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ASYMPTOTIC EXPANSIONS FOR MODIFIED BESSEL FUNCTIONS OF LARGE ORDER

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In this report asymptotic expansions for modified Bessel functions of large order will be derived by applying the saddle-point method to well-known integral representations for $K_\nu(x)$ and $I_\nu(x)$, in case $\nu > 0$ and $x \geq 0$. Moreover, error bounds for the remainders will be given. Finally, a comparison is made between these error bounds and the real errors that are made by approximating the Bessel functions by parts of their asymptotic series.

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1. INTRODUCTION

In this report, we consider asymptotic expansions of the modified Bessel functions $K_\nu(x)$ and $I_\nu(x)$ for $\nu > 0$ and $x \geq 0$. These functions are solutions of the second-order differential equation

$$(1.1) \qquad x^2 \frac{d^2w}{dx^2} + x\frac{dw}{dx} - (x^2+\nu^2)\, w = 0.$$

There exists a simple relation between the two functions, namely

$$(1.2) \qquad K_\nu(x) = \tfrac{1}{2}\pi \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)},$$

where the right-hand side of the equation is replaced by its limiting value if ν is an integer or zero. Several asymptotic expansions for $K_\nu(x)$ and $I_\nu(x)$ are known. For instance,

$$(1.3) \qquad I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}\{1 - \frac{\mu-1}{8x} + \frac{(\mu-1)(\mu-9)}{2!(8x)^2} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8x)^3} + \ldots\},$$

ν fixed, $|x|$ large and $|\arg x| < \frac{1}{2}\pi$, $\mu = 4\nu^2$,

and

$$(1.4) \qquad K_\nu(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\{1 + \frac{\mu-1}{8x} + \frac{(\mu-1)(\mu-9)}{2!(8x)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8x)^3} + \ldots\},$$

ν fixed, $|x|$ large and $|\arg x| < \frac{3}{2}\pi$, $\mu = 4\nu^2$.

We are especially interested in asymptotic expansions for Bessel functions of large order. Our starting-point is an article of LUKE [1]. He started with the following asymptotic expansions

$$(1.5) \qquad K_\nu(\nu z) = \left(\frac{\pi u}{2\nu}\right)^{\frac{1}{2}} e^{-\nu\eta}\, V(\nu,u)$$

with

$$(1.6) \qquad V(\nu,u) \sim \sum_{k=0}^{\infty} (-1)^k \frac{U_k(u)}{\nu^k}, \quad |\nu| \to \infty, \ \varepsilon - \frac{\pi}{2} \leq \arg \nu \leq \frac{3\pi}{2} - \varepsilon, \ \varepsilon > 0,$$

uniformly in z, $\delta - \frac{3\pi}{2} \leq \arg z \leq \frac{\pi}{2} - \delta$, $\delta > 0$,

and

$$(1.7) \qquad I_\nu(\nu z) = (u/2\pi\nu)^{\frac{1}{2}} e^{\nu\eta}\, S(\nu,u),$$

with

(1.8) $$S(\nu,u) \sim \sum_{k=0}^{\infty} \frac{U_k(u)}{\nu^k}, \quad |\nu| \to \infty, \ |\arg \nu| \le \frac{\pi}{2} - \delta, \ \delta > 0, \text{ uniformly in } z,$$

$$|\arg z| \le \frac{\pi}{2} - \varepsilon, \ \varepsilon > 0,$$

where

(1.9) $$\eta = u^{-1} + \ell n \left(\frac{uz}{u+1}\right), \quad u = (1+z^2)^{-\frac{1}{2}},$$

and

(1.10) $$U_k = U_k(u) = u^k \sum_{r=0}^{k} a_{r,k} u^{2r}, \quad U_0 = 1.$$

These expansions were deduced earlier by OLVER [4] by a differential equation approach. The results, however, were already known by Debye. The terms U_k satisfy a difference-differential-integral equation, which is hard to deal with numerically. To avoid this problem, Luke rearranged the series S and V. He deduced a differential equation for V by taking the derivative with respect to η. Next he substituted a formal series

(1.11) $$V = \sum_{k=0}^{\infty} c_k(\nu) u^k, \quad c_0(\nu) = 1$$

in this equation. The differential equation was formally solved when the coefficients c_k satisfied the recurrence relation

(1.12) $$8\nu k c_k = -(2k-1)^2 c_{k-1} + (2k-1)(2k-5) c_{k-3}, \ c_0 = 1, \ c_k = 0, \ k < 0.$$

In this way he found a rather simple asymptotic expansion for $K_\nu(\nu z)$:

(1.13) $$K_\nu(\nu z) \sim \left(\frac{\pi u}{2\nu}\right)^{\frac{1}{2}} e^{-\nu\eta} \sum_{k=0}^{\infty} c_k(\nu) u^k, \ |z| \to \infty, \ |\arg z| \le \frac{\pi}{2} - \varepsilon, \ \varepsilon > 0,$$

uniformly for all ν, $|\nu| > \nu_0 > 0$, ν_0 fixed but arbitrary, $|\arg \nu| \le \frac{\pi}{2} - \delta$, $\delta > 0$, with u, η given in (1.9).
Analogously, he found

(1.14) $$S = \sum_{k=0}^{\infty} (-1)^k c_k(\nu) u^k, \quad c_0(\nu) = 1$$

and

$$(1.15) \qquad I_\nu(\nu z) \sim [u/(2\pi\nu)]^{\frac{1}{2}} e^{\nu\eta} \sum_{k=0}^{\infty} (-1)^k c_k(\nu) u^k, \quad |z| \to \infty,$$

$$\varepsilon - \frac{3\pi}{2} \le \arg z \le \frac{\pi}{2} - \varepsilon,\ \varepsilon > 0, \text{ uniformly for all } \nu,\ |\nu| > \nu_0 > 0,$$

$$\nu_0 \text{ fixed but arbitrary},\ \delta - \frac{\pi}{2} \le \arg \nu \le \frac{3\pi}{2} - \delta,\ \delta > 0.$$

In the next chapters we will derive the asymptotic series (1.13) and (1.15), starting from integral representations for $K_\nu(x)$ and $I_\nu(x)$, respectively. But, as we said before, we restrict ourselves to $\nu>0$ and $z\ge0$. Moreover, we will give estimates for error bounds.

In chapter 2, analytical aspects of our method will be considered, whereas in chapter 3 numerical aspects in relation to error bounds for the remainders are worked out. Moreover, we will pay attention to the usefulness of the asymptotic series for approximating $K_\nu(\nu z)$ and $I_\nu(\nu z)$ in this chapter. In chapter 4, we look at some aspects connected with the coefficients c_n and the related coefficients a_{2n}, which satisfy the recurrence relation

$$(1.16) \qquad a_{2n} = \{a_{2n-6} - (2n-1)(2n-3)a_{2n-2}\}/(8\nu n(2n-3)),$$

$$a_0 = 1,\ a_{2n} = 0,\ n < 0.$$

<u>REMARK</u>. It turns out that the asymptotic expansions (1.13) and (1.15) are not uniform in ν. At least, we are not able to prove that they are, because the error bounds we use ((3.4) and (3.18)) contain parameters (σ_n and τ_n, respectively) that are not bounded in ν, i.e. not bounded in ν for all values of n. For instance, $\sigma_2(\nu) \sim 1.18\nu$, but $\sigma_3(\nu) = 1$ for all ν. It's a pity that Luke derives his results in a formal way and that no error bounds for the remainders are given in his article. Therefore it is hard to say whether Luke's conclusions with respect to the uniformity in ν of the asymptotic expansions are defensible.

2. ANALYTICAL DERIVATION OF ASYMPTOTIC SERIES FOR $K_\nu(x)$ AND $I_\nu(x)$ FOR LARGE VALUES OF ν

Our aim is to derive asymptotic expansions for $K_\nu(x)$ and $I_\nu(x)$ for large values of ν. As we already said in chapter 1, we restrict ourselves to $\nu > 0$, $x \geq 0$. We will derive asymptotic series for $K_\nu(x)$ and $I_\nu(x)$, which are valid for $x \to \infty$ and $\nu \geq \nu_0 > 0$. To this end, we introduce the variable

$$\lambda = \sqrt{1 + x^2/\nu^2} = \sqrt{1+z^2}. \tag{2.1}$$

Observe that λ is equal to u^{-1} as defined in (1.9).

We start from the well-known integral representations

$$K_\nu(x) = \int_0^\infty e^{-x\operatorname{ch}t}\operatorname{ch}(\nu t)dt, \tag{2.2}$$

and

$$I_\nu(x) = \frac{1}{2\pi i}\int_C e^{x\operatorname{ch}t-\nu t}dt, \tag{2.3}$$

where C is a contour as drawn in figure 1.

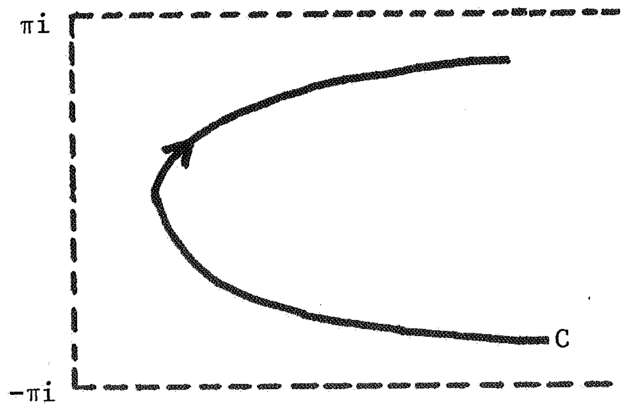

Figure 1. Contour for (2.3)

To obtain asymptotic expansions of these integrals, we apply the saddle-point method. It will turn out that the asymptotic expansions for $\lambda \to \infty$ resulting from this method are equivalent to (1.13) and (1.15), respectively. Details of the derivation are given in §2.1 for $K_\nu(x)$ and in §2.2 for $I_\nu(x)$.

2.1 The function $K_\nu(x)$

Write (2.2) as

$$K_\nu(x) = \tfrac{1}{2}\int_0^\infty e^{-xcht+\nu t}\,dt + \tfrac{1}{2}\int_0^\infty e^{-xcht-\nu t}\,dt.$$

Changing t to -t in the second integral yields

$$(2.4) \qquad K_\nu(x) = \tfrac{1}{2}\int_{-\infty}^\infty e^{-xcht+\nu t}\,dt.$$

Define

$$(2.5) \qquad \phi(t) = xcht - \nu t.$$

Then $\phi'(t) = xsht - \nu$ and $\phi''(t) = xcht$. It follows that the integrand is maximal at the point $t = t_0 = \text{arcsh}\,(\nu/x) = \text{arcsh}\,(1/z)$. Therefore we write

$$K_\nu(x) = \tfrac{1}{2}e^{-\phi(t_0)}\int_{-\infty}^\infty e^{-[\phi(t)-\phi(t_0)]}\,dt.$$

For reasons of symmetry we define

$$(2.6) \qquad \tau = t - t_0.$$

Further, let

$$(2.7) \qquad \eta = \phi(t_0)\,/\,\nu\,.$$

Then simple calculations yield the representation

$$(2.8) \qquad K_\nu(x) = \tfrac{1}{2}e^{-\nu\eta}\int_{-\infty}^\infty \exp\,[-\nu(2\lambda sh^2(\tau/2) + sh\,\tau - \tau)]\,dt.$$

With

$$(2.9) \qquad y = 2\sqrt{\nu}\;sh(\tau/2)$$

we can write (2.8) as

$$(2.10)\qquad K_\nu(x) = \frac{1}{2\sqrt{\nu}}\, e^{-\nu\eta} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda y^2} f(y)dy,$$

where

$$f(y) = \sqrt{\nu}\,\frac{d\tau}{dy}\, e^{-\nu(\mathrm{sh}\tau-\tau)},\quad \frac{d\tau}{dy} = \frac{1}{\sqrt{\nu+\frac{1}{4}y^2}}\,.$$

It follows that

$$(2.11)\qquad f(y) = \frac{1}{\sqrt{1+y^2/4\nu}}\,\exp\,[-\nu(\mathrm{sh}\tau-\tau)].$$

Observe that η defined in (2.7) is the same as that of (1.9).
To apply Laplace's method on (2.10), we want to know the Taylor series of f for small values of y, that is , for $|y|< 2\sqrt{\nu}$. Straightforward calculation is quite cumbersome, due to the difficult relation between y and τ. It is much more attractive to derive a differential equation for f and to search for a formal solution. In appendix A, we prove that f satisfies the differential equation

$$(2.12)\qquad y(4\nu+y^2)\, f'' + (y^2-8\nu)\, f' - (y^4+1)\, yf = 0.$$

Substitute into (2.12) the formal series

$$f(y) = \sum_{n=0}^{\infty} a_n y^n \text{ with } a_0 = f(0) = 1.$$

Equating coefficients yields successively
$a_1 = 0$, $a_2 = -\frac{1}{8\nu}$, a_3 is indefinite, $a_4 = \frac{3}{128\nu^2}$, $a_5 = -\frac{1}{5\nu}\,a_3$, and, for $n \geq 5$,

$$(2.13)\qquad -a_{n-5} + n(n-2)\, a_{n-1} + 4\nu(n-2)(n+1)a_{n+1} = 0.$$

For every choice of a_3, we have a regular solution of (2.12). In fact, $a_3 = f^{(3)}(0)/3!$, but it is obvious to choose $a_3 = 0$. For we use the Taylor series to derive an asymptotic expansion for $K_\nu(x)$ and terms like $a_{2n+1}y^{2n+1}$ lead to integrals of the form

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda y^2}\, y^{2n+1} dy = 0.$$

The consequence of the choice $a_3 = 0$ is that all odd coefficients a_{2k+1} vanish. Henceforth we consider the even part of f. For simplicity we call this f again. So

$$(2.14) \qquad f(y) = \sum_{n=0}^{\infty} a_{2n} y^{2n} = \frac{1}{\sqrt{1+y^2/4\nu}} \operatorname{ch}(\nu(\operatorname{sh}\tau - \tau)),$$

where the coefficients a_{2n} satisfy the recurrence relation

$$(2.15) \qquad 8\nu n(2n-3)\, a_{2n} + (2n-1)(2n-3) a_{2n-2} - a_{2n-6} = 0$$

with initial conditions

$$(2.16) \qquad a_0 = 1,\ a_{2n} = 0,\ n < 0.$$

The asymptotic expansion for $K_\nu(x)$ as $\lambda \to \infty$ follows immediately by substituting (2.14) into (2.10):

$$(2.17) \qquad K_\nu(x) \sim \frac{1}{2\sqrt{\nu}}\, e^{-\nu\eta} \sum_{n=0}^{\infty} a_{2n} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda y^2} y^{2n} dy$$

$$= \frac{1}{2\sqrt{\nu}}\, e^{-\nu\eta} \sum_{n=0}^{\infty} a_{2n}\, \Gamma(n+\tfrac{1}{2}) (\tfrac{1}{2}\lambda)^{-n-\frac{1}{2}}$$

$$= \left(\frac{\pi}{2\lambda\nu}\right)^{\frac{1}{2}} e^{-\nu\eta} \sum_{n=0}^{\infty} c_n \lambda^{-n}, \qquad \lambda \to \infty,$$

with $c_n = a_{2n} \dfrac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} 2^n$.

From (2.15) we derive, by straightforward calculation, a recurrence relation for the coefficients c_n:

$$(2.18) \qquad 8\nu n c_n + (2n-1)^2 c_{n-1} - (2n-1)(2n-5) c_{n-3} = 0$$

with initial conditions

$$(2.19) \qquad c_0 = 1,\ c_n = 0,\ n < 0.$$

The first coefficients are $c_0 = 1$, $c_1 = -\frac{1}{8\nu}$, $c_2 = \frac{9}{128\nu^2}$.

The formulas (2.17) and (2.18) correspond to (1.13) and (1.12), respectively. So we derived the results of Luke in quite a simple way.

2.2 The function $I_\nu(x)$

We start from (2.3). Analogously to the former case we write, by using (2.6) and (2.7),

$$(2.20) \qquad I_\nu(x) = \frac{e^{\nu\eta}}{2\pi i} \int_C \exp[\nu(2\lambda \operatorname{sh}^2(\tau/2) + \operatorname{sh}\tau - \tau)]\, d\tau = \frac{e^{\nu\eta}}{2\pi i} \int_C g(\nu,\lambda,\tau)\, d\tau,$$

with $\qquad g(\nu,\lambda,\tau) = \exp[\nu(2\lambda \operatorname{sh}^2(\tau/2) + \operatorname{sh}\tau - \tau)]$.

An analysis of the function ϕ of (2.5) for complex values of t shows that $\phi'(t)$ has zeros at the points

$$t = (-1)^k \operatorname{arcsh}(\tfrac{\nu}{x}) + k\pi i = (-1)^k t_0 + k\pi i, \quad k \in \mathbb{Z}.$$

The only saddle-points we have to deal with are $t_{-1} = -t_0 - \pi i$, t_0, and, $t_1 = -t_0 + \pi i$. Calculations show that the lines of steepest descent are $\operatorname{Im} \tau = -\pi$, $\operatorname{Re} \tau = 0$ and $\operatorname{Im} \tau = \pi$, respectively. Further, the points $\tau = \pm \pi i$ are singular points of the function f of (2.11). Therefore we choose the contour L as drawn in figure 2.

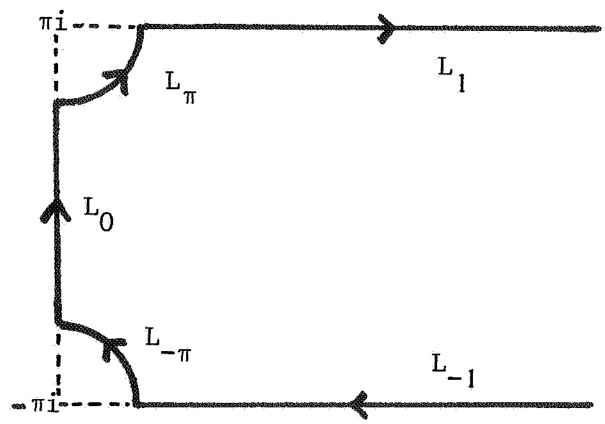

Figure 2. Contour for (2.20).

This contour L consists of 5 parts, viz.

(i) L_{-1} : $\text{Im}\,\tau = -\pi,\ \delta < \text{Re}\,\tau < \infty$

(ii) $L_{-\pi}$: $\tau = -\pi i + \delta e^{i\phi},\ 0 \le \phi \le \frac{\pi}{2}$

(iii) L_0 : $\text{Re}\,\tau = 0,\ -\pi + \delta \le \text{Im}\,\tau \le \pi - \delta$

(iv) L_π : $\tau = \pi i + \delta e^{i\phi},\ -\frac{\pi}{2} \le \phi \le 0$

(v) L_1 : $\text{Im}\,\tau = \pi,\ \delta < \text{Re}\,\tau < \infty$,

with $0 < \delta < \pi$ arbitrary. Let us look at these parts more specifically.

(i) Substitute $\tau = -\pi i + \sigma$, then $\text{sh}\,\tau = -\text{sh}\,\sigma$, $\text{sh}(\tau/2) = -i\text{ch}(\frac{1}{2}\sigma)$, and

(2.21) $$\frac{e^{\nu\eta}}{2\pi i}\int_{L_{-1}} g(\nu,\lambda,\tau)d\tau = \frac{e^{\nu\eta}}{2\pi i}\int_{\infty}^{\delta} \exp[\nu(-2\lambda\text{ch}^2(\tfrac{1}{2}\sigma) - \text{sh}\,\sigma - \sigma + \pi i)]d\sigma.$$

(ii) Let $\tau = -\pi i + \delta e^{i\phi}$, $0 \le \phi \le \frac{\pi}{2}$, $0 < \delta \le \frac{\pi}{4}$ be arbitrary. In appendix B, we prove that

$$\left|\int_{L_{-\pi}} g(\nu,\lambda,\tau)d\tau\right| = O(e^{-\lambda\nu}),\quad \lambda > 0.$$

(iii) Substitute $\tau = it$, then $\text{sh}(\tau/2) = i\sin(t/2)$, and

(2.22) $$\frac{e^{\nu\eta}}{2\pi i}\int_{L_0} g(\nu,\lambda,\tau)d\tau = \frac{e^{\nu\eta}}{2\pi}\int_{-\pi+\delta}^{\pi-\delta} \exp[\nu(-2\lambda\sin^2(t/2) + i(\sin t - t)]dt.$$

(iv) Analogously to (ii), this yields

$$\left|\int_{L_\pi} g(\nu,\lambda,\tau)d\tau\right| = O(e^{-\lambda\nu}),\quad \lambda > 0.$$

(v) Let $\tau = \pi i + \sigma$, then

(2.23) $$\frac{e^{\nu\eta}}{2\pi i}\int_{L_1} g\,(\nu,\lambda,\tau)d\tau = \frac{e^{\nu\eta}}{2\pi i}\int_{\delta}^{\infty} \exp[\nu(-2\lambda\text{ch}^2(\tfrac{1}{2}\sigma) - \text{sh}\sigma - \sigma - \pi i)]d\sigma.$$

Taking together (2.21) and (2.23) we get from this analysis that

(2.24) $$I_\nu(x) = \frac{e^{\nu\eta}}{2\pi i}\{i\int_{-\pi+\delta}^{\pi-\delta} \exp[\nu(-2\lambda\sin^2(t/2)+i(\sin t - t)]\,dt$$

$$-2i\sin(\nu\pi)\int_{\delta}^{\infty}\exp[\nu(-2\lambda \mathrm{ch}^2(\tfrac{1}{2}\sigma) - \mathrm{sh}\,\sigma - \sigma)]\,d\sigma$$

$$+\,\mathcal{O}(e^{-\lambda\nu})\}, \qquad \lambda > 0.$$

In appendix C, we prove that

$$\int_{\delta}^{\infty}\exp[\nu(-2\lambda \mathrm{ch}^2(\tfrac{1}{2}\sigma) - \mathrm{sh}\,\sigma - \sigma)]\,d\sigma = \mathcal{O}(e^{-\lambda\nu}),\ \lambda > 0.$$

Further, we choose

(2.25) $$\delta = \pi/4.$$

This yields

(2.26) $$I_\nu(x) = \frac{e^{\nu\eta}}{2\pi}\left\{\int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}\exp[\nu(-2\lambda\sin^2(t/2)+i(\sin t - t)]dt + \mathcal{O}(e^{-\lambda\nu})\right\}, \quad \lambda > 0.$$

Into the integral, we substitute

(2.27) $$y = 2\sqrt{\nu}\sin(t/2).$$

Then we have

$$\frac{dy}{dt} = \sqrt{\nu}\cos(t/2) = \sqrt{\nu}\sqrt{1-y^2/4\nu}$$

and

(2.28) $$I_\nu(x) = \frac{e^{\nu\eta}}{2\pi}\left\{\int_{-2\sqrt{\nu}\cos(\pi/8)}^{2\sqrt{\nu}\cos(\pi/8)} e^{-\frac{1}{2}\lambda y^2} g(y)\,dy + \mathcal{O}(e^{-\lambda\nu})\right\}, \quad \lambda > 0,$$

with

(2.29) $$g(y) = e^{\nu i(\sin t - t)}\frac{dt}{dy} = e^{\nu i(\sin t - t)}\frac{1}{\sqrt{\nu}\sqrt{1-y^2/4\nu}}.$$

Comparing (2.9), (2.11) to (2.27), (2.29) yields

$$g(y) = f(-iy)/\sqrt{\nu}.$$

So finally we have

$$(2.30) \qquad I_\nu(x) = \frac{e^{\nu\eta}}{2\pi\sqrt{\nu}}\left\{\int_{-2\sqrt{\nu}\cos(\pi/8)}^{2\sqrt{\nu}\cos(\pi/8)} e^{-\frac{1}{2}\lambda y^2} f(-iy)dy + O(e^{-\lambda\nu})\right\}, \ \lambda > 0.$$

Analogously to the former case we replace f by its even part, so

$$(2.31) \qquad f(-iy) = \sum_{n=0}^{\infty} (-1)^n a_{2n} y^{2n}.$$

The asymptotic expansion for $I_\nu(x)$ as $\lambda \to \infty$ is given by

$$(2.32) \qquad I_\nu(x) \sim \frac{e^{\nu\eta}}{2\pi\sqrt{\nu}} \sum_{n=0}^{\infty} (-1)^n a_{2n} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda y^2} y^{2n} dy$$

$$= \frac{e^{\nu\eta}}{\sqrt{2\pi\lambda\nu}} \sum_{n=0}^{\infty} (-1)^n c_n \lambda^{-n}, \qquad \lambda \to \infty.$$

The formula (2.32) corresponds to (1.15).

REMARKS. (i) We emphasize that we derived the asymptotic expansions for $\nu > 0$ and $x \geq 0$, whereas the expansions given in (1.13) and (1.15) have a much larger domain of validity.

(ii) In chapter 4 we discuss numerical aspects of the recurrence relation (2.18), especially whether it is stable or not. Luke doesn't pay attention to such questions.

3. NUMERICAL ASPECTS

In this chapter, we first discuss error bounds for the remainders and thereafter we compare these error bounds with the real errors that are made by approximating the modified Bessel functions by the first terms of their respective asymptotic series. The function $K_\nu(x)$ is considered in §3.1 and $I_\nu(x)$ in §3.2.

3.1 The function $K_\nu(x)$

We define

$$(3.1.\quad \begin{cases} R_0(y) = f(y) \\ f(y) = \sum_{s=0}^{n-1} a_{2s}\, y^{2s} + a_{2n} y^{2n} R_n(y)\,,\ n > 0, \end{cases}$$

provided that $a_{2n} \neq 0$. So

$$R_n(y) = \frac{f(y) - \sum_{s=0}^{n-1} a_{2s}\, y^{2s}}{a_{2n} y^{2n}}\,,\quad R_n(0) = 1.$$

Next, we write

$$(3.2)\qquad 2\sqrt{\nu}\, e^{\nu\eta}\, K_\nu(x) = \sum_{s=0}^{n-1} a_{2s} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda y^2}\, y^{2s}\, dy + E_n(x,\nu).$$

Then (3.2) yields, with (2.10) and (3.1),

$$(3.3)\qquad E_n(x,\nu) = a_{2n} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda y^2}\, y^{2n} R_n(y)\, dy.$$

In §3.1.1 we prove that there always is an upper bound of the form

$$(3.4)\qquad |R_n(y)| \le e^{\frac{1}{2}\sigma_n y^2}\,,\quad y \in \mathbb{R}\,,\ (n \ge 0),\ \text{where } \sigma_n \text{ doesn't depend on } \lambda.$$

Substituting (3.4) into (3.3) we get

$$(3.5)\qquad |E_n(x,\nu)| \le |c_n| \left(\frac{2\pi}{\lambda-\sigma_n}\right)^{\frac{1}{2}} (\lambda-\sigma_n)^{-n}\,,\quad \lambda > \sigma_n.$$

From (3.5) and (3.2), we draw the conclusion

$$(3.6)\qquad K_\nu(x) = \left(\frac{\pi}{2\lambda\nu}\right)^{\frac{1}{2}} e^{-\nu\eta} \left\{ \sum_{s=0}^{n-1} c_s\, \lambda^{-s} + O(\lambda^{-n}) \right\}\,,\quad \lambda \to \infty\,,\quad n = 0, 1, 2, \ldots\,.$$

When σ_n does not depend on ν, we have in (3.6) an expansion which is uniformly valid with respect to $\nu, \nu \ge \nu_0 > 0$. In the next subsection it will turn out, however, that, for some values of n, σ_n is heavily dependent on ν.

In §3.1.1 we prove (3.4) and we consider some aspects of the calculation of the σ_n's in (3.6). In §3.1.2 we compare the error bound $\frac{1}{2\sqrt{\nu}} e^{-\nu\eta}|E_n(x,\nu)|$ to the real error that is made by approximating $K_\nu(x)$ by the first n terms of its asymptotic series.

3.1.1 Upper bounds for $|R_n(y)|$

In this subsection we will show that there exists an upper bound for $|R_n(y)|$ like (3.4). Further we consider some aspects of the numerical evaluation of σ_n. We give some results in table 1 and (for a slightly modified bound) in table 2.

We start with (2.14) and (3.1). On a compact interval it is trivial that there exists an upper bound of the form $e^{\frac{1}{2}\sigma_n y^2}$.
Moreover,

$$\lim_{y\to\infty} \frac{\ell n|f(y)|}{\frac{1}{2}y^2} = 1.$$

The algebraic terms in $R_n(y)$ can't disturb the convergence, so

$$\text{(3.7)} \qquad \lim_{y\to\infty} \frac{\ell n|R_n(y)|}{\frac{1}{2}y^2} = 1.$$

From these considerations we conclude that there is an upper bound for $|R_n(y)|$ like (3.4).

For n = 0 we can derive analytically that

$$\text{(3.8)} \qquad |R_0(y)| = |f(y)| \le e^{\frac{1}{2}\sqrt{17}\,y^2}.$$

PROOF. Let $\nu > 0$ be arbitrary. We prove (3.8) by cutting the real axis into pieces. We make use of the formulas

$$\text{(2.9)} \qquad y = 2\sqrt{\nu}\,\mathrm{sh}(\tau/2)$$

and

$$\text{(2.14)} \qquad f(y) = \frac{1}{\sqrt{1+y^2/4\nu}}\,\mathrm{ch}(\nu(\mathrm{sh}\,\tau-\tau)).$$

(i) Let $|y| \le \frac{1}{2}\sqrt{\nu}$,

then $|\tau| = 2 \text{ arcsh}\left(\frac{|y|}{2\sqrt{\nu}}\right) \le 2 \text{ arcsh } (\frac{1}{4}) < \frac{1}{2}.$

From [3], p.88, formula (4.6.31) we know

$$\text{arcsh } z = z - \frac{1}{2.3} z^3 - \frac{1.3}{2.4.5} z^5 + \frac{1.3.5}{2.4.6.7} z^7 + \dots , \quad |z| < 1,$$

so

$$\text{arcsh } |z| \le |z| + |z|^3 + |z|^5 + \dots = \frac{|z|}{1-|z|^2}, \quad |z| < 1.$$

This yields

$$|\tau| \le 2 \left|\frac{y}{2\sqrt{\nu}}\right| \frac{1}{1-\left|\frac{y}{2\sqrt{\nu}}\right|^2} \le \frac{16}{15} \frac{|y|}{\sqrt{\nu}},$$

and

$$|\tau|^3 \le (\frac{16}{15})^3 \frac{1}{\nu\sqrt{\nu}} |y|^3.$$

Further $\text{sh } s = \sum_{\ell=0}^{\infty} \frac{s^{2\ell+1}}{(2\ell+1)!}, \quad |s| < \infty,$

so

$$|\text{sh } s - s| \le \frac{|s|^3}{3!} + \frac{|s|^5}{5!} + \dots \le \frac{1}{3!} \sum_{\ell=1}^{\infty} |s|^{2\ell+1} = \frac{1}{3!} \frac{|s|^3}{1-|s|^2}$$

$$\le \frac{2}{9} |s|^3,$$

if $|s| < \frac{1}{2}$.

With the help of the above results we get

$$|R_0(y)| = |f(y)|$$
$$\le \text{ch}\,(\nu\,\text{sh}\,\tau - \nu\tau)$$
$$\le \text{ch}\,(\tfrac{2}{9}\,\nu|\tau|^3)$$
$$\le \text{ch}\,(\tfrac{2}{9}\,\nu\,\frac{1}{\nu\sqrt{\nu}}\,|y|^3(\tfrac{16}{15})^3)$$
$$\le \exp(\tfrac{2}{9}(\tfrac{16}{15})^3\,\frac{1}{\sqrt{\nu}}\,|y|^3)$$
$$\le \exp(\tfrac{1}{2}.\,\tfrac{2}{9}(\tfrac{16}{15})^3\,y^2).$$

(ii) Let

$$|y| \ge \tfrac{1}{2}\sqrt{\nu}\,,$$

then

$$|f(y)| \le \text{ch}(\nu\,\text{sh}\,\tau) \le e^{\nu\,\text{sh}\,\tau} \le e^{\frac{1}{2}\sqrt{17}\,y^2}.$$

For

$$\text{sh}\,\tau = 2\text{sh}(\tfrac{1}{2}\tau)\,\text{ch}(\tfrac{1}{2}\tau) = 2\,\frac{y}{2\sqrt{\nu}}\,\sqrt{1+y^2/4\nu}$$

and

$$\nu\,\text{sh}\,\tau = 2\sqrt{\nu}\,y\,\sqrt{1+y^2/4\nu} = \tfrac{1}{2}\,y^2\,\sqrt{\frac{4\nu}{y^2}+1} \le \tfrac{1}{2}\sqrt{17}\,y^2.$$

(iii) Combination of the results of (i) and (ii) leads to the conclusion that for all $y \in \mathbb{R}$

$$|R_0(y)| \le e^{\frac{1}{2}\sqrt{17}\,y^2}. \qquad \square$$

For arbitrary n it is not easy to find a value for σ_n in an analytical way, due to the fact that the coefficients a_{2n} are quite intractable. Therefore we compute values of σ_n numerically. Define, for arbitrary n,

$$(3.9)\qquad \begin{cases} h_n(y) = \dfrac{2\ln|R_n(y)|}{y^2}, \quad y \neq 0 \\[2ex] h_n(0) = \lim\limits_{y\to 0} h_n(y) = 2\,\dfrac{a_{2n+2}}{a_{2n}}. \end{cases}$$

Then

$$\sigma_n = \sup_y \; h_n(y)$$

is the best estimate we can get. It is this value of σ_n that we compute. From (3.7) is is clear that $\sigma_n \geq 1$. It will turn out that the estimate $\sqrt{17}$ for σ_0 is much too coarse.

The evaluation of σ_n however is not quite trivial. In the first place loss of significant digits plays a rôle in the calculation of

$$f(y) - \sum_{s=0}^{n-1} a_{2s}\, y^{2s}.$$

Therefore we replace (for small values of y) this expression by a number of subsequent terms of the Taylor series belonging to f. The neighbourhood of 0 and the number of terms we take are determined by the restriction that we demand a relative precision of 10^{-10} with respect to f. The environment of 0 and the number of terms are dependent on ν and n of course. Secondly, overflow plays a rôle in the evaluation of $\mathrm{ch}(\nu\,\mathrm{sh}\,\tau - \nu\tau)$ if τ tends to infinity. We solve this problem as follows. On the one hand we take a term $e^{\nu\mathrm{sh}\tau}$ apart and on another we consider the interval $0 \leq y \leq 150\sqrt{\nu}$ only. This choice is motivated by the following argument. If $\tau \geq 10$, then

$$h(\tau) \leq \frac{\ln(\mathrm{ch}(\nu\,\mathrm{sh}\,\tau))}{2\nu\,\mathrm{sh}^2(\tau/2)} \leq \frac{\nu\,\mathrm{sh}\,\tau}{2\nu\,\mathrm{sh}^2(\tau/2)} = \coth(\tau/2) \leq 1.001.$$

Further, $0 \leq \tau \leq 10$ implies

$$0 \leq y \leq 2\sqrt{\nu}\,\mathrm{sh}\,5 \;<\; 150\,\sqrt{\nu}.$$

From (3.7) and (3.9) it is obvious that the value of σ_n is highly dependent on the value of

$$(3.10)\qquad (\mathrm{RC})_n = \frac{a_{2n+2}}{a_{2n}},$$

where $h_n(0) = 2(RC)_n$. We distinguish three possibilities.

(i) $(RC)_n < 0$.

In this case we have $h_n(0) < 0$ and so $h_n(y) < 0$ in a neighbourhood of 0. The maximum of h_n is attained outside this neighbourhood.

(ii) $0 < 2\,(RC)_n < 1$.

In this case h_n is maximal at a point outside the neighbourhood of 0 where $h_n(y) < 1$.

(iii) $2(RC)_n > 1$.

Now the maximum of $h_n(y)$ is found in the neighbourhood of 0 where $h_n(y) > 1$.

We have evaluated the value of σ_n for the following values of ν and n : $n = 0, 1, \ldots, 10$, $\nu = 1, 2, 5, 10, 25, 50, 100$. It turns out that in the cases (i) and (ii) $\sigma_n = 1$ ($= \lim_{y\to\infty} h_n(y)$). The results for case (iii) are given in table 1.

Table 1.

n	ν	$(RC)_n$	σ_n
2	2	1.08	2.16
	5	2.92	5.84
	10	5.90	11.81
	25	14.81	29.61
	50	29.61	59.25
		59.25	118.51
5	25	0.84	1.68
	50	1.69	3.40
	100	3.40	6.81
8	100	0.68	1.36

Let us consider the case n = 2 in particular. For the values of ν mentioned in the table it is obvious that

$$\sigma_2(\nu) \sim h_2(0) = 2(RC)_2 = 2^{a}6/a_4 \sim 1.18\,\nu .$$

This is in agreement with what we predicted. We return to this subject in chapter 4. Observe that uniformity with respect to ν cannot be proved for n = 2.

The values of σ_n in table 1 give rise to upper bounds for $|R_n(y)|$ that are not realistic outside a small neighbourhood of 0. Therefore we modify the upper bound a litte in those cases in which $2(RC)_n > 1$. We then search for upper bounds of the form

$$(3.11) \qquad |R_n(y)| \leq M_n e^{\frac{1}{2}\sigma_n y^2} .$$

The problem is to find a suitable combination of M_n and σ_n, that is to find a value of M_n so that $\sigma_n \approx 1$. We evaluated σ_2 for several values of M_2 and ν. The results are given in table 2.

Table 2.

M_2	ν	σ_2
2	5	1.27
	10	2.70
	25	6.85
	50	13.74
3	5	1
	10	1.62
	25	4.16
	50	8.35
	100	16.73
5	100	9.44
8	100	5.71

We conclude from the table that, especially for large values of ν, the increase of M_n decreases the value of σ_n quite slowly to an acceptable value (1 à 1.5). So this modification does not bring us the improvement we expected (at least not for large values of ν). We restrict ourselves to (3.4) in the next subsection, but we have to take into consideration that the error bound for the remainder, which is only valid for $\lambda > \sigma_n$, is in some cases just valid for quite large values of λ. This problem, however, is easy to avoid as we will see in the next subsection.

3.1.2 Approximations for $K_\nu(x)$

In this subsection we compare $K_\nu(x)$ with an approximation by several terms of its asymptotic series. For convenience, we introduce the following definitions:

$$\text{(3.12)} \qquad \text{knmin1}(x,\nu) = \left(\frac{\pi}{2\lambda\nu}\right)^{\frac{1}{2}} e^{-\nu\eta} \sum_{s=0}^{n-1} c_s \lambda^{-s}$$

and

$$\text{(3.13)} \qquad \text{errorbn}(x,\nu) = \left(\frac{\pi}{2(\lambda-\sigma_n)\nu}\right)^{\frac{1}{2}} e^{-\nu\eta} \, |c_n| (\lambda-\sigma_n)^{-n}, \quad \lambda > \sigma_n.$$

In words: knmin1(x,ν) is an approximation for $K_\nu(x)$ consisting of the first n terms of its asymptotic expansion, and errorbn(x,ν) is an error bound for the remainder when $K_\nu(x)$ is approximated by knmin1(x,ν).

We look especially at the influence of increasing n on the real error $K_\nu(x)$ - knmin1(x,ν), and at the ratio between the error bound errorbn(x,ν) and the real error.

For the numerical evaluations we use the numal-procedure bess kaplusn from [5] as an exact representation of $K_\nu(x)$. We consider all combinations of n = 2, 3, 5, 8 and ν = 1, 5, 10, 25 for several values of λ. The results are given in table 3 and 4. In table 3 we compare the cases n = 2 and n = 3 for ν = 1, 5, 10, 25. For each ν, five values of λ are given for which we evaluate

(i) the relative accuracy r.a. = $(K_\nu(x)$ - knmin1$(x,\nu))/K_\nu(x)$,

(ii) the ratio r = $|$ errorbn(x,ν) / $(K_\nu(x)$ - knmin1$(x,\nu))|$.

In table 4 we do the same for n = 5 and n = 8. For some values of λ, the ratio is unreliable. Firstly, this is due to loss of digits and secondly, to the requirement that $\lambda > \sigma_n$. For these values of λ the ratio is not given in the tables.

Table 3

		n = 2		n = 3	
	λ	r.a.	r	r.a.	r
ν = 1	10	$8.16*10^{-4}$	1.12	$1.11*10^{-4}$	1.74
	50	$2.92*10^{-5}$	1.02	$1.04*10^{-6}$	1.20
	100	$7.17*10^{-6}$	1.01	$1.32*10^{-7}$	1.058
	500	$2.82*10^{-7}$	1.001	$1.08*10^{-9}$	1.011
	1000	$7.05*10^{-8}$	1.0005	$1.35*10^{-10}$	1.006
ν = 5	10	$6.70*10^{-5}$	3.67	$3.91*10^{-5}$	1.50
	60	$9.70*10^{-7}$	1.04	$2.89*10^{-7}$	1.067
σ_2 = 5.84	100	$3.22*10^{-7}$	1.015	$4.09*10^{-8}$	1.040
	600	$8.00*10^{-9}$	1.0004	$1.90*10^{-10}$	1.006
	1000	$2.85*10^{-9}$	1.00002	$4.11*10^{-11}$	1.0036
ν = 10					
	10	$2.71*10^{-5}$		$2.01*10^{-5}$	1.47
σ_2 = 11.81	50	$4.47*10^{-7}$	1.235	$1.65*10^{-7}$	1.077
	100	$9.10*10^{-8}$	1.055	$2.07*10^{-8}$	1.038
	500	$2.98*10^{-9}$	1.002	$1.66*10^{-10}$	1.0075
	1000	$7.23*10^{-10}$	1.0005	$2.08*10^{-11}$	1.0034
ν = 25					
	8	$1.75*10^{-5}$		$1.58*10^{-5}$	1.60
σ_2 = 29.61	40	$2.00*10^{-7}$	10.2	$1.30*10^{-7}$	1.09
	80	$3.38*10^{-8}$	1.65	$1.62*10^{-8}$	1.05
	400	$8.33*10^{-10}$	1.02	$1.30*10^{-10}$	1.01
	800	$1.92*10^{-10}$	1.01	$1.63*10^{-11}$	1.004

From table 3 it is clear that the approximation of $K_\nu(x)$ by knmin1(x,ν) is reliable in 3 digits as soon as $\lambda > 10$ (with exception of $n = 2$, $\nu = 1$: 2 digits). Further it is obvious that, when ν and n are fixed, the approximation becomes better if λ tends to infinity, exactly what we expected. It also appears that, when λ and n are fixed, the approximation becomes better if ν increases. The approximation for $n = 3$ is better than the one for $n = 2$, as we expected too. Further it is obvious from table 3 that the ratio of errorbn(x,ν) and the real error decreases to 1 quite quickly when λ increases. Comparing the just mentioned ratio for $n = 2$ and $n = 3$ yields the following results. For small values of ν, say less than 8 (then $\sigma_2(\nu) < 10$), the differences between the ratios for $n = 2$ and $n = 3$ are quite small (less than 0.1 in general). If $\nu \geq 10$ then the ratio for $n = 3$ is much smaller than that for $n = 2$, especially for small values of λ, assuming that the last can be evaluated at all (think of $\lambda > \sigma_n$!). All together, we prefer an approximation of $K_\nu(x)$ by three terms to one by two terms for the following reasons:

(i) the approximation is somewhat better,

(ii) the approximation and the error bound are realistic for $\lambda \geq 1$ (for $\sigma_3(\nu) = 1$ for all $\nu > 0$),

(iii) the error bound for the remainder for $n = 3$ is less than or equal to the one for $n = 2$.

From table 4, it is obvious that the approximation of $K_\nu(x)$ by knmin1(x,ν) is reliable in at least 7 digits for $\lambda > 5$ à 10. For increasing λ the accuracy becomes better very quickly. For instance, for $\nu = 25$ and $n = 8$ the relative accuracy of the approximation has the size of the machine precision (10^{-14}) for $\lambda \geq 12$. Naturally the approximation for $n = 8$ is better than that for $n = 5$, as we expected. Both for $n = 5$ and $n = 8$, the ratio is hard to determine. On the one hand this is due to the fast increase of the relative accuracy and on another to the fact that we deal with parts of asymptotic series that are not useful for small values of $\lambda(\sim 1)$. In some cases the requirement that $\lambda > \sigma_n$ also plays a rôle.

Table 4.

	λ	n = 5 r.a.	n = 5 r	n = 8 r.a.	n = 8 r
$\nu = 1$	1	$2.60*10^{-1}$		$-2.44*10^{0}$	
	5	$1.40*10^{-4}$	4.67	$-1.56*10^{-5}$	10.43
	10	$5.41*10^{-6}$	2.16	$-8.39*10^{-8}$	3.28
	20	$1.87*10^{-7}$	1.47	$-3.84*10^{-10}$	1.82
	50	$2.04*10^{-9}$	1.17	$-1.75*10^{-13}$	
$\nu = 5$	2	$2.71*10^{-4}$	27.45	$3.18*10^{-5}$	395
	4.1	$7.44*10^{-6}$	3.69	$1.36*10^{-7}$	11.66
	10	$7.73*10^{-8}$	1.59	$1.15*10^{-10}$	2.53
	20	$2.33*10^{-9}$	1.25	$4.70*10^{-13}$	1.58
	40	$7.11*10^{-11}$	1.11		
$\nu = 10$	2.24	$3.62*10^{-5}$	11.49	$1.62*10^{-6}$	130
	4.12	$1.31*10^{-6}$	2.63	$1.15*10^{-8}$	9.43
	6	$1.63*10^{-7}$	1.76	$4.98*10^{-10}$	4.21
	8	$3.67*10^{-8}$	1.47	$5.14*10^{-11}$	2.87
	10	$1.15*10^{-8}$	1.34	$8.71*10^{-12}$	2.28
$\nu = 25$	2.24	$4.91*10^{-6}$	439	$7.26*10^{-8}$	76.2
	4.12	$1.50*10^{-7}$	5.68	$4.29*10^{-10}$	6.63
$\sigma_5 = 1.68$	8	$3.55*10^{-9}$	1.71	$1.67*10^{-12}$	2.32
	12	$3.96*10^{-10}$	1.30		
	20	$2.58*10^{-11}$	1.11		

CONCLUSIONS

It's very well possible to approximate $K_\nu(x)$ by several terms of its asymptotic series, even for small values of λ. The error bound for the remainder, given in (3.13), is a reasonable estimate for the absolute value of the real error. In case of small values of λ, you must see to it that $\sigma_n = 1$. If $\sigma_n > 1$ for a certain value of n, you must increase n by 1, and then σ_n becomes 1. It will be shown in chapter 4 that this is a good solution for this problem in general. Finally, the asymptotic expansion (3.6) is not uniform with respect to $\nu, \nu \geq \nu_0 > 0$, because the parameters σ_n depend on ν.

3.2 The function $I_\nu(x)$

We deal with the function $I_\nu(x)$ in very much the same way as with $K_\nu(x)$. Define

$$(3.14)\qquad \begin{cases} S_0(y) = f(-iy) \\ f(-iy) = \sum_{s=0}^{n-1} (-1)^s\, a_{2s} y^{2s} + (-1)^n\, a_{2n} y^{2n}\, S_n(y), \quad n > 0, \end{cases}$$

provided that $a_{2n} \neq 0$. So

$$(3.15)\qquad S_n(y) = \frac{f(-iy) - \sum_{s=0}^{n-1} (-1)^s\, a_{2s}\, y^{2s}}{(-1)^n a_{2n} y^{2n}}, \quad S_n(0) = 1, \quad n \geq 0.$$

Further, we write

$$(3.16)\qquad 2\pi\sqrt{\nu}\, e^{-\nu\eta}\, I_\nu(x) = \sum_{s=0}^{n-1} (-1)^s a_{2s} \int_{-2\sqrt{\nu}\cos(\pi/8)}^{2\sqrt{\nu}\cos(\pi/8)} e^{-\frac{1}{2}\lambda y^2}\, y^{2n}\, dy + F_n(x,\nu).$$

Then

$$(3.17)\qquad F_n(x,\nu) = (-1)^n\, a_{2n} \int_{-2\sqrt{\nu}\cos(\pi/8)}^{2\sqrt{\nu}\cos(\pi/8)} e^{-\frac{1}{2}\lambda y^2}\, y^{2n}\, S_n(y)\, dy.$$

Analogously to (3.4) we look for an upper bound for $|S_n(y)|$ of the form

$$(3.18)\qquad |S_n(y)| \le e^{\frac{1}{2}\tau_n y^2} \quad \text{on } D = [-2\sqrt{\nu}\cos(\tfrac{\pi}{8})],\ n \ge 0.$$

Substitution of (3.18) into (3.17) yields

$$(3.19)\qquad |F_n(x,\nu)| \le |c_n| \left(\frac{2\pi}{\lambda-\tau_n}\right)^{\frac{1}{2}} (\lambda-\tau_n)^{-n}, \quad \lambda > \tau_n.$$

From (3.16) and (3.19) it is obvious now that

$$(3.20)\qquad I_\nu(x) = \frac{e^{\nu\eta}}{\sqrt{2\pi\lambda\nu}} \left\{ \sum_{s=0}^{n-1} (-1)^s c_s \lambda^{-s} + O(\lambda^{-n}) \right\}, \quad \lambda \to \infty.$$

In subsection 3.2.1 we evaluate the values of τ_n. Moreover another upper bound for $|S_n(y)|$ is discussed. In subsection 3.2.2 we compare the error bound $(2\pi\nu^{\frac{1}{2}})^{-1} e^{\nu\eta} |F_n(x,\nu)|$ to the real error that is made by approximating $I_\nu(x)$ by the first terms of its asymptotic series.

3.2.1 Upper bounds for $|S_n(y)|$

An error bound of the form (3.4) always exists because D is a finite interval. We evaluated τ_n for the same values of ν and n as we used in the former case for σ_n. It appears that τ_n is almost always less than 1, sometimes much smaller than 1. There are three exceptions, namely

$$\tau_2(5) = 1.77,\ \tau_2(10) = 3.36 \quad \text{and} \quad \tau_2(25) = 8.27.$$

In neither of the cases we examined it yields $-2(RC)_n > 1$.

In this case we have to deal with a finite interval D. Therefore we can find a simpler upper bound for $|S_n(y)|$, namely

$$(3.21)\qquad |S_n(y)| \le D_n, \quad n \ge 0.$$

Numerical computations show that $D_n \gg 1$ in many cases. Such values of D_n give rise to error bounds for the remainder that are much larger than the absolute value of the next term of the asymptotic series. (For the same reason we don't want estimates like (3.11) with $M_n > 2$.) A favourable exception to the general rule is the case n = 3 where D_n lies between 1 and

1.5 for moderate values of ν.

3.2.2 Approximations for $I_\nu(x)$

In this subsection we compare the error bound for the remainder and the real error for approximations of $I_\nu(x)$ by the first terms of its asymptotic series. We introduce the following notations

(3.22) $$\text{inminl}(x,\nu) = \frac{e^{\nu\eta}}{\sqrt{2\pi\lambda\nu}} \sum_{s=0}^{n-1} (-1)^s c_s \lambda^{-s}$$

and

(3.23) $$\text{ferrorbn}(x,\nu) = \frac{e^{\nu\eta}}{\sqrt{2\pi(\lambda-\tau_n)\nu}} |c_n| (\lambda-\tau_n)^{-n}, \quad \lambda > \tau_n.$$

In words: inminl(x,ν) is an approximation for $I_\nu(x)$ with the first n terms of the asymptotic series of $I_\nu(x)$, and ferrorbn(x,ν) is an error bound for the remainder when $I_\nu(x)$ is approximated by inminl(x,ν).

For the numerical evaluations we use the numal-procedure bess iaplusn, cf. [5], to compare inminl(x,ν) with. We consider the same values of n and ν as in the former case. The results are given in table 5 and 6. In table 5 we compare the cases n = 2 and n = 3 for ν = 1, 5, 10, 25. In table 6 we do the same for n = 5 and n = 8. In the tables we use the abbreviations

(i) r.a. = $(I_\nu(x)$ - inminl(x,ν)) / $I_\nu(x)$ (the relative accuracy)

(ii) r. = | ferrorbn(x,ν)/($I_\nu(x)$ - inminl(x,ν))| (the ratio).

With respect to the ratio r the same remark can be made as in subsection 3.1.2.

Table 5

		n = 2		n = 3	
	λ	r.a.	r	r.a.	r
ν = 1	10	$5.20*10^{-4}$	1.72	$-1.68*10^{-4}$	1.13
	50	$2.69*10^{-5}$	1.10	$-1.13*10^{-6}$	1.02
	100	$6.88*10^{-6}$	1.05	$-1.38*10^{-7}$	1.01
	500	$2.80*10^{-7}$	1.009	$-1.09*10^{-9}$	1.003
	1000	$7.02*10^{-8}$	1.004	$-1.35*10^{-10}$	1.001
ν = 5	10	$-1.42*10^{-5}$	3.18	$-4.20*10^{-5}$	1.39
	60	$5.89*10^{-7}$	1.43	$-1.91*10^{-7}$	1.05
τ_2 = 1.77	100	$2.40*10^{-7}$	1.225	$-4.12*10^{-8}$	1.03
	600	$7.62*10^{-9}$	1.03	$-1.90*10^{-10}$	1.007
	1000	$2.77*10^{-9}$	1.02	$-4.11*10^{-11}$	1.002
ν = 10	10	$-1.39*10^{-5}$	1.38	$-2.08*10^{-5}$	1.41
	50	$1.14*10^{-7}$	2.93	$-1.67*10^{-7}$	1.07
τ_2 = 3.36	100	$4.95*10^{-8}$	1.55	$-2.08*10^{-8}$	1.03
	500	$2.65*10^{-9}$	1.08	$-1.66*10^{-10}$	1.006
	1000	$6.82*10^{-10}$	1.04	$-2.08*10^{-11}$	1.003
ν = 25	8	$-1.43*10^{-5}$		$-1.60*10^{-5}$	1.58
	40	$-6.00*10^{-8}$	2.09	$-1.30*10^{-7}$	1.09
τ_2 = 8.27	80	$1.30*10^{-9}$	17.7	$-1.63*10^{-8}$	1.04
	400	$5.73*10^{-10}$	1.29	$-1.30*10^{-10}$	1.009
	800	$1.60*10^{-10}$	1.125	$-1.63*10^{-11}$	1.004

Table 6

	λ	n = 5: r.a	n = 5: r	n = 8: r.a.	n = 8: r
ν = 1	1	$1.58*10^{-1}$		$6.36*10^{0}$	
	5	$-3.71*10^{-4}$	1.68	$-6.07*10^{-5}$	2.61
	10	$-8.34*10^{-6}$	1.36	$-1.86*10^{-7}$	1.45
	20	$-2.31*10^{-7}$	1.18	$-5.50*10^{-10}$	1.25
	50	$-2.22*10^{-9}$	1.07	$-4.12*10^{-13}$	
ν = 5	2	$4.55*10^{-5}$	161	$3.08*10^{-5}$	351
	4.1	$-3.18*10^{-6}$	8.53	$1.52*10^{-7}$	10.28
	10	$-5.80*10^{-8}$	2.11	$1.22*10^{-10}$	2.37
	20	$-2.03*10^{-9}$	1.43	$4.92*10^{-13}$	
	40	$-6.63*10^{-11}$	1.19	$2.09*10^{-14}$	
ν = 10	2.24	$1.58*10^{-5}$	26.1	$5.31*10^{-7}$	395
	4.12	$-1.86*10^{-8}$	184	$7.63*10^{-9}$	14.15
	6	$-3.92*10^{-8}$	7.29	$3.85*10^{-10}$	5.43
	8	$-1.39*10^{-8}$	3.88	$4.26*10^{-11}$	3.45
	10	$-5.47*10^{-9}$	2.81	$7.50*10^{-12}$	2.66
ν = 25	2.24	$3.61*10^{-6}$	7.34	$-1.74*10^{-8}$	317
	4.12	$6.56*10^{-8}$	3.35	$6.93*10^{-11}$	41
	8	$3.05*10^{-10}$	11.4	$8.04*10^{-13}$	
	12	$-4.85*10^{-11}$	7.48		
	20	$-9.40*10^{-12}$	2.50		

From table 5 it is clear that the approximation of $I_\nu(x)$ by inmin1(x,ν) is reliable in at least 5 digits as $\lambda > 10$. Of course the approximation is better for larger values of λ. In general, inmin1(x,ν) with $n = 3$ is a better approximation than inmin1(x,ν) with $n = 2$. Nevertheless there are exceptions. We'll try to give an explanation for this phenomenon now. For $n = 3$ $I_\nu(x) -$ inmin1$(x,\nu) < 0$ for all values of λ, but for $n = 2$ $I_\nu(x) -$ inmin1$(x,\nu) < 0$ for small values of λ and the real error is positive for larger values of λ. (Aside, the larger the value of ν, the larger the values of λ for which the real error is still negative.) The values of λ for which the approximation for $n = 3$ is worse than the approximation for $n = 2$ always lie in the neighbourhood of that value of λ for which $I_\nu(x) -$ inmin1(x,ν) changes sign. This just described behaviour of the real error for $n = 2$ is also responsible for the irregular convergence of the ratio to 1 for $\lambda \to \infty$. For $n = 3$ the convergence is quite quick. The influence of values of $\tau_2 \neq 1$ on the behaviour of the ratio is neglectable, because those values are small (except for $\tau_2(25) = 8.27$).

From table 6 it is obvious that for $\nu \geq 5$ the approximations of $I_\nu(x)$ are good in at least 5 digits for $\lambda > 2$, and in at least 8 digits for $\lambda > 10$. For both approximations the sign of $I_\nu(x) -$ inmin1(x,ν) changes. The relative accuracy tends to the machine precision quite quickly, so it is difficult to show the convergence of the ratio to 1 for $\lambda \to \infty$.

CONCLUSIONS

The function inmin1(x,ν) is a good approximation for $I_\nu(x)$ for quite moderate values of n, even for small values of λ. The error bound for the remainder given by (3.23) is a reasonable estimate for the real error, but in some cases it is hard to show this. This is mainly due to the irregular behaviour of the real error which is probably a consequence of the appearance of the terms $(-1)^s$ in the asymptotic series. Finally, it is hard to say whether the parameters τ_n of (3.18) are uniformly bounded with respect to ν, due to the irregular behaviour of these parameters as a function of ν. So it remains an open question whether the asymptotic expansion (3.20) is uniform with respect to ν, $\nu \geq \nu_0 > 0$ or not.

4. THE COEFFICIENTS a_{2n} AND c_n

In this chapter we consider the coefficients a_{2n} and c_n in more detail. For convenience, we recall the recurrence relations which generate the rows, (see (2.15), (2.16), (2.18) and (2.19)). The coefficients a_{2n} satisfy the recurrence relation

(4.1) $$8\nu n\,(2n-3)a_{2n} + (2n-1)(2n-3)a_{2n-2} - a_{2n-6} = 0$$

with initial conditions

(4.2) $$a_0 = 1,\ a_{2n} = 0 \quad n < 0.$$

For the coefficients c_n we have

(4.3) $$8\nu n c_n + (2n-1)^2 c_{n-1} - (2n-1)(2n-5)c_{n-3} = 0$$

with initial conditions

(4.4) $$c_0 = 1,\ c_n = 0,\ n < 0.$$

From these relations no simple formula can be deduced for the coefficients a_{2n} and c_n, respectively. Therefore we resort to numerical calculation. We'll determine the coefficients by forward recursion. This is allowed only when the solution we are searching for (that is, the solution which satisfies (4.2) ((4.4)) is dominant with respect to other solutions of (4.1)((4.3)). In §4.1 we will show that this is indeed the case. In §4.2 we consider the inner structure of the coefficients a_{2n} in more detail. We will show that if $(RC)_n > 0$ then $(RC)_{n+1} < 0$. It will appear, however, that there still are some open questions with respect to the coefficients a_{2n}.

4.1 Computing of the coefficients a_{2n} and c_n.

From (4.1) and (4.3) is is clear that the coefficients a_{2n} and c_n satisfy a third-order difference equation. Such a difference equation has three independent solutions. We will show now that the solution that satisfies (4.2) ((4.4)) is a dominant solution.

Our starting-point is the coefficient a_{2n}. This is the 2n-th coefficient from the Taylor series of the function

$$f(y) = \frac{1}{\sqrt{1+y^2/4\nu}} \; ch(\nu\; sh\, \tau - \nu\tau),$$

where

$$y = 2\sqrt{\nu}\; sh\; (\tau/2).$$

We may write, using Cauchy's formula,

$$a_{2n} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z^{2n+1}}\, dz,$$

where C_r is a circle with centre 0 and radius $r < 2\sqrt{\nu}$.
Now, let $\varepsilon > 0$. Choose $r = R$ so that $2\sqrt{\nu} - \varepsilon < R < 2\sqrt{\nu}$. Then

$$|a_{2n}| = \left| \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z^{2n+1}}\, dz \right| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(Re^{i\theta})|}{R^{2n+1}}\, R\, d\theta = O(R^{-2n}).$$

So

(4.5) $$a_{2n} = O\left(\left(\frac{1}{2\sqrt{\nu}}\right)^{2n}\right) = O\left(\left(\frac{1}{4\nu}\right)^{n}\right), \quad n \to \infty,$$

and

(4.6) $$c_n = a_{2n} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})} 2^n = O\left(\frac{\Gamma(n+\frac{1}{2})}{(2\nu)^n}\right), \quad n \to \infty.$$

In principle it may be possible that

$$|a_{2n}| << M \left(\frac{1}{4\nu}\right)^n ,$$

but further analysis shows that $a_{2n} \sim Mn^{-\frac{1}{2}}\left(\frac{1}{4\nu}\right)^n$.

It will appear below that the dominant solution of (4.3) also shows the behaviour (4.6), so it is allowed to apply forward recursion to calculate the coefficients c_n (and, of course, a_{2n})

We write (4.3) as

$$(4.7)\qquad \frac{8\nu x}{(2x-1)(2x-5)}\, y(x) + \frac{2x-1}{2x-5}\, y(x-1) - y(x-3) = 0,$$

where c(n) is replaced by y(x).

According to [3] we find asymptotic series for the solutions of (4.7) by substituting the formal series

$$(4.8)\qquad y(x) = \rho_0^x\, x^r\{1+\sum_{k=1}^{\infty} \gamma_k x^{-k}\}.$$

It is clear that (4.7) has the form

$$(\frac{2\nu}{x} + O(x^{-2}))\, y(x) + (1+\frac{2}{x} + O(x^{-2}))\, c(x-1) - c(x-3) = 0.$$

According to [3] we find ρ_0 from the characteristic equation

$$(4.9)\qquad -1 + \rho^2 = 0,$$

so $\rho_0 = \pm 1$. We examine the roots separately.

(i) $\rho_1 = +1$. In this case we find, by equalizing coefficients, that $r = -1-\nu$.

Then we have the asymptotic solution

$$y_1(x) = x^{-1-\nu}\{1 + \sum_{k=1}^{\infty} \gamma_k\, x^{-k}\}.$$

(ii) $\rho_2 = -1$. Now $r = -1+\nu$ and this yields

$$y_2(x) = (-1)^x\, x^{-1+\nu}\{1 + \sum_{k=1}^{\infty} \delta_k\, x^{-k}\}.$$

The equation (4.9) gives rise to two independent solutions of (4.7). A third solution we find as follows. Substitute

$$y(x) = z(x)\, \Gamma(x+\tfrac{1}{2})$$

into equation (4.7). This yields

(4.10) $$2\nu \frac{x}{x-\frac{1}{2}} z(x) + z(x-1) - \frac{1}{(x-\frac{1}{2})(x-3/2)} z(x-3) = 0.$$

Substituting a formal series of the form (4.8) yields the characteristic equation

$$2\nu\rho^3 + \rho^2 = 0$$

with solutions $\rho = -\frac{1}{2\nu}$ and $\rho = 0$ (2times). We examine the root $\rho = -\frac{1}{2\nu}$. It will turn out that this root gives the third independent solution of (4.7).

(iii) $\rho_3 = -\frac{1}{2\nu}$. Now $r = -\frac{1}{2}$ and

$$z(x) = (-\frac{1}{2\nu})^x x^{-\frac{1}{2}} \{1 + \sum_{k=1}^{\infty} \varepsilon_k x^{-k}\},$$

so

$$y_3(x) = (-\frac{1}{2\nu})^x x^{-\frac{1}{2}} \Gamma(x+\tfrac{1}{2})\{1 + \sum_{k=1}^{\infty} \varepsilon_k x^{-k}\}.$$

From (i), (ii) and (iii) it follows that the three independent solutions of (4.7), or, equally, (4.3) have the following asymptotic behaviour

(i) $c_1(x) \sim x^{-1-\nu}$ $\quad x \to \infty$

(ii) $c_2(x) \sim x^{-1+\nu}$ $\quad x \to \infty$

(iii) $c_3(x) \sim (-\frac{1}{2\nu})^x x^{-\frac{1}{2}} \Gamma(x+\frac{1}{2})$ $\quad x \to \infty$.

It is obvious that (iii) is the dominant solution, and it is just the solution we need.

Concluding, we can say that we may apply forward recursion for the calculation of the coefficients a_{2n} and c_n.

4.2 The inner structure of the coefficients a_{2n}.

The coefficients a_{2n} that satisfy (4.1) with initial conditions (4.2) are polynomials in powers of $\frac{1}{\nu}$ of the n-th degree. This can be proved by complete induction. We write, for all $n \geq 0$,

(4.11) $$a_{2n} = p_0^{(n)} + \frac{p_1^{(n)}}{\nu} + \ldots + \frac{p_n^{(n)}}{\nu^n}.$$

From (4.2) it follows

$$(4.12)\quad \begin{cases} a_0 = 1, & \text{so } p_0^{(0)} = 1 \\ a_2 = -\frac{1}{8\nu}, & \text{so } p_1^{(1)} = -\frac{1}{8},\ p_0^{(1)} = 0 \\ a_4 = \frac{3}{128\nu^2}, & \text{so } p_2^{(2)} = \frac{3}{128},\ p_1^{(2)} = 0,\ p_0^{(2)} = 0. \end{cases}$$

Substitution of (4.11) into (4.1) gives the following recurrence relations for coefficients $p_i^{(j)}$ for $n \geq 3$

$$(4.13)\quad 8n(2n-3)\ p_\ell^{(n)} = p_{\ell-1}^{(n-3)} - (2n-1)(2n-3)\ p_{\ell-1}^{(n-1)} \qquad \ell = 1,\dots, n-2,$$

$$(4.14)\quad 8n(2n-3)\ p_{n-1}^{(n)} = -(2n-1)(2n-3)\ p_{n-2}^{(n-1)} ,$$

$$(4.15)\quad 8n(2n-3)\ p_n^{(n)} = -(2n-1)(2n-3)\ p_{n-1}^{(n-1)} .$$

From (4.14) we may conclude, with (4.11), that

$$(4.16)\quad p_{n-1}^{(n)} = 0 , \quad \text{for all } n \geq 0.$$

Complete induction yields

$$p_0^{(n)} = 0 ,\ n \geq 1 ,$$
$$p_1^{(n)} = 0 ,\ n \geq 4 ,$$

and, in general,

$$(4.17)\quad p_i^{(n)} = 0 \text{ for } n \geq 3i + 1.$$

Now, let's consider the first term $p_i^{(n)} \neq 0$ in the polynomial representation of a_{2n}. There are three possibilities.

(i) $n = 3\ell$. Then

$$a_{2n} = a_{6\ell} = \frac{p_\ell^{(n)}}{\nu^\ell} + \dots + \frac{p_n^{(n)}}{\nu^n} .$$

In this case the coefficient $p_\ell^{(n)}$ is dependent on $a_0 = p_0^{(0)} = 1$ only, because the third term in (4.13) is equal to zero. This can be proved by complete induction, using (4.17). Therefore the sign of $p_\ell^{(n)}$ is always positive.

(ii) $n = 3\ell + 1$. Then

$$a_{2n} = a_{6\ell+2} = \frac{p_{\ell+1}^{(n)}}{\nu^{\ell+1}} + \dots + \frac{p_n^{(n)}}{\nu^n} .$$

Now, $p_{\ell+1}^{(n)}$ is dependent on $p_\ell^{(n-1)}$ and $p_\ell^{(n-3)}$, thus on $p_0^{(0)}$ respectively $p_1^{(1)}$, and the sign of $p_{\ell+1}^{(n)}$ is negative.

(iii) $n = 3\ell + 2$. Then

$$a_{2n} = a_{6\ell+4} = \frac{p_{\ell+2}}{\nu^{\ell+2}} + \dots + \frac{p_n^{(n)}}{\nu^n} .$$

Now $p_{\ell+2}^{(n)}$ is dependent on $\rho_{\ell+1}^{(n-1)}$ and $p_{\ell+1}^{(n-3)}$, thus on $p_1^{(1)}$ and $p_2^{(2)}$, and the sign of $p_{\ell+2}^{(n)}$ is positive.

Suppose that the first non-zero coefficient in the polynomial representation of a_{2n} is dominant to the other terms in that representation (for all $n \geq 0$). Then we can predict the behaviour of σ_n from the signs of these non-zero terms. For in that case $(RC)_n = a_{2n+2}/a_{2n}$ is mainly determined by these terms. We can conclude immediately for what values of n $(RC)_n$ is positive, namely for $n = 3\ell + 2$. Then $(RC)_n = \mathcal{O}(\nu)$. This agrees with what we found in chapter 3 for $n = 2$ (ℓ=0). In principle it yields that $(RC)_n > 0$ for n = 2, 5, 8, 11, 14, ... Unfortunately when n increases, the other terms in the polynomial representation of a_{2n} begin to play a more important rôle, and then the foregoing reasoning is not valid anymore. However, for larger values of ν, this effect is less important. We can illustrate this by the following examples:

(i) $\nu = 3$ $(RC)_n > 0$ for n = 2, 5, 11.

(ii) $\nu = 9$ $(RC)_n > 0$ for n = 2, 5, 8, 12, ...

(iii) $\nu = 100$ $(RC)_n > 0$ for n = 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 33, ...

From (4.1) it is clear that for large values of n the last term may be neglected. So

$$\lim_{n\to\infty} \frac{a_{2n}}{a_{2n-2}} = -\frac{1}{4\nu} . \tag{4.18}$$

From (4.18) it is clear that $(RC)_n < 0$ on the long term and thus $\sigma_n = 1$ for large values of n. Unfortunately, (4.18) is not uniform in ν.

Numerical experiments concerning the value of $(RC)_n$ suggest some conjectures that we can't prove. These conjectures are: For a fixed integer value of ν there are ν values of n for which $(RC)_n > 0$ and the largest one is $n_{max} = 2$ (if $\nu = 1$), and, in general, $n_{max} = 5\nu - 4$ $(\nu \geq 2)$.

Appendix A

In this appendix we derive the differential equation (2.12) for the function f (see (2.11)).

We know that

$$f(y) = \sqrt{\nu}\ \frac{d\tau}{dy}\ e^{-\nu(\text{sh}\tau-\tau)}\ ,\quad y = 2\sqrt{\nu}\ \text{sh}(\tau/2).$$

Then

$$\text{(A1)}\qquad f'(y) = \sqrt{\nu}\,\frac{d^2\tau}{dy^2}\ e^{-\nu(\text{sh}\ \tau-\tau)} + \sqrt{\nu}\ \frac{d\tau}{dy}\ \frac{d}{d\tau}\,[e^{-\nu(\text{sh}\ \tau-\tau)}]\ \frac{d\tau}{dy}$$

$$= \frac{d^2\tau}{dy^2}\ \frac{dy}{d\tau}\ f(y) - \nu(\text{ch}\ \tau-1)\ \frac{d\tau}{dy}\ f(y)$$

Now $\frac{d^2\tau}{dy^2} = -\frac{\frac{1}{4}y}{\nu+\frac{1}{4}y^2}\ \frac{d\tau}{dy}$ and $\text{ch}\ \tau - 1 = 2\ \text{sh}^2(\tau/2)$, so

$$\text{(A2)}\qquad f'(y) = -f(y)\left[\frac{\frac{1}{4}y}{\nu+\frac{1}{4}y^2} + \frac{\frac{1}{2}y^2}{\sqrt{\nu+\frac{1}{4}y^2}}\right].$$

We can write (A2), with

$$\text{(A3)}\qquad A = \nu + \tfrac{1}{4}y^2,$$

as

$$\text{(A4)}\qquad A\,f' = -\tfrac{1}{4}\,yf - \tfrac{1}{2}\,y^2\,f\sqrt{A}\,.$$

Now take the derivative of (A4). This yields

$$A'f' + A\,f'' = -\tfrac{1}{4}f - \tfrac{1}{4}yf' - yf\ \sqrt{A} - \tfrac{1}{2}\,y^2\ f'\sqrt{A} - \tfrac{1}{4}\,y^2\,f\ \frac{A'}{A}\ \sqrt{A}$$

$$= -\tfrac{1}{4}f - \tfrac{1}{4}\,yf' + y\ \frac{A\,f' + \frac{1}{4}yf}{\frac{1}{2}y^2} + \tfrac{1}{4}\,y^2\ \frac{A'}{A}\ \left[\frac{Af' + \frac{1}{4}yf}{\frac{1}{2}y^2}\right] +$$

$$+ \tfrac{1}{2} y^2 \sqrt{A}\, f\, [\frac{\frac{1}{4}y}{A} + \frac{\frac{1}{2}y^2}{\sqrt{A}}]$$

$$= -\tfrac{1}{4} f - \tfrac{1}{4} yf' + (Af'+\tfrac{1}{4}yf)(\frac{2}{y} + \tfrac{1}{2}\frac{A'}{A}) - \frac{\frac{1}{8}y^3}{A}(\frac{A f'+\frac{1}{4}yf}{\frac{1}{2}y^2}) +$$

$$+ \tfrac{1}{4} y^4 f.$$

This yields

$$A f'' + f'(A' + \tfrac{1}{4}y - \frac{2A}{y} - \tfrac{1}{2}A' + \tfrac{1}{4}y) + f(-\tfrac{1}{4} + \tfrac{1}{4}y\,\frac{2}{y} + \tfrac{1}{4}y\,\tfrac{1}{2}\frac{A'}{A} +$$
$$- \tfrac{1}{4}\,\tfrac{1}{4}\,\frac{y^2}{A} + \tfrac{1}{4}y^4)$$

so

(A5) $$y(4\nu+y^2)\, f'' + (y^2 - 8\nu)\, f' - (y^4 + 1)\, y f = 0.$$

(A5) corresponds to (2.12). □

Appendix B

In this appendix we prove that

(B1) $$\left| \int_{L_{-\pi}} \exp[\nu(2\lambda\, sh^2(\tau/2) + sh\,\tau - \tau)]\, d\tau \right| = O(e^{-\lambda\nu}),\ \lambda > 0,$$

where $L_{-\pi}$ is given by

(B2) $$z = -\pi i + \delta e^{i\phi},\quad 0 \le \phi \le \frac{\pi}{2}.$$

With (B2) we get

$$sh\,\tau = sh(-\pi i+\delta e^{i\phi}) = -\, sh(\delta e^{i\phi}),\ sh(\tau/2) = -i\, ch(\delta e^{i\phi}).$$

Then

(B3) $$\int_{L_{-\pi}} \exp[\nu(2\lambda sh^2(\tau/2) + sh\,\tau-\tau)]\, dt = \int_0^{\pi/2} \exp[\nu(-2\lambda ch^2(\delta e^{i\phi})]$$
$$* \exp[-\nu(sh(\delta e^{i\phi}) - \delta e^{i\phi})]$$
$$* \exp[-\nu\pi i]\, i\,\delta e^{i\phi} d\phi.$$

Now it is easy to see that

(B4) $$|e^{-\delta e^{i\phi}}| = |e^{-\delta\cos\phi - i\delta\sin\phi}| = e^{-\delta\cos\phi}.$$

Further

(B5) $$|e^{-sh(\delta e^{i\phi})}| = |e^{-\frac{1}{2}(e^{\delta e^{i\phi}} - e^{-\delta e^{i\phi}})}|$$

$$= |e^{-\frac{1}{2}(e^{\delta\cos\phi + i\delta\sin\phi} - e^{-\delta\cos\phi - i\delta\sin\phi})}|$$

$$= |e^{-\frac{1}{2}e^{\delta\cos\phi}(\cos(\delta\sin\phi) + i\sin(\delta\sin\phi))}|$$

$$|e^{\frac{1}{2}e^{-\delta\cos\phi}(\cos(\delta\sin\phi) - i\sin(\delta\sin\phi))}|$$

$$= e^{\frac{1}{2}[e^{-\delta\cos\phi}\cos(\delta\sin\phi) - e^{\delta\cos\phi}\cos(\delta\sin\phi)]}$$

$$= e^{-sh(\delta\cos\phi)\cos(\delta\sin\phi)},$$

and

(B6) $$|e^{ch^2(\delta e^{i\phi})}| = |e^{\frac{1}{2} + \frac{1}{4}e^{2\delta e^{i\phi}} + \frac{1}{4}e^{-2\delta e^{i\phi}}}|$$

$$= |e^{\frac{1}{2} + \frac{1}{4}e^{2\delta(\cos\phi + i\sin\phi)} + \frac{1}{4}e^{-2\delta(\cos\phi + i\sin\phi)}}|$$

$$= e^{\frac{1}{2} + \frac{1}{4}e^{2\delta\cos\phi}\cos(2\delta\sin\phi) + \frac{1}{4}e^{-2\delta\cos\phi}\cos(2\delta\sin\phi)}$$

$$= e^{\frac{1}{2}[1 + ch(2\delta\cos\phi) - \cos(2\delta\sin\phi)]}.$$

Substitution of (B4), (B5) and (B6) into (B3) yields

(B7) $$\left| \int_{L_{-\pi}} \exp[\nu(2\lambda sh^2(\tau/2) + sh\tau - \tau)]\, d\tau \right|$$

$$\leq \delta \int_0^{\pi/2} e^{-\lambda\nu[1+ch(2\delta\cos\phi)\cos(2\delta\sin\phi)]} e^{-\nu sh(\delta\cos\phi)\cos(\delta\sin\phi)} e^{\nu\delta\cos\phi} d\phi$$

$$\leq \delta \int_0^{\pi/2} e^{-\lambda\nu} .1.\, e^{\nu\delta}\, d\phi = \frac{\pi\delta}{2}\, e^{-\nu(\lambda-\delta)} = \mathcal{O}(e^{-\lambda\nu}),\quad \lambda > 0$$

provided that $0 < \delta \le \pi/4$. (If $0 < \delta \le \frac{\pi}{4}$ then $\mathrm{ch}(2\delta\cos\phi)\cos(2\delta\sin\phi) \ge 0$.)

□

Appendix C

In this appendix we show that

$$\text{(C1)} \qquad \text{Int} = \int_\delta^\infty \exp[\nu(-2\lambda\,\mathrm{ch}^2(\sigma/2) - \mathrm{sh}\,\sigma - \sigma)]\, d\sigma = O(e^{-\lambda\nu}), \quad \lambda > 0,\ \delta > 0.$$

Substitute

$$w = 2\sqrt{\nu}\,\mathrm{ch}(\sigma/2)$$

into (C1). This yields

$$\text{(C2)} \qquad \text{Int} = \int_{2\sqrt{\nu}\mathrm{ch}(\delta/2)}^\infty e^{-\frac{1}{2}\lambda w^2}\, e^{-\nu\,\mathrm{sh}\sigma - \nu\sigma}\, \frac{d\sigma}{dw}\, dw$$

Now

$$\text{(C3)} \qquad \frac{dw}{d\sigma} = \sqrt{\nu}\,\mathrm{sh}(\sigma/2)$$

and

$$\text{(C4)} \qquad \mathrm{sh}\,\sigma = 2\,\mathrm{sh}\,\frac{\sigma}{2}\,\mathrm{ch}\,\frac{\sigma}{2} = 2\sqrt{\mathrm{ch}^2(\tfrac{\sigma}{2}) - 1}\;\mathrm{ch}(\sigma/2) = \frac{w}{\sqrt{\nu}}\sqrt{\frac{w^2}{4\nu} - 1} \ge 0$$

Substituting (C3) and (C4) into (C2) we get

$$\text{Int} \le \int_{2\sqrt{\nu}\mathrm{ch}(\delta/2)}^\infty e^{-\frac{1}{2}\lambda w^2}\, \nu^{-\frac{1}{2}} \left(\frac{w^2}{4\nu} - 1\right)^{-\frac{1}{2}} dw$$

$$\le \frac{1}{\sqrt{\nu}}\, \frac{1}{\sqrt{\mathrm{ch}^2(\delta/2) - 1}} \int_{2\sqrt{\nu}\mathrm{ch}(\delta/2)}^\infty e^{-\frac{1}{2}\lambda w^2}\, dw$$

$$\le \frac{1}{\sqrt{\nu}}\, \frac{1}{\mathrm{sh}(\delta/2)} \int_{2\sqrt{\nu}\mathrm{ch}(\delta/2)}^\infty e^{-\lambda\sqrt{\nu}\mathrm{ch}(\delta/2) w}\, dw$$

$$= \frac{1}{\sqrt{\nu}}\, \frac{1}{\mathrm{sh}(\delta/2)}\, \frac{1}{\lambda\sqrt{\nu}\,\mathrm{ch}(\delta/2)}\, e^{-2\lambda\nu\,\mathrm{ch}(\delta/2)} = O(e^{-\lambda\nu}), \lambda>0, \delta>0. \quad \square$$

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