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Distances and Cuts in Planar Graphs

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We prove the following theorem. Let $G=(V,E)$ be a planar bipartite graph, embedded in the euclidean plane. Let O and I be two of its faces. Then there exist pairwise edge-disjoint cuts $C_1,\ldots,C_t$ so that for each two vertices v,w with $v,w \in O$ or $v,w \in I$, the distance from v to w in G is equal to the number of cuts C_j separating v and w.

This theorem is dual to a theorem of Okamura on plane multicommodity flows, in the same way as a theorem of Karzanov is dual to one of Lomonosov.

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1. INTRODUCTION.

We prove the following theorem:

THEOREM. *Let* $G = (V,E)$ *be a planar bipartite graph, embedded in the euclidean plane. Let* O *and* I *be two of the faces. Then there exist pairwise edge-disjoint cuts* $\delta(X_1),\ldots,\delta(X_t)$ *so that for each two vertices* v,w *with* $v,w \in O$ *or* $v,w \in I$, *the distance of* v *to* w *in* G *is equal to the number of cuts* $\delta(X_j)$ *separating* v *and* w.

[Here, for $X \subseteq V$, $\delta(X) := \{e \in E \mid |e \cap X| = 1\}$, while $\delta(X)$ *separates* v and w if $|\{v,w\} \cap X| = 1$.]

Note that for any graph G, whatever collection of pairwise edge-disjoint cuts $\delta(X_j)$ we take, for any two vertices v,w of G, the distance from v to w is always at least as large as the number of these cuts separating v and w. The point in the theorem is that we can get equality, under the conditions given.

This theorem is 'dual' to a theorem of Okamura [1983] on plane multicommodity flows, in the same way as the results of Karzanov [1985] are dual to those of Lomonosov [1976,1979] on multicommodity flows, as we shall explain in Section 2 below. The theorem extends a result of Hurkens, Schrijver and Tardos [1986], dual to a theorem of Okamura and Seymour [1981]; this result restricts v,w to belong to only one fixed face.

The theorem cannot be generalized to the obvious extension with more than two faces, as is shown by the complete bipartite graph $K_{2,3}$. This graph also shows that we cannot allow in the theorem above pairs v,w with $v \in O$ and $w \in I$.

2. RELATION TO MULTICOMMODITY FLOWS.

In this section we discuss a relation of the theorem above with multicommodity flow problems. Let G=(V,E) be an undirected graph. Let $\{r_1,s_1\},\ldots,\{r_k,s_k\}$ be pairs of vertices. Suppose we wish to decide if

(1) there exist pairwise edge-disjoint paths $P_1,\ldots,P_k$ so that P_i connects r_i and s_i $(i=1,\ldots,k)$.

Clearly, the following 'cut condition' is a necessary condition:

(2) each cut $\delta(X)$ separates at most $|\delta(X)|$ of the pairs r_i,s_i.

Now Lomonosov [1976,1979] (extending earlier work by Menger [1927], Hu [1963], Rothschild and Whinston [1966], Papernov [1976], Seymour [1980]), Okamura [1983] (extending earlier work by Okamura and Seymour [1981]), and Seymour [1981] showed the following three results, each of which uses the following 'parity condition':

(3) for each vertex v, $|\delta(\{v\})| + |\{i \mid v \in \{r_i, s_i\}\}|$ is even.

Lomonosov's theorem: *If*

(4) *the graph* $H := (\{r_1, s_1, \ldots, r_k, s_k\}, \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ *has at most four vertices, or is isomorphic to* C_5 *(the circuit with five vertices), or contains two vertices* v,w *so that* $\{v,w\} \cap \{r_i, s_i\} \neq \emptyset$ *for all* $i=1,\ldots,k$,

then the cut condition (2) *and the parity condition* (3) *together imply* (1).

Okamura's theorem: *If*

(5) G *is planar, so that there are two of its faces,* O *and* I, *with for each* $i=1,\ldots,k$: $r_i, s_i \in O$ *or* $r_i, s_i \in I$,

then the cut condition (2) *and the parity condition* (3) *together imply* (1).

Seymour's theorem: *If*

(6) *the graph* $(V, E \cup \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ *is planar,*

then the cut condition (2) *and the parity condition* (3) *together imply* (1).

A consequence of these results is that, if (4),(5) or (6) holds, and if moreover the cut condition (2) holds, then there exist paths $P_1', P_1'', \ldots, P_k', P_k''$ so that both P_i' and P_i'' connect r_i and s_i $(i=1,\ldots,k)$, and so that each edge of G is in at most two of the paths $P_1', P_1'', \ldots, P_k', P_k''$. (This follows by duplicating each edge of G and each pair $\{r_i, s_i\}$, after which (2) and (3) hold.)

Hence, if (4), (5) or (6) holds, and if $c \in \mathbb{Q}_+^E$ (a 'capacity function') and $d \in \mathbb{Q}_+^k$ (a 'demand function') so that

(7) for each $X \subseteq V$, $\sum(c_e \mid e \in \delta(X)) \geq \sum(d_i \mid i=1,\ldots,k;$ X separates r_i and $s_i)$,

then there exist paths $P_1^1,\ldots,P_1^{t_1},P_2^1,\ldots,P_2^{t_2},\ldots,P_k^1,\ldots,P_k^{t_k}$ (where each P_i^j connects r_i and s_i) and rationals $\lambda_1^1,\ldots,\lambda_1^{t_1},\lambda_2^1,\ldots,\lambda_2^{t_2},\ldots,\lambda_k^1,\ldots,\lambda_k^{t_k} \geq 0$ so that:

(8)
$$\sum_{i=1}^{k}\sum_{\substack{j=1\\ e\in P_i^j}}^{t_i} \lambda_i^j \leq c_e \qquad (e \in E),$$

$$\sum_{j=1}^{t_i} \lambda_i^j = d_i \qquad (i=1,\ldots,k)$$

(a 'multicommodity flow'). (For (5) this is a result of Papernov [1976].) (This result follows from the result in the previous paragraph, by observing that we may take, without loss of generality, c and d to be integral, and hence we can replace each edge e of G by c_e parallel edges, and each pair $\{r_i,s_i\}$ by d_i parallel pairs, after which we apply the previous result.)

In polyhedral terms, this statement is equivalent to: if (4),(5) or (6) holds, then the cone of vectors $(d;c) \in \mathbb{Q}^k \times \mathbb{Q}^E$ defined by the linear inequalities:

(9)
$$\text{(i)}\quad \sum(c_e \mid e \in \delta(X)) \geq \sum(d_i \mid i \in \rho(X)) \qquad (X \subseteq V),$$
$$\text{(ii)}\quad d_i \geq 0 \qquad (i=1,\ldots,k),$$
$$\text{(iii)}\quad c_e \geq 0 \qquad (e \in E),$$

(where $\rho(X) := \{i=1,\ldots,k \mid X$ separates r_i and $s_i\}$), is equal to the cone generated by the following vectors:

(10)
$$\text{(i)}\quad (\varepsilon_i;\chi^P) \qquad (i=1,\ldots,k;\ P\ r_i\text{-}s_i\text{-path}),$$
$$\text{(ii)}\quad (0;\varepsilon_e) \qquad (e \in E).$$

(Here ε_i denotes the i-th unit basis vector in $\mathbb{Q}^k$; ε_e denotes the e-th unit basis vector in $\mathbb{Q}^E$; χ^P is the *incidence vector* of P in $\mathbb{Q}^E$, i.e., $\chi^P(e)=1$ if $e \in P$ and $=0$ otherwise.)

By polarity, this last statement is equivalent to: if (4),(5) or (6) holds, then the cone of vectors $(b;\ell) \in \mathbb{Q}^k \times \mathbb{Q}^E$ defined by the linear inequalities:

(11)
$$\text{(i)}\quad b_i + \sum_{e\in P} \ell_e \geq 0 \qquad (i=1,\ldots,k;\ P\ r_i\text{-}s_i\text{-path}),$$
$$\text{(ii)}\quad \ell_e \geq 0 \qquad (e \in E),$$

is equal to the cone generated by the following vectors:

$$
\begin{array}{llll}
(12) & \text{(i)} & (-\chi^{\rho(X)}; \chi^{\delta(X)}) & (X \subseteq V), \\
 & \text{(ii)} & (\varepsilon_i; 0) & (i=1,\ldots,k), \\
 & \text{(iii)} & (0; \varepsilon_e) & (e \in E).
\end{array}
$$

Note that (11)(i) just means that $-b_i$ is a lower bound for the distance from r_i to s_i, taking ℓ as a length function. So the statement is equivalent to: if (4),(5) or (6) holds, then for any 'length function' $\ell: E \to \mathbb{Q}_+$, there exist subsets $X_1,\ldots,X_t$ of V and rationals $\mu_1,\ldots,\mu_t \geq 0$, so that

$$
\begin{array}{llll}
(13) & \text{(i)} & \sum(\mu_j \mid j=1,\ldots,t;\ i \in \rho(X_j)) \geq \text{dist}_\ell(r_i,s_i) & (i=1,\ldots,k), \\
 & \text{(ii)} & \sum(\mu_j \mid j=1,\ldots,t;\ e \in \delta(X_j)) \leq \ell_e & (e \in E).
\end{array}
$$

[Here, dist_ℓ denotes the distance, taking ℓ as length function. Note that equality in (i) can be derived from (ii).]

Now Karzanov [1985] showed that if (4) holds, and if ℓ is integral, we can take the μ_j half-integral. In fact, he showed that if ℓ is integral so that each circuit of G has an even length, we can take the μ_j integral (thus extending work of Hu [1973] and Seymour [1978]). Equivalently, if G is bipartite and (4) holds, then there exist pairwise edge-disjoint cuts $\delta(X_1),\ldots,\delta(X_t)$ so that for each $i=1,\ldots,k$, the distance from r_i to s_i is equal to the number of cuts $\delta(X_j)$ separating r_i and s_i. (The equivalence follows in one way by taking $\ell_e=1$ for each edge e, and in the other way by replacing each edge e of length ℓ_e by a path consisting of ℓ_e edges.)

The theorem to be proved in this paper is similar, but now with respect to Okamura's condition (5), instead of Lomonosov's cobdition (4). Note that, in a similar way as above, a fractional version of Okamura's theorem can be derived from our theorem.

3. PROOF OF THE THEOREM.

Suppose that the theorem is not true, and let G be a counterexample with

$$
(14) \qquad \sum_{F \neq O, I} 2^{e(F)} \text{ as small as possible,}
$$

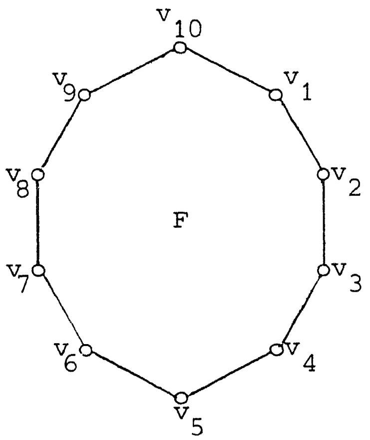
where the sum ranges over all faces $F \neq O,I$, and where $e(F)$ denotes the number of edges surrounding F. We may assume that O is the unbounded face.

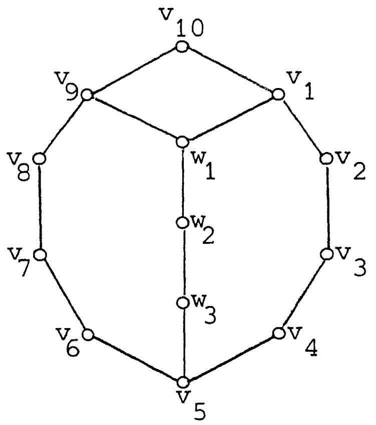
G has no multiple edges: otherwise, either the circuit C formed by them is a face, in which case we can delete one of the edges, thereby decreasing the sum (14) , or C contains edges both in its interior and in its exterior, in which case the graph formed by C and its interior or the graph formed by C and its exterior yields a counterexample with smaller sum (14).

We first show:

Claim 1. Each face $F \neq O, I$ forms a quadrangle (i.e., $e(F)=4$).

Proof of Claim 1. Let F be some face forming a k-gon, with $k \neq 4$. Since G is bipartite and has no parallel edges, k is even and $k \geq 6$. We make a counterexample with smaller sum (14) as follows. Let $v_1, \ldots, v_k$ be the vertices surrounding F. Add, in the interior of F, new vertices $w_1, \ldots, w_{\frac{1}{2}k-2}$ and new edges $\{v_1, w_1\}, \{v_{k-1}, w_1\}$, $\{w_i, w_{i+1}\}$ $(i=1, \ldots, \frac{1}{2}k-3)$ and $\{w_{\frac{1}{2}k-2}, v_{\frac{1}{2}k}\}$. E.g., for k=10,

(15) 

becomes 

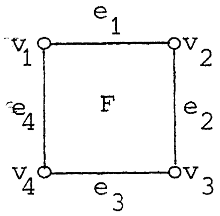
.

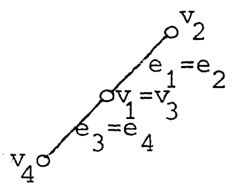
Note that this modifaction does not change the distance from v to w, for any two vertices of the original graph. Therefore, after this modification we have again a counterexample to the theorem, with, however, smaller sum (14) (since $2^k > 2^{k-2}+2^{k-2}+2^4$), contradicting our assumption. $\square$

Next we show:

Claim 2. Let F be a face, with $F \neq O, I$, and let $e_1=\{v_1, v_2\}$, $e_2=\{v_2, v_3\}$, $e_3=\{v_3, v_4\}$, $e_4=\{v_4, v_1\}$ be the four edges surrounding F. Then there exist vertices v,w, with $v,w \in O$ or $v,w \in I$, and a shortest path from v to w which uses both e_1 and e_2.

Proof of Claim 2. Suppose no such v,w exist. Identify v_1 and v_3, e_1 and e_2, and e_3 and e_4. So

(16) 

becomes 

.

After this modification, all distances between vertices v,w on O and between vertices v,w on I, are unchanged. Hence, the new graph is again a counterexample. However, the sum (14) has decreased, contradicting our assumption. $\square$

Now we define *dual paths* $Q_1,\ldots,Q_t$, i.e., paths (including circuits) in the (planar) dual graph of G. These dual paths are determined by the following properties: each edge of the graph occurs exactly once in $Q_1,\ldots,Q_t$; if $F(\neq O,I)$ is surrounded by the edges e_1,e_2,e_3,e_4 (in this order), then e_1,F,e_3 (or e_3,F,e_1) will occur in exactly one of the Q_j; the faces O and I only occur as beginning or end faces in $Q_1,\ldots,Q_t$.

More precisely, $Q_1,\ldots,Q_t$ are all sequences of form

$$(17) \qquad (F_0,e_1,F_1,e_2,\ldots,F_{k-1},e_k,F_k)$$

satisfying: (i) for i=1,...,k: e_i is an edge separating the faces F_{i-1} and F_i; (ii) for i=1,...,k-1: $F_i\notin\{O,I\}$ and e_i and e_{i+1} are opposite edges of F_i; (iii) either $F_0=F_k\notin\{O,I\}$ and e_1 and e_k are opposite edges of F_0, or $F_0,F_k\in\{O,I\}$. If $F_0=F_k\notin\{O,I\}$, we identify all possible sequences obtained from (17) by cyclically shifting it or by reversing it. If $F_0,F_k\in\{O,I\}$, we identify (17) with its reverse. Clearly, in this way the edges of G are partitioned into dual paths and circuits.

Consider now some fixed Q_g, represented by (17). Let for each i=1,...,k, v_i and w_i be vertices so that $e_i=\{v_i,w_i\}$ and so that if we would orient the edges surrounding F_i clockwise, then e_i is oriented from v_i to w_i. Then also $f_i := \{v_i,v_{i+1}\}$ and $g_i := \{w_i,w_{i+1}\}$ are edges of G (i=1,...,k-1). So

$$(18) \qquad (v_1,f_1,v_2,f_2,\ldots,v_{k-1},f_{k-1},v_k)$$

is the path along Q_g 'on the right side', and

$$(19) \qquad (w_1,g_1,w_2,g_2,\ldots,w_{k-1},g_{k-1},w_k)$$

is the path along Q_g 'on the left side'.

<u>Claim 3</u>. For all $i,j\in\{1,\ldots,k\}$: $\text{dist}(v_i,v_j) = \text{dist}(w_i,w_j)$, where dist denotes distance.

Proof of Claim 3. Suppose to the contrary that $dist(v_i,v_j) \neq dist(w_i,w_j)$ for some i,j. Choose such i,j so that $i<j$ and $|j-i|$ is as small as possible. By symmetry, we may assume that $dist(v_i,v_j) < dist(w_i,w_j)$. As G is bipartite, $j-i \geqslant dist(w_i,w_j) \geqslant dist(v_i,v_j)+2 \geqslant 2$.

Let

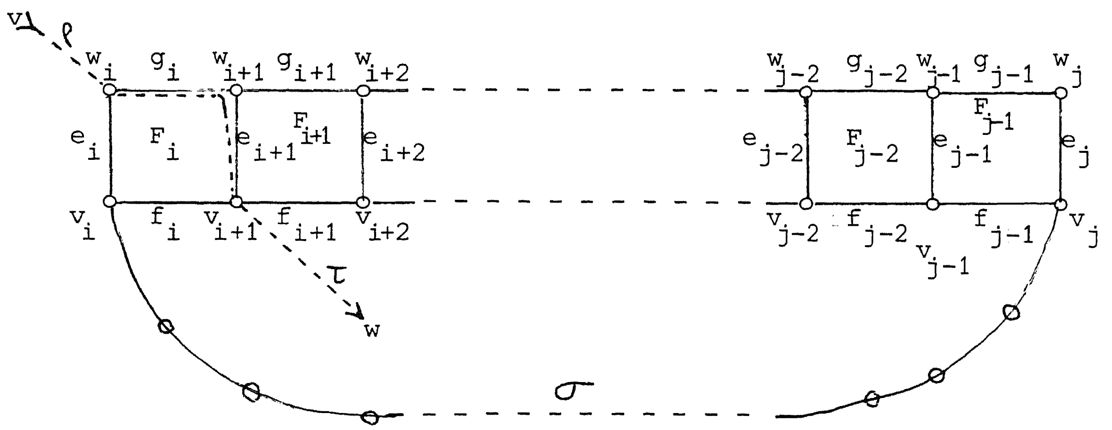
(20) (v_i,σ,v_j)

be a shortest v_i-v_j-path, for some string σ. Since $dist(w_i,w_j) \geqslant dist(v_i,v_j)+2$, it follows that

(21) $(w_i,e_i,v_i,\sigma,v_j,e_j,w_j)$

is a shortest w_i-w_j-path. Consider the circuit

(22) $C := (v_i,\sigma,v_j,f_{j-1},v_{j-1},\ldots,f_{i+1},v_{i+1},f_i,v_i)$.



C is a simple circuit, i.e., no vertex occurs twice in (22), except for the beginning and end vertex. Indeed, all vertices in (20) are distinct, as it is a shortest path. Moreover, all vertices $v_i,v_{i+1},\ldots,v_j$ are distinct, except possibly $v_i=v_j$: if $v_p=v_q$ with $i \leqslant p<q \leqslant j$, then $dist(v_p,v_q)=0 < dist(w_p,w_q)$ (since G has no parallel edges), and hence, by the minimality of $j-i$, $q-p \geqslant j-i$; that is $p=i$ and $q=j$. Suppose finally, $\sigma = (\sigma',v_q,\sigma'')$ for some strings σ',σ'' and $i+1 \leqslant q \leqslant j-1$. Then $dist(v_i,v_q)+dist(v_q,v_j) = dist(v_i,v_j)$ (as v_q is on the shortest v_i-v_j-path (20)), and hence, $dist(v_i,v_q) + dist(v_q,v_j) = dist(v_i,v_j) < dist(w_i,w_j) \leqslant dist(w_i,w_q)+dist(w_q,w_j)$. Therefore, $dist(v_i,v_q) < dist(w_i,w_q)$ or $dist(v_q,v_j) < dist(w_q,w_j)$, contradicting the minimality of $j-i$.

By Claim 2, there exist vertices v and w, either both on O or both on I, and a shortest v-w-path P with:

(23) $P = (v,\rho,w_i,e_i,v_i,f_i,v_{i+1},\tau,w)$,

where ρ and τ are strings. Hence, also the path

(24) $P' := (v,\rho,w_i,g_i,w_{i+1},e_{i+1},v_{i+1},\tau,w)$

is a shortest v-w-path. Since $O,I \notin \{F_i,\ldots,F_{j-1}\}$ (as $1 \leqslant i < j \leqslant k$), we are in one of the following three cases.

<u>Case 1</u>. (v,ρ) and (v_i,σ,v_j) have a vertex in common, say u:

(25) $(v,\rho) = (\rho',u,\rho'')$,
$(v_i,\sigma,v_j) = (\sigma',u,\sigma'')$,

for (possibly empty) strings $\rho',\rho'',\sigma',\sigma''$. Then

(26) $(\rho',u,(\sigma')^{-1},e_i,w_i,g_i,w_{i+1},e_{i+1},v_{i+1},\tau,w)$

also would be a shortest v-w-path, since (w_i,e_i,σ',u) is a shortest w_i-u-path (as it is part of (21)). But then

(27) $(\rho',u,(\sigma')^{-1},e_{i+1},v_{i+1},\tau,w)$

would be an even shorter v-w-path, which is a contradiction.

<u>Case 2</u>. (τ,w) and (v_i,σ,v_j) have a vertex in common, say u:

(28) $(\tau,w) = (\tau',u,\tau'')$,
$(v_i,\sigma,v_j) = (\sigma',u,\sigma'')$,

for (possibly empty) strings $\tau',\tau'',\sigma',\sigma''$. So

(29) $(w_i,g_i,w_{i+1},e_{i+1},v_{i+1},\tau',u)$

is not longer than

(30) (w_i,e_i,σ',u)

(since (29) is part of the shortest path P'). Hence, substituting (30) by

(29) in (21),

(31) $$(w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', u, \sigma'', v_j, e_j, w_j)$$

is a shortest w_i-w_j-path. In particular, $\text{dist}(v_{i+1}, v_j) < \text{dist}(w_{i+1}, w_j)$, contradicting the minimality of j-i.

<u>Case 3</u>. $(\tau, w) = (\tau', v_p, e_p, w_p, \tau'')$ for some p with $i+1 \leq p \leq j-1$, and certain (possibly empty) strings τ', τ''. Substitution in P gives:

(32) $$P = (v, \rho, w_i, g_i, w_{i+1}, e_{i+1}, v_{i+1}, \tau', v_p, e_p, w_p, \tau'').$$

Since P is a shortest v-w-path, it follows that $\text{dist}(v_{i+1}, v_p) < \text{dist}(w_{i+1}, w_p)$, contradicting the minimality of j-i. $\square$

A consequence of Claim 3 is that Q_g will have no self-intersections: if $F_i = F_j$ with $i \neq j$, then $v_i = v_{j+1}, w_i \neq w_{j+1}$, or $v_{i+1} = v_j, w_{i+1} \neq w_j$, as one easily checks. This contradicts Claim 3.

Next we show:

<u>Claim 4</u>. Each Q_g connects O and I.

<u>Proof of Claim 4</u>. Suppose Q_g does not connect O and I, for some g=1,...,t. Then either Q_g connects O with O, or connects I with I, or is a circuit. That is, the edges in G form a cut $\delta(X)$, for some $X \subseteq V$.

I. We first show: for each $v, w \in V$: for each v-w-path P there exists a v-w-path P' so that:

(33) $$\text{length}(P') - \text{int}(P', Q_g) \leq \text{length}(P) - \text{int}(P, Q_g), \text{ and}$$
$$\text{int}(P', Q_g) \leq 1,$$

where $(\text{int}(.., Q_g)$ denotes the number of edges in .. in common with Q_g.

This is shown by induction on length(P). If $\text{int}(P, Q_g) \geq 2$, there exist i,j so that $P = (\rho, v_i, e_i, w_i, \sigma, w_j, e_j, v_j, \tau)$ for strings ρ, σ, τ, where σ does not have any edge in common with Q_g (we use the notation introduced before Claim 3; maybe v_i, v_j and w_i, w_j are interchanged). Since by Claim 3, $\text{dist}(w_i, w_j) = \text{dist}(v_i, v_j)$, there exists a path $\tilde{P}$ with $\text{length}(\tilde{P}) \leq \text{length}(P) - 2$ and $\text{int}(\tilde{P}, Q_g) \geq \text{int}(P, Q_g) - 2$. Applying the induction hypothesis to $\tilde{P}$, implies the statement above.

II. Now contract all edges occurring in Q_g. This gives a smaller graph G'. For the new distance function dist' in G' we have:

(34) $\quad \text{dist}'(v,w)=\text{dist}(v,w)-1,\quad$ if X separates v and w,
$\quad \text{dist}'(v,w)=\text{dist}(v,w),\quad$ otherwise.

To see this, it suffices to show that $\text{dist}'(v,w)\geqslant \text{dist}(v,w)-1$ for all v,w (by the bipartiteness of G and G'). Let Π be a shortest v-w-path in G'. It corresponds to a v-w-path P in G with $\text{length}(P)-\text{int}(P,Q_g) = \text{length}(\Pi)$. Hence, by I above, there exists a v-w-path P' in G so that $\text{length}(P') - \text{int}(P',Q_g)\leqslant \text{length}(\Pi)$ and $\text{int}(P',Q_g)\leqslant 1$. Hence, $\text{dist}(v,w)\leqslant \text{length}(P')\leqslant$ $\leqslant \text{length}(\Pi)+1 = \text{dist}'(v,w)+1$.

By induction, in G' there exist cuts $\delta(X_1),\ldots,\delta(X_{t'})$ so that for all pairs of vertices v,w, both on O or both on I:

(35) $\quad \text{dist}'(v,w) = \left|\{i=1,\ldots,t' \mid X_i \text{ separates } v \text{ and } w\}\right|.$

So by (34), taking $X_{t'+1}:=X$, in G we have for all such v,w:

(36) $\quad \text{dist}(v,w) = \left|\{i=1,\ldots,t'+1 \mid X_i \text{ separates } v \text{ and } w\}\right|.$

As $\delta(X_1),\ldots,\delta(X_{t'+1})$ are pairwise disjoint, G is not a counterexample to the theorem, contradicting our assumption. $\square$

Our final claim will finish off the counterexample:

<u>Claim 5</u>. No two distinct Q_i and Q_j have a face $F\neq O,I$ in common.

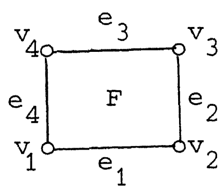
<u>Proof of Claim 5</u>. Suppose to the contrary

(37) $\quad Q_i = (O,\sigma,F,\varphi),$
$\quad Q_j = (O,\tau,F,\psi),$

for strings σ,φ,τ,ψ and face $F\neq O,I$ $(i\neq j)$. We may assume that (σ,F) and (τ,F) do not have any other face in common than F (by taking (37) so that σ and τ have minimal length).

Consider the face F:

(38)



We may assume that e_1 is the last symbol of σ, and that e_2 is the last symbol of τ. By Claim 2, there exist vertices v,w, both on O or both on I, and a shortest v-w-path P using e_2 and e_3:

(39) $$P = (v,\pi,v_2,e_2,v_3,e_3,v_4,\rho,w).$$

As P is a shortest v-w-path, with $v,w \in O$ or $v,w \in I$, P has at most one edge in common which each of the Q_g (g=1,...,k) (by Claim 3). Since P crosses both Q_i and Q_j at F, while the vertex v_2 is contained in the set of vertices contained in the circuit (O,σ,F,τ^{-1},O), P should have also its beginning vertex v inside of this circuit. So v is on O, and hence also w is on O.

Since P has exactly one edge in common with Q_i, it follows that P is homotopic (in the space obtained from the euclidean plane by deleting the interiors of O and I) to the v-w-path P' which follows the boundary of O and which contains the first edge of Q_i. Similarly, P is homotopic to the v-w-path P" which follows the boundary of O and which contains the first edge of Q_j. Since v is inside of the circuit (O,σ,F,τ^{-1},O), while w is outside of it, P' is not homotopic to P", a contradiction. □

Claim 5 implies that there are no faces other than O and I (any other face would belong to two different Q_i and Q_j). So G is a simple circuit, for which the theorem trivially holds. □□□

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