $\delta(I)$ have m vertices and m edges. Number them in clockwise order by $0,1,2,\ldots,2m-2,2m-1$, where vertices get an odd number and edges an even number. Define

(8) $w'_j :=$ vertex numbered $j-2mt$, with $t:=\lfloor j/2m \rfloor$, for $j\in\mathbb{Z}$, j odd;
$w'_j :=$ edge numbered $j-2mt$, with $t:=\lfloor j/2m \rfloor$, for $j\in\mathbb{Z}$, j even.

Fix a dual path $D = (O,w_0,\ldots,w'_0,I)$ not containing the same face twice. We say that a path, curve or dual path Q is of *type* (p,q), with $p = (p_1,p_2)$ and $q = (q_1,q_2)$ both in $\mathbb{Z}\times\{0,1\}$, if:

(9) (i) if $p_2=q_2=0$, Q is homotopic to the path $(w_{p_1},\ldots,w_{q_1})$ (if $p_1\leq q_1$ subscripts go up with steps 1; if $p_1>q_1$ subscripts go down with steps 1);

(ii) if $p_2=0$, $q_2=1$, Q is homotopic to the path $(w_{p_1},\ldots,w_0,D,w'_0,\ldots,w'_{q_1})$;

(iii) if $p_2=1$, $q_2=0$, Q is homotopic to the path $(w'_{p_1},\ldots,w'_0,D^{-1},w_0,\ldots,w_{q_1})$;

(iv) if $p_2=q_2=1$, Q is homotopic to the path $(w'_{p_1},\ldots,w'_{q_1})$.

So if Q is of type $((p_1,0),(q_1,0))$, it is also of type $(p_1+2n,0),q_1+2n,0))$, and conversely. Similarly, if $p_2=q_2=1$. Moreover, if Q is of type $((p_1,0),(q_1,1))$, it is also of type $(p_1+2n),0),(q_1+2m,1))$, and conversely.

So dual paths are of types (p,q) with p_1 and q_1 even, while paths in the graph itself, and the curves C_i above, are of types (p,q) with p_1 and q_1 odd.

The number cr(Q,C) can be expressed in the types of Q and C. Let Q be a dual path and let C be a curve from $V\cap\delta(I\cup O)$ to $V\cap\delta(I\cup O)$. Let Q have type (p,q) and let C have type (c,d). Then if $c_2=d_2=0$:

(10) if $p_2=q_2=0$, $p_1\leq q_1$, then: $cr(Q,C) = \left|\{j\in\mathbb{Z} \mid p_1+2nj<c_1<q_1+2nj<d_1 \text{ or } c_1<p_1+2nj<d_1<q_1+2nj\}\right|$;

if $p_2=q_2=0$, $p_1>q_1$, then: $cr(Q,C) = |\{j\in\mathbb{Z} \mid q_1+2nj<c_1<p_1+2nj <d_1 \text{ or } c_1<q_1+2nj<d_1<p_1+2nj\}|$;
if $p_2=0, q_2=1$, then: $cr(Q,C) = |\{j\in\mathbb{Z} \mid c_1<p_1+2nj<d_1\}|$;
if $p_2=1, q_2=0$, then: $cr(Q,C) = |\{j\in\mathbb{Z} \mid c_1<q_1+2nj<d_1\}|$;
if $p_2=q_2=1$, then: $cr(Q,C) = 0$.

If $c_2=0$, $d_2=1$:

(11) if $p_2=q_2=0$, then: $cr(Q,C) = |\{j\in\mathbb{Z} \mid p_1+2nj<c_1<q_1+2nj\}|$;
if $p_2=0, q_2=1$, then: $cr(Q,C) = |\{j\in\mathbb{Z} \mid (p_1+2nj<c_1 \text{ and } d_1<q_1+2mj) \text{ or } (c_1<p_1+2nj \text{ and } q_1+2mj<d_1)\}|$;
if $p_2=1, q_2=0$, then: $cr(Q,C) = |\{j\in\mathbb{Z} \mid (q_1+2nj<c_1 \text{ and } d_1<p_1+2mj) \text{ or } (c_1<q_1+2nj \text{ and } p_1+2mj<d_1)\}|$;
if $p_2=q_2=1$, then: $cr(Q,C) = |\{j\in\mathbb{Z} \mid p_1+2mj<d_1<q_1+2mj\}|$.

The case $c_2=1, d_2=0$ is similar to (11), and the case $c_2=d_2=1$ is similar to (10).

Now choose a dual path Q with equality in (2). Let Q have type (p,q). We choose Q,p,q so that:

(12) (Case 1) $p_2=q_2=0$, $p_1<q_1$, q_1-p_1 is as small as possible.
(Case 2) If Case 1 does not apply: $p_2=q_2=1$, $p_1<q_1$, q_1-p_1 is as small as possible.
(Case 3) If Cases 1 and 2 do not apply: $p_2=0, q_2=1$, and $q_1n-p_1m > \max\{d_1n-c_1m \mid \exists i\in\{1,\ldots,k\}: C_i \text{ is of type } ((c_1,0),(d_1,1))\}$.
(Case 4) If Cases 1,2 and 3 do not apply: $p_2=0$, $q_2=1$, and q_1n-p_1m is as large as possible.

By symmetry with Case 1, we may assume we are not in Case 2.

Since Q has equality in (2), there exists $i\in\{1,\ldots,k\}$ so that $cr(Q,C_i)\geqslant 1$. Let C_i have type (c,d). If we are in Case 1, we choose C_i,c,d so that:

(13) (Case 1a) $c_2=d_2=0$, $c_1<p_1<d_1<q_1$ and c_1 is as large as possible.
(Case 1b) If Case 1a does not apply: $c_2=d_2=0$, $p_1<c_1<q_1<d_1$ and d_1 is as small as possible.
(Case 1c) If Cases 1a and 1b do not apply: $c_2=1, d_2=0$, $p_1<d_1<q_1$ and c_1 is as small as possible.

By symmetry with Case 1a, we may assume we are not in Case 1b.

If we are in Case 3, we choose C_i,c,d so that:

(14) (Case 3a) $c_2=d_2=0$, $c_1<p_1<d_1$ and c_1 is as large as possible.
(Case 3b) If Case 3a does not apply: $c_2=d_2=1$, $c_1<q_1<d_1$ and d_1 is as small as possible.
(Case 3c) If Cases 3a and 3b do not apply: $c_2=1$, $d_2=0$, $c_1<q_1$, $p_1<d_1$ and c_1 is as small as possible.

By symmetry with Case 3a, we may assume we are not in Case 3b.

If we are in Case 4, we choose C_i,c,d so that:

(15) (Case 4a) $c_2=0$, $c_1<p_1$, if $d_2=0$ then $p_1<d_1$, if $d_2=1$ then $q_1<d_1$, and c_1 is as large as possible.
(Case 4b) If Case 4a does not apply: $c_2=d_2=1$, $c_1<q_1<d_1$ and d_1 is as small as possible.
(Case 4c) If Cases 4a and 4b do not apply: $c_2=1$, $d_2=0$, $c_1<q_1$, $p_1<d_1$, and c_1 is as small as possible.

By symmetry with Case 4a, we may assume we are not in Case 4b.

Concluding, six cases to consider are left: Cases 1a,1c,3a,3c,4a and 4c. Without loss of generality, i=1. Now let C_1' and C_1'' be curves of types $(c,(p_1+\delta,0))$ and $((p_1+\delta,0),d)$, respectively, where $\delta=1$ if $c = (p_1-1,0)$, and $\delta = -1$ otherwise. We show that condition (2) is maintained with respect to the curves $C_1',C_1'',C_2,\ldots,C_k$. Showing this will finish the proof, since in the new configuration (5) has been decreased, so by induction we know the existence of edge-disjoint paths $P_1'\simeq C_1'$, $P_1''\simeq C_1''$, $P_2\simeq C_2$, ... $,P_k\simeq C_k$ in G. Taking $P_1 := P_1'P_1''$, we obtain edge-disjoint paths $P_1\simeq C_1,\ldots,P_k\simeq C_k$.

Suppose (2) does not hold with respect to $C_1',C_1'',C_2,\ldots,C_k$. Hence there exists a dual path R so that

(16) $$e(R) - cr(R,C_1')-cr(R,C_1'') - \sum_{i=2}^{k} cr(R,C_i)$$

is negative. Since (2) holds for $C_1,\ldots,C_k$, we know:

(17) $$\begin{aligned} e(R)-cr(R,C_1')-cr(R,C_1'')-\sum_{i=2}^{k} cr(R,C_i) &\geq \\ cr(R,C_1)-cr(R,C_1')-cr(R,C_1'') &\geq c_1-p_1-\delta \quad \text{(Cases 1a,3a,4a)}, \\ \text{,,} \qquad &\geq p_1-d_1+\delta \quad \text{(Cases 1c,3c,4c)}. \end{aligned}$$

Hence we may assume that we have chosen R so that (16) is as small as possible. Under this condition, we assume we have chosen R so that

(18) $$cr(R,C_1')+cr(R,C_1'')-cr(R,C_1)$$

is as small as possible; note that (18) necessarily is nonnegative, and in fact, is positive as R satisfies (2) with respect to $C_1,\ldots,C_k$ but not with respect to $C_1',C_1'',C_2,\ldots,C_k$.

Let R have type (r,s). We may assume we have chosen r and s so that:

(19) in Case 1a: $r_2=s_2=0$, $c_1<r_1<p_1+\delta<s_1<d_1$ and s_1 is as large as possible;
in Case 1c: <u>(Case 1c1)</u> $r_2=s_2=0$ and $r_1<p_1+\delta<s_1<d_1$; or
<u>(Case 1c2)</u> $r_2=1, s_2=0$, $r_1<c_1$ and $p_1+\delta<s_1<d_1$;
in Case 3a: $r_2=s_2=0$ and $c_1<r_1<p_1+\delta<s_1<d_1$;
in Case 3c: <u>(Case 3c1)</u> $r_2=s_2=0$ and $r_1<p_1+\delta<s_1<d_1$; or
<u>(Case 3c2)</u> $r_2=1$, $s_2=0$, $r_1<c_1$ and $p_1+\delta<s_1<d_1$;
in Case 4a: <u>(Case 4a1)</u> $d_2=0$, $r_2=s_2=0$, $c_1<r_1<p_1+\delta<s_1<d_1$; or
<u>(Case 4a2)</u> $d_2=1, s_2=0$, $r_2=0$, $c_1<r_1<p_1+\delta<s_1$; or
<u>(Case 4a3)</u> $d_2=1, s_2=1$, $r_2=0$, $c_1<r_1<p_1+\delta$ and $d_1<s_1$;
in Case 4c: <u>(Case 4c1)</u> $r_2=s_2=0$ and $r_1<p_1+\delta<s_1<d_1$; or
<u>(Case 4c2)</u> $r_2=1$, $s_2=0$, $r_1<c_1$ and $p_1+\delta<s_1<d_1$.

Since

(20)
$r_1\leq p_1\leq s_1<q_1$ (if $r_2=p_2=s_2=q_2=0$),
$p_1\leq s_1<q_1$ (if $r_2=1$, $p_2=s_2=q_2=0$),
$r_1\leq p_1\leq s_1$ (if $r_2=p_2=s_2=0$, $q_2=1$),
$r_1<q_1$ and $p_1\leq s_1$ (if $r_2=1$, $p_2=s_2=0$, $q_2=1$),
$r_1\leq p_1$ and $q_1\leq s_1$ (if $r_2=p_2=0$, $s_2=q_2=1$),

there exist dual paths Q' and R' so that:

(21) $e(Q') + e(R') = e(Q) + e(R)$,
Q' has type (p,s), R' has type (r,q).

We now first show that for each i=1,...,k:

(22) $$cr(Q',C_i) + cr(R',C_i) = cr(Q,C_i) + cr(R,C_i).$$

For let C_i have type (a,b). Then:

(23) in Case 1a: $a_2=b_2=0$ and $r_1<a_1<p_1\leq s_1<b_1<q_1$ is not possible (by our choice of C_1 in (13));

in Case 1c1: $a_2=b_2=0$ and $r_1<a_1<p_1\leq s_1<b_1<q_1$ is not possible (otherwise we would be in Case 1a);

in Case 1c2: $a_2=b_2=0$, $a_1<p_1\leq s_1<b_1<q_1$ is not possible (otherwise we would be in Case 1a) and $a_2=1$, $b_2=0$, $a_1<r_1$, $p_1\leq s_1<b_1<q_1$ is not possible (by our choice of C_1 in (13));

in Case 3a: $a_2=b_2=0$, $r_1<a_1<p_1\leq s_1<b_1$ is not possible (by our choice of C_1 in (14)) and $a_2=0,b_2=1,r_1<a_1<p_1,q_1<b_1$ is not possible (by our choice of C_1 in (14));

in Case 3c1: $a_2=b_2=0,r_1<a_1<p_1\leq s_1<b_1$ is not possible (otherwise we would be in Case 3a) and $a_2=0,b_2=1,r_1<a_1<p_1,q_1<b_1$ is not possible (as $q_1n-p_1m > b_1n-a_1m$, by (12));

in Case 3c2: $a_2=b_2=0,a_1<p_1\leq s_1<b_1$ is not possible (otherwise we would be in Case 3a), $a_2=0,b_2=1,a_1<p_1,q_1<b_1$ is not possible (as $q_1n-p_1m > b_1n-a_1m$, by (12)) , $a_2=1,b_2=0,a_1<r_1,s_1<b_1$ is not possible (by our choice of C_1 in (14)), and $a_2=b_2=1,a_1<r_1,q_1<b_1$ is not possible (otherwise we would be in Case 3b);

in Case 4a1: $a_2=b_2=0,r_1<a_1<p_1\leq s_1<b_1$ is not possible (by our choice of C_1 in (15)) and $a_2=0,b_2=1,r_1<a_1<p_1,q_1<b_1$ is not possible (by our choice of C_1 in (15));

in Case 4a2: $a_2=b_2=0,r_1<a_1<p_1\leq s_1<b_1$ is not possible (by our choice of C_1 in (15)) and $a_2=0,b_2=1,r_1<a_1<p_1,q_1<b_1$ is not possible (by our choice of C_1 in (15));

in Case 4a3: $a_2=0,b_2=1,r_1<a_1<p_1,q_1<b_1<s_1$ is not possible (by our choice of C_1 in (15));

in Case 4c1: $a_2=b_2=0,r_1<a_1<p_1\leq s_1<b_1$ is not possible (otherwise we would be in Case 4a) and $a_2=0,b_2=1,r_1<a_1<p_1,q_1<b_1$ is not possible (otherwise we would be in Case 4a);

in Case 4c2: $a_2=b_2=0,a_1<p_1\leq s_1<b_1$ is not possible (otherwise we would be in Case 4a), $a_2=0,b_2=1,a_1<p_1,q_1<b_1$ is not possible (otherwise we would be in Case 4a), $a_2=1,b_2=0,a_1<r_1,s_1<b_1$ is not possible (by our choice of C_1 in (15)), and $a_2=b_2=1,a_1<r_1,q_1<b_1$ is not possible (otherwise we would be in Case 4b).

This shows (22).

Moreover,

(24) $$cr(R',C_1')+cr(R',C_1'')-cr(R',C_1) \geqslant cr(R,C_1')+cr(R,C_1'')-cr(R,C_1)-2.$$

To show this, again we distinguish cases:

(25) Case 1a: $cr(R,C_1')+cr(R,C_1'')-cr(R,C_1) = 2\cdot|\{j\in\mathbb{Z} \mid c_1<r_1+2nj<p_1+\delta<s_1+2nj<d_1\}|$. Now $c_1>p_1-2n$ or $d_1>q_1-2n$ (otherwise we could replace c_1,d_1 by c_1+2n,d_1+2n, contradicting the maximality of c_1 (cf. (13))). If $c_1>p_1-2n$ then $cr(R,C_1')+cr(R,C_1'')-cr(R,C_1)=2$, and hence (24) follows trivially. If $d_1>q_1-2n$ then:

$$\begin{aligned}
&cr(R',C_1')+cr(R',C_1'')-cr(R',C_1) = \\
&2\cdot|\{j\in\mathbb{Z} \mid c_1<r_1+2nj<p_1+\delta<q_1+2nj<d_1\}| \geq \\
&2\cdot|\{j\in\mathbb{Z} \mid c_1<r_1+2nj<p_1+\delta<s_1+2nj<d_1\}|-2 = \\
&cr(R,C_1')+cr(R,C_1'')-cr(R,C_1)-2.
\end{aligned}$$

The inequality here can be seen as follows: if $j\in\mathbb{Z}$ and $c_1<r_1+2nj<p_1+\delta<s_1+2nj<d_1$ then $j\leq 0$ (since otherwise we can replace r_1,s_1 by r_1+2n,s_1+2n, contradicting the choice of r,s in (19)); if moreover $j\leq -1$, then $c_1<r_1+2nj<p_1+\delta<s_1+2nj<q_1+2nj<d_1$.

Case 1c1: we have

$$\begin{aligned}
&cr(R',C_1')+cr(R',C_1'')-cr(R',C_1) = \\
&2\cdot|\{j\in\mathbb{Z} \mid r_1+2nj<p_1+\delta<q_1+2nj<d_1\}| \geq \\
&2\cdot|\{j\in\mathbb{Z} \mid r_1+2nj<p_1+\delta<s_1+2nj<d_1\}|-2 = \\
&cr(R,C_1')+cr(R,C_1'')-cr(R,C_1)-2.
\end{aligned}$$

The inequality here can be seen as follows: if $j\in\mathbb{Z}$ and $r_1+2nj<p_1+\delta<s_1+2nj<d_1$ then $j\leq\lfloor(d_1-s_1)/2n\rfloor$; if $j\leq\lfloor(d_1-s_1)/2n\rfloor-1$ then for $j' := j-\lfloor(q_1-s_1)/2n\rfloor$ we have $r_1+2nj'\leq r_1+2nj<p_1+\delta<s_1+2nj<q_1+2nj'<d_1$.

Case 1c2: we have

$$\begin{aligned}
&cr(R',C_1')+cr(R',C_1'')-cr(R',C_1) = \\
&2|\{j\in\mathbb{Z} \mid r_1+2nj<c_1 \text{ and } p_1+\delta<q_1+2nj<d_1\}| \geq \\
&2|\{j\in\mathbb{Z} \mid r_1+2nj<c_1 \text{ and } p_1+\delta<s_1+2nj<d_1\}|-2 = \\
&cr(R,C_1')+cr(R,C_1'')-cr(R,C_1)-2.
\end{aligned}$$

The inequality here can be seen as follows: if $j\in\mathbb{Z}$, $r_1+2nj<c_1$ and $p_1+\delta<s_1+2nj<d_1$, then $j\leq\lfloor(d_1-s_1)/2n\rfloor$; if $j\leq\lfloor(d_1-s_1)/2n\rfloor-1$, then for $j' := j-\lfloor(q_1-s_1)/2n\rfloor$ we have $r_1+2nj'\leq r_1+2nj<c_1$ and $p_1+\delta<s_1+2nj\leq q_1+2nj'<d_1$.

Cases 3a and 4a: we now have

$$cr(R,C_1')+cr(R,C_1'')-cr(R,C_1) \leq$$
$$2\cdot|\{j\in\mathbb{Z} \mid c_1<r_1+2nj<p_1+\delta\}| \leq 2,$$

because c_1 is chosen as large as possible (cf. (14) and (15)), so $c_1>p_1-2n$. (24) follows.

Cases 3c1 and 4c1: now

$$cr(R,C_1')+cr(R,C_1'')-cr(R,C_1) =$$
$$2\cdot|\{j\in\mathbb{Z} \mid r_1+2nj<p_1+\delta<s_1+2nj<d_1\}| \leq 2,$$

because c_1 is chosen as small as possible (cf. (14) and (15)), so $d_1<p_1+2n$. (24) follows.

Cases 3c2 and 4c2: we have

$$cr(R,C_1')+cr(R,C_1'')-cr(R,C_1) =$$
$$2\cdot|\{j\in\mathbb{Z} \mid r_1+2nj<c_1 \text{ and } p_1+\delta<s_1+2nj<d_1\}| \leq 2,$$

because c_1 is chosen as small as possible (cf (14) and (15)), so $d_1<p_1+2n$. Again, (24) follows.

This shows (24).

Next we show:

(26) $$e(Q') \geq \sum_{i=1}^{k} cr(Q',C_i)+2.$$

Since Q' has type (p,s) we know by our choice of Q that

(27) $$e(Q') > \sum_{i=1}^{k} cr(Q',C_i).$$

In Case 1 this follows from our choice of Q (as $s_1<q_1$). In Cases 3, 4a1,4a2, 4c1 and 4c2 this follows since Case 1 does not apply (as $p_2=s_2=0$). In Case 4a3 this follows from our choice of Q in (12), since $s_2=1$, $s_1>q_1$, whence $s_1n-p_1m>q_1n-p_1m$.

Moreover,

(28) $$e(Q') \equiv \sum_{i=1}^{k} cr(Q',C_i) \pmod 2.$$

If Q' goes from O to O, that is, if $p_2=s_2=0$, this follows from the fact that (1) is even for each vertex v. If Q' does not go from O to O, we are in Case 4a3, in which case $p_2=0, s_2=1, q_2=1$. Hence, again as (1) is even for each v,

$$(29) \qquad e(Q')-\sum_{i=1}^{k} cr(Q',C_i) \equiv e(Q)-\sum_{i=1}^{k} cr(Q,C_i)=0 \qquad (\text{mod } 2).$$

This shows (26).

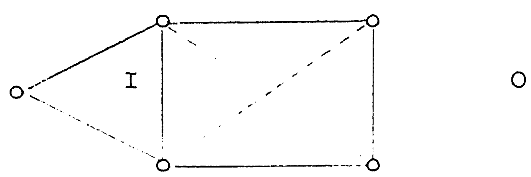
By (21), (22), (24), (26) and since Q has equality in (2) we know:

$$
\begin{aligned}
(30) \qquad & e(R')-cr(R',C_1')-cr(R',C_1'')-\sum_{i=2}^{k} cr(R',C_i) = \\
& e(R)+e(Q)-e(Q')-cr(R',C_1')-cr(R',C_1'')-\sum_{i=2}^{k} cr(R',C_i) \leq \\
& e(R)+\sum_{i=1}^{k} cr(Q,C_i)-\sum_{i=1}^{k} cr(Q',C_i)-2-cr(R',C_1')-cr(R',C_1'')-\sum_{i=2}^{k} cr(R',C_i) = \\
& e(R)-2-\sum_{i=1}^{k} cr(R,C_i)+cr(R',C_1)-cr(R',C_1')-cr(R',C_1'') \leq \\
& e(R)-cr(R,C_1')-cr(R,C_1'')-\sum_{i=2}^{k} cr(R,C_i).
\end{aligned}
$$

Since we have taken R with (16) minimal, we have equality throughout in (30). Hence we have equality in (24), contradicting the minimality of (18). □

3. FURTHER REMARKS.

The condition that (1) is even for each vertex v, cannot be deleted in the theorem, as is shown by the following graph:



Here dotted lines represent curves.

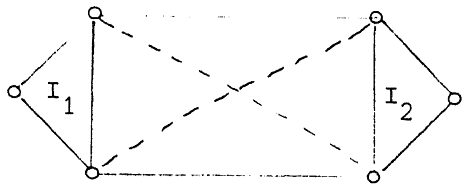
The theorem above generalizes the following theorem of Okamura and Seymour [3]: Let G = (V,E) be a planar graph, embedded in $\mathbb{R}^2$, let O denote the interior of the unbounded face, and let $r_1,\dots,r_k,s_1,\dots,s_k$ be vertices in $V \cap \delta(O)$ so that $\deg_G(v) + |\{i \mid r_i=v\}| + |\{i \mid s_i=v\}|$ is even for each vertex v. Then there exist pairwise edge-disjoint paths $P_1,\dots,P_k$ in G so that P_i connect r_i and s_i $(i=1,\dots,k)$, if and only if for each subset W of V we have:

(31) $$\left|\{e\in E \mid |e\cap W| = 1\}\right| \geq \left|\{i\in \{1,\ldots,k\} \mid |\{r_i,s_i\}\cap W| =1\}\right|.$$

The fact that Okamura and Seymour's theorem follows from our theorem can be seen by putting in O two new vertices, joined by two parallel edges, enclosing a region in O, which will be the face I.

There is another extension of Okamura and Seymour's theorem, due to Okamura [2], which resembles our theorem, but which is different: Let $G = (V,E)$ be a planar graph embedded in $\mathbb{R}^2$, let O be the interior of the unbounded face, let I be the interior of some other face, let $r_1,\ldots,r_m$, $s_1,\ldots,s_m \in V\cap\delta(O)$ and let $r_{m+1},\ldots,r_k,s_{m+1},\ldots,s_k \in V\cap\delta(I)$, so that $\deg_G(v) + |\{i \mid r_i=v\}|+|\{i \mid s_i=v\}|$ is even for each vertex v. Then there exist pairwise edge-disjoint paths $P_1,\ldots,P_k$ in G so that P_i connects r_i and s_i $(i=1,\ldots,k)$, if and only if (31) holds for each $W\subseteq V$.

The obvious extension of our theorem to more than one "hole" does not hold, as is shown by the following example:



Again, dotted lines represent curves C_1 and C_2. Now for each dual path Q from $\{I_1,I_2,O\}$ to $\{I_1,I_2,O\}$ we have $e(Q)\geq cr(Q,C_1)+cr(Q,C_2)$. However, there are no edge-disjoint paths P_1 and P_2 so that $P_1\simeq C_1$ and $P_2\simeq C_2$ in $\mathbb{R}^2\setminus(I\cup O)$. Kaufmann and Mehlhorn [1] showed that an extension to arbitrarily many holes holds in the case of so-called grid graphs.

We finally show that the proof of Section 2 yields a polynomial-time algorithm to find edge-disjoint paths $P_1\simeq C_1,\ldots,P_k\simeq C_k$ as required, if they exist.

Indeed, the proof gives a polynomial time construction of the paths, provided we are able to identify a dual path Q with equality in (2), so that (12) holds. To this end, we choose for the dual path D in our proof some fixed *shortest* path in the dual graoh from O to I (shortest in the sense of having the smallest number of edges). Let the C_i be given by their types with respect to this path D. Let

(32) $$K := e(D) + \max\{|d_1-c_1| \mid i\in\{1,\ldots,k\},\ j\in\{0,1\}: C_i \text{ has type } ((c_1,j),(c_2,j))\}.$$

Now, if (2) holds for each dual path Q, then:

(33) if there exists a dual path Q having equality in (2) of type $((p_1,0),(q_1,0))$, there exists also one with $|q_1-p_1| \leq K$.

Proof. If $q_1 > p_1+2n$ there exist dual paths Q' and Q" of types $((p_1,0),(q_1-2n,0))$ and $((p_1,0),(q_1+2n,0))$ respectively, so that $e(Q')+e(Q") = 2.e(Q)$. If moreover $q_1-p_1 > K$, then $cr(Q',C_i)+cr(Q",C_i) = 2.cr(Q,C_i)$ for $i=1,\dots,k$. Since

(34)
$$e(Q') \geq \sum_{i=1}^{k} cr(Q',C_i),$$
$$e(Q") \geq \sum_{i=1}^{k} cr(Q",C_i),$$
$$e(Q) = \sum_{i=1}^{k} cr(Q,C_i),$$

it follows that also Q' has equality in (2). Repeating this procedure we finally find a dual path as required. □

Similarly, one shows:

(35) if there exists a dual path Q having equality in (2) of type $((p_1,1),(q_1,1))$, there exists also one with $|q_1-p_1| \leq K$.

Moreover:

(36) if there exists a dual path Q having equality in (2) of type $((p_1,0),(q_1,1))$, there exists also one with $(q_1/2m)-(p_1/2n) \leq K$.

Proof. If $(q_1/2m)-(p_1/2n) > K$, Q intersects D more than e(D) times. Hence Q contains some face in D more than once. Let there be j rotations of Q in between. Then there exist dual paths Q' and Q" of types $((p_1,0),(q_1-2mj,1))$ and $((p_1,0),(q_1+2mj,1))$ respectively, so that $e(Q')+e(Q") = 2.e(Q)$. Note that $cr(Q',C_i)+cr(Q",C_i) \geq 2.cr(Q,C_i)$ for $i=1,\dots,k$. Hence we have also equality in (2) for Q' and Q". Repeating this procedure we finally find a dual path as required. □

Now we construct the following graph $\tilde{G} = (\tilde{V},\tilde{E})$. First orient the edges in D so that they all have the same orientation with respect to D. Let $\tilde{V} := V \times \{1,\dots,K\}$, and let $\tilde{E}$ be defined as follows:

(37) $(v,i),(w,j) \in \tilde{E} \Leftrightarrow i=j, \{v,w\} \in E$ and $\{v,w\}$ does not belong to D, or $j=i+1$, $\{v,w\} \in E$ and $\{v,w\}$ occurs in D and is directed from v to w, or $j=i-1$, $\{v,w\} \in E$ and $\{v,w\}$ occurs in D and is directed from w to v.

Clearly, $\tilde{G}$ is a planar graph, and an embedding of $\tilde{G}$ easily follows from an embedding of G by "unfolding" G. So edges on the boundary of the unbounded face of $\tilde{G}$ come from edges of G on $\delta(I \cup O)$ or from edges incident with one of the faces in D.

By (33), (35) and (36) above, if (2) is satisfied, we can find a dual path required by (12), by finding the shortest paths in the dual graph of $\tilde{G}$ from each edge e' on the boundary of the unbounded face of $\tilde{G}$ to any other such edge e" (except if e' and e" do not come from an edge in G on $\delta(I \cup O)$). This can be done in polynomial time, by applying Dijkstra's algorithm a polynomial number of times. So each of these pairs e',e" yields a shortest dual path $Q_{e',e''}$. We next select those $Q_{e',e''}$ for which $e(Q_{e',e''}) = \sum_{i=1}^{k} cr(Q_{e',e''}, C_i)$. We can now make our choice described in (12).

Note that if (2) is not satisfied for some dual path Q, the above algorithm automatically gets stuck, as (2) is a necessary condition. So we do not need to test (2) in advance, but may just <u>assume</u> that the input of our algorithm satisfies (2).

The algorithm has running time bounded by a polynomial in $|V|+|E|+\sum_{i=1}^{k}(|c_{i,1}|+|d_{i,1}|)$, where $((c_{i,1},c_{i,2}),(d_{i,1},d_{i,2}))$ is the given type of C_i.

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