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The econometric modeling of time series by linear stochastic models has been criticized by R.E. Kalman. Instead he proposes to formulate this modeling problem as a stochastic realization problem. In this note Kalman's approach is followed and in a non-dynamic framework generalized to multivariate stochastic realization problems. The special case of the three-variate Gaussian stochastic realization problem is investigated in some detail. In a dynamic context the stochastic realization problem is posed of representing an observed process such that the inherent causality or dependency relation between the components is made explicit.

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Note: This report has been submitted elsewhere.

1. Introduction

The purpose of this paper is to formulate certain stochastic realization problems that are motivated by econometrics.

A general problem of econometrics, as of other areas of science and engineering, is to represent observations by a model. R.E. Kalman [24,25,26,27] has been voicing a critique of modeling in econometrics. His suggestion is to formulate this modeling problem as a realization problem of system theory.

R.E. Kalman's criticism of econometric modeling may be summarized by:

1. one should not impose conditions on the model that cannot be falsified by the data;
2. that a definition of identifiability that has been proposed by econometricians [33] and that is still used in econometrics is besides the point.

Econometric modeling and the above criticism lead to the following system theoretic questions:

1. how to model an observation vector such that the dependencies among the components are made explicit while keeping the model falsifiable?;
2. how to model an observed stochastic process such that the dependencies between the components of the observed process are made explicit?

A difficulty in econometrics is that the above questions are mixed up and that there the solution of question 1 is adapted to solve question 2. In the opinion of the author the above questions have to be separated. As yet it is not clear that the solution to question 1 may be used to solve question 2.

For the modeling of an observed random vector several models have been used. Some of these models are: the regression model, the error-in-variables model, the factor analysis model and the confluence analysis model. Factor analysis has been introduced by the psychologist C. Spearman in

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1904 [49], generalized by L.L. Thurstone in 1931 [50] and developed by psychologists and statisticians. Confluence analysis has been introduced by R. Frisch in 1934 [14] and partly developed by O. Reiersøl [33,43,44,45,46]. Econometricians such as T.C. Koopmans [31] have considered these models, but have apparently become bogged down by what they call the "identifiability" problem for these models [32,33,45,46]. Yet there still is much to say for the factor or confluence analysis model. The key property of this model is the conditional independence of the components of the observation vector given the factor. On the basis of this property one may define a rather general factor model. This model then includes the model of latent structure analysis introduced by the sociologist P.F. Lazersfeld [36] and further developed by him and others [37]. This approach will be followed below.

In this paper stochastic realization problems will be formulated for an observation vector in a non-dynamic context. The questions in this problem are the existence of a factor model that represents the given observation vector and the classification of all minimal such models. Only a special case of these stochastic realization problems will be discussed in some detail.

In a dynamic context where an observed stochastic process has to be modeled, the criticism of econometrics leads to a stochastic realization problem for a Gaussian process. The problem is to represent an observed process such that the inherent causality relation is made explicit. Several classes of stochastic dynamic systems with which this may be done are discussed.

To properly motivate the stochastic realization problems to be posed below, section 2 contains a lengthy discussion of stochastic models. However, because the terminology in the literature is not standard, this discussion is necessary for the understanding of the proposed problems. In this paper no proofs are given, they are deferred to a later publication.

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2. Problem formulations

In this section several stochastic realization problems are motivated and formulated. First some notation and terminology is introduced.

Definitions and notations

Let (Ω,F,P) denote a complete probability space, consisting of a set Ω, a σ-algebra F and a probability measure P. Let

$$\underline{F} = \left\{\begin{matrix} G \text{ a } \sigma\text{-algebra of subsets of } \Omega \mid \\ G \subset F, \text{ completed with all the null sets of } F \end{matrix}\right\}.$$

and for $G \in \underline{F}$

$$L^+(G) = \left\{ x : \Omega \rightarrow R_+ \mid x \text{ is } G-\text{measurable} \right\}.$$

If $y:\Omega \rightarrow R^k$ is a random variable then $F^y \in F$ is the σ-algebra generated by y. If $F_1, F_2 \in F$ then $(F_1 \vee F_2) \in F$ denotes the smallest σ-algebra that contains both F_1 and F_2, and that is completed with the null sets of F. The notation $(F_1, F_2, ..., F_n) \in I$ is used to indicate that $F_1, F_2, ..., F_n$ are *independent σ-algebras.* The set R^n will be equipped with the σ-algebra B_n of the Lebesgue measurable sets, together denoted by (R^n, B_n).

The following notation is needed. Let

$$Z_+ = \{ 1, 2, 3, \cdots \}, \; N = \{0, 1, 2, \cdots \},$$

and for $n \in Z_+$,

$$Z_n = \{1, 2, ..., n\}, \; N_n = \{0, 1, 2, ..., n\}.$$

If $n \in Z_+$ and $Q \in R^{n \times n}$, then Q^T denotes the transpose of Q; if Q is symmetric then $Q \geqslant 0$ denotes that Q is *positive definite* and $Q > 0$ that it is *strictly positive definite.*

The notation $x \in G(\mu, Q)$ will denote that the random variable x has a Gaussian distribution with mean μ and variance Q. Furthermore, $(x_1,x_2,...,x_m) \in G(\mu,Q)$ denotes that with $x^T=(x_1,...,x_m)$, $x \in G(\mu,Q)$.

Modeling of an observation vector

The question here is, given a probability distribution on a vector, to represent this distribution by a model. This vector will be referred to as the observation vector and its components as the observations. For the following discussion one limits attention to a random vector with a Gaussian distribution. Below four models for the representation of such an observed vector are discussed.

In general one distinguishes exact and approximate modeling. In exact modeling there must be equality between the given probability distribution and the distribution of the observed random vector that is associated with the model. In approximate modeling the given probability distribution must be approximated in a specified sense by the probability distribution on the observed vector that is associated with the model. Below attention is limited to exact modeling.

DEFINITION 2.1. *The* regression model. *The observation vector* $y: \Omega \to R^k$, $y \in G(0,Q_y)$, *is defined by*

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; y_1:\Omega \to R^{k_1}, \, y_2:\Omega \to R^{k_2}, \; k_1+k_2=k,$$

$$y_2 = Hy_1+w$$

where $w:\Omega \to R^{k_2}$, $w \in G(0,Q_w)$, (y_1,w) *are independent and* $H \in R^{k_2 \times k_1}$. *Here* y_1 *is called the* regressor *or* independent variable *and* y_2 *the* dependent *variable.*

References for this model are [38,p.81;40,p.72].

The main criticism of the regression model is that it imposes a partition and causality structure that cannot be determined from the data. Thus the question whether one can determine the indices $k_1,k_2 \in N$ from the variance matrix of y has a negative answer. In fact for any pair of indices $k_1,k_2 \in N$, with $k_1+k_2=k$, and any partition of the vector y one can write a regression model.

The interpretation of causality is that somehow y_2 is caused by y_1 and the error w. Definitions of a causality relation between random variables or stochastic processes have been proposed in the econometric literature. The two concepts most often mentioned are Granger causality [17,18,19], and Sims causality [47,48]; see also the references [8,13,42,58] and the references quoted there. None of these definitions are useful in this context.

Another interpretation is that the variable y_2 consists of y_1 and the error w, while the variable y_1 does not contain an error.

DEFINITION 2.2. *The* errors-in-variables model (restricted sense). *The observation vector* $y:\Omega \to R^k$ *is defined by*

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; y_1:\Omega \to R^{k_1}, \; y_2:\Omega \to R^{k_2}, \; k_1+k_2=k,$$

$$x = Gu,$$

$$y_1 = u+w_1,$$

$$y_2 = x+w_2,$$

$w_1,u:\Omega \to R^{k_1}$, $w_2,x:\Omega \to R^{k_2}$, $G \in R^{k_2 \times k_1}$, (u,w_1,w_2) *are independent and Gaussian distributed.*

The errors-in-variables model has been introduced into econometrics by A. Wald [55,p.286]; actually there the additional condition is imposed that Q_{w_1}, Q_{w_2} are diagonal. See also [40,p.72]. The model discussed by K. Pearson [41] can be considered as a special case of definition 2.2. In this model both observation components are subject to errors and in that sense it counters the criticism of the regression model. A point of criticism remains, one still assumes a partition of the observation vector into (y_1,y_2) that in general cannot be determined from the data. The errors-in-variables model 2.2 has a serious difficulty that will be discussed in section 3.

DEFINITION 2.3. *The* factor analysis model. *The observation vector* $y:\Omega\rightarrow R^k$, *with components* $(y_1, ..., y_k)$, *is defined by*

$$y = H\,x + w,$$

or, equivalently, as

$$y_i = H_i x + w_i,\ \ i \in Z_m,$$

where $x:\Omega\rightarrow R^n$, $x \in G(0,Q_y)$ *called the* factor , $H \in R^{k\times n}$, *called the* matrix of factor loadings, $w:\Omega\rightarrow R^k$, $w\in G(0,Q_w)$, Q_w *diagonal and (x,w) independent. Here* Q_w *diagonal is equivalent to* $(w_1, \ldots, w_k)$ *independent.*

Factor analysis is a subject that has apparently been introduced by the psychologist C. Spearman in 1904 [49], and that has been generalized by L.L. Thurstone in 1931 [50]. In [49] the condition is imposed that $n=1$.

DEFINITION 2.4. *The* confluence analysis model (the errors-in-variables model (general sense)). *The observation vector* $y:\Omega\rightarrow R^k$, $y\in G$, *is defined by*

$$y=u+w, \quad Su=0,$$

where $u,w:\Omega\rightarrow R^k$, $w\in G(0,Q_w)$, *with* Q_w *diagonal,* $u\in G(0,Q_u)$, *(u,w) independent and* $S\in R^{p\times k}$ *of full rank.*

The factor analysis model 2.3 and the confluence analysis model are easily seen to be equivalent. This fact is well known [5,p.20;33;46]. In the econometrics literature model 2.4 is also called the errors-in-variables model [38,Ch.10,§2,p.386]. Moreover, the models 2.2 and 2.4 are not always distinguished. Because of the equivalence of the models 2.3 and 2.4, in the following attention will be restricted to the factor analysis model.

The model 2.4 has been introduced by the econometrician R. Frisch in 1934 [14] in a subject termed confluence analysis, in fact for the case $p=1$. Confluence analysis has been developed partly by O. Reiersøl [43,44]. Frisch's reason for proposing the confluence analysis model has apparently been the inappropriate use of the regression model.

By which of the above four models should one represent an observed random vector? Assume given a probability distribution on an observed random vector. Because exact modeling has been adopted one has to determine a model in the appropriate class such that the probability distribution of the observed vector associated with the model equals the given probability distribution. As criterion for the above question we now pose that one must be able to falsify the model on the basis of the given probability distribution.

The factor analysis model and equivalently the confluence analysis model satisfy the above criterion with the modifications stated below. Therefore the factor analysis model will be used in the sequel. The regression model is rejected because it imposes a causality assumption that cannot be determined from the given probability distribution. Here the indices k_1,k_2 as defined in 2.1 cannot be determined from the data. In the factor analysis model no causality assumption is imposed, all observation components are treated equally. The modification that one must impose on the factor analysis model to make it falsifiable are the restriction to a minimal factor and the dividing out by an equivalence class. This will be discussed below.

What is particular about the factor analysis model? To answer this question the concept of conditional independence is needed.

DEFINITION 2.5. *The* multivariate conditional independence relation *for an (m+1)-tuple of σ-algebras* $F_1, F_2, ..., F_m, G \in \underline{F}$ *is defined by the condition that, with for all* $i \in Z_m$ $y_i \in L^+(F_i)$,

$$E[y_1 \cdots y_m \mid G] = E[y_1 \mid G] \cdots E[y_m \mid G].$$

Then one says that $F_1, F_2, ..., F_m$ *are* conditionally independent *given* G. *Notation* $(F_1, F_2, ..., F_m \mid G) \in CI$.

DEFINITION 2.6. *The* multivariate Gaussian conditional independence relation *for an (m+1)-tuple of σ-algebras* $F^{y_1}, \ldots, F^{y_m}, F^x$ *generated by* $y_1 : \Omega \to R^{k_1}$, ..., $y_m : \Omega \to R^{k_m}$, $x : \Omega \to R^n$, *is defined by the conditions*

$$(F^{y_1}, F^{y_2}, ..., F^{y_m} \mid F^x) \in CI,$$

$$(y_1, y_2, ..., y_m, x) \in G.$$

Notation $(F^{y_1}, ..., F^{y_m} \mid F^x) \in CIG$.

It will be shown in section 3 that random variables $(y_1, ..., y_m, x)$ satisfy

$$(F^{y_1}, ..., F^{y_m} \mid F^x) \in CIG$$

iff they satisfy the specification of the factor analysis model, namely

$$y = H x + w,$$

with Q_w diagonal, or, equivalently, as

$$y_i = H_i x + w_i,$$

in which $(w_1, \ldots, w_m)$ are independent. The interpretation of this condition is that the factor x represents all the information such that conditioned on it the observations are independent. In other words, every observation consists of a systematic part represented by the factor and an error that is particular for the observation component. From a modeling viewpoint this is a natural property. This point has been stressed in [2,p.112;4,p.11]. The concept of factor is a generalization of the concept of state in stochastic dynamic systems [12,53,54].

In the following the conditional independence relation will be used to describe the dependencies of the components of the observations. One may generalizalize the factor analysis model as follows.

DEFINITION 2.7. *a. A* factor model *is a collection of sets* $Y_1, ..., Y_m, X$ *and random variables* $y_1 : \Omega \to Y_1, \ldots, y_m : \Omega \to Y_m$, $x : \Omega \to X$, *defined by the condition that*

$$(F^{y_1}, ..., F^{y_m} \mid F^x) \in CI.$$

The random variables $y_1, \ldots, y_m$ *are called the* observations *and the random variable* x *is called the* factor.

b. A Gaussian factor model *is a factor model such that for* $i \in Z_m$, $Y_i = R^{d_i}$, $X = R^n$ *and* $(y_1, ..., y_m, x)$ *are jointly Gaussian.*

c. A finite factor model *is a factor model such that* $Y_1, ..., Y_m, X$ *are finite sets.*

The Gaussian factor model contains the factor analysis model, the confluence analysis model and the errors-in-variables model 2.2, with m=2, as special cases. A factor model like 2.7 has been used by T.W. Anderson in [4,p.10] and D.J. Bartholomew in [6,7].

Factor models for random variables taking values in discrete sets have been used in latent structure analysis. This subject has been introduced by the sociologist P. Lazersfeld [36]. The concept of latent variable in sociology and statistics is identical to the concept of factor, see [4,p.10;6,p.295;39,p.551]. The term latent variable is used differently in econometrics [1,p.1323;32,p.127;46,p.123].

Stochastic realization problems
The modeling problem of econometrics will now be formulated as a static stochastic realization problem. Before presenting a formal problem formulation an informal discussion is given.

The problem is, given an observed random vector with Gaussian distribution $G(0,Q)$ on R^k, to show existence of a factor analysis model with parameters (k,n,H,Q_x,Q_w) such that the distribution of the observation vector y that is associated with this model equals the given distribution $G(0,Q)$. If such a model exists then it will be called a *stochastic realization* of the given observed vector. In general a stochastic realization, if it exists, is not unique. Two stochastic realizations will be called *equivalent* if they are associated with the same distribution for the observed vector. For any observed vector there is a class of equivalent stochastic realizations. This class must be reduced by imposing a minimality condition. To formulate this condition a definition follows.

DEFINITION 2.8. *The* minimal multivariate conditional independence relation *for an (m+1)-tuple of σ-algebras $F_1,...,F_m,G \in F$ is defined by the conditions:*
1. $(F_1,F_2,...,F_m \mid G) \in CI$;
2. *if* $(F_1, ...,F_m \mid H) \in CI$, *and if* $H \subset G$, *then* $H = G$.
Notation $(F_1,...,F_m \mid G) \in CI_{\min}$.

In analogy with the definitions 2.4 and 2.5 one may define a minimal multivariate Gausssian conditional independence relation that will be denoted by

$$(F^{y_1}, ..., F^{y_m} \mid F^x) \in CIG_{\min}.$$

Minimality in this relation is equivalent to minimality of the dimension of the random variable x. The minimal two-variate conditional independence relation is discussed in [53,54] and the minimal two-variate Gaussian case in [52].

In the stochastic realization problem a major question is to *classify* , or to describe, all equivalent minimal stochastic realizations. It is a fundamental fact of stochastic realization theory for Gaussian processes that in general a minimal stochastic realization is not *unique.* The above mentioned classification question is therefore important. Once one has obtained the classification of all minimal stochastic realizations of a given probability distribution one can obtain a falsifiable model by dividing out the equivalence relation.

Identifiability in econometrics. It is illustrative to compare the above formulated system theoretic approach for the stochastic realization problem to the econometric approach to modeling. We would like to concentrate attention on the socalled "identifiability" problem of econometrics. R.E. Kalman [26, p. 119] has criticized the econometric concept of "identifiability" and this criticism should be widely known.

The econometrics literature usually refers for definitions of "identifiability" of a model to the paper of T.C. Koopmans and O. Reiersøl of 1950 [33]. In that paper a realization problem is posed in abstract terms [33,p.169]. There two structures are defined to be equivalent if these have the same distribution on the observations. But,

> "We then say that a model $\mathfrak{S}$ identifies a parameter $\theta(S)$ in a structure S_0, if that parameter has the same value in all structures S_0^*, contained in G and equivalent to S_0" [33,p.169].

There follows,

> "... a new group of *identification* problems: to determine which of the parameters or other characteristics of a given structure are identifiable by (or "within") a given model" [33,p.169].

Attention is thus concentrated on finding parameters that may be determined from the distribution of the observations. See also [32,p.133;45,p.376;46,p.125]. If the parameters of a certain model cannot all be determined from that distribution, then the model is termed "unidentifiable". Current practice in econometrics has not changed much since [1,p.1332-1335;16,p.993;20;28;38,p.63,p.646]. "Identifiability" of a parameter is sometimes also defined as the existence of a consistent parameter estimator for this parameter [1,p.1335;38]. The question of minimality of a realization is hardly ever posed and neither is the question of classification of all minimal realizations.

By analogy consider the standard deterministic linear realization problem where one wants to represent a linear input-output map by an element of the class of time-invariant finite-dimensional linear systems. Such systems are described by the parameters k,n,m, respectively the dimension of the output, state and input space, and matrices A,B,C,D [56,57]. According to the above definition of "identifiability" only the parameter D is "identifiable". The parameters n,A,B,C are "unidentifiable". Note that no condition of minimality is imposed.

Stochastic realization problems continued

PROBLEM 2.9. *The* weak multivariate Gaussian stochastic realization problem. *Given* $m \in Z_+$, $k_1, \ldots, k_m \in Z_+$ *and a Gaussian probability measure* $G(0,W)$ *on* R^r, $r = \sum_{i=1}^{i=m} k_i$.

a. Does there exist a Gaussian factor model, say with $n \in Z_+$, *a Gaussian measure* $G(0,Q)$ *on* R^{r+n} *and for* $i \in Z_m$, $y_i : \Omega \rightarrow R^{k_i}$, $x : \Omega \rightarrow R^n$ *random variables with*

$$(F^{y_1},\ldots,F^{y_m} \mid F^x) \in CIG,$$

$$(y_1, \ldots, y_m, x) \in G(0,Q),$$

such that

$$(y_1, \ldots, y_m) \in G(0,W)?$$

If so, call (n, G(0,Q)) a weak Gaussian stochastic realization *of* $G(0,W)$.
b. What is the minimal $n \in Z_+$ *for which the question a. has a solution? Call a stochastic realization for which n is minimal, a* minimal stochastic realization.
c. Classify all minimal weak Gaussian stochastic realizations of $G(0,W)$.

The motivation of problem 2.9 comes from psychology, sociology and econometrics. It may also be relevant for game and team theory. Problem 2.9 is a problem of mathematics. It should not be taken as a suggestion to apply it to the representation problem discussed earlier with arbitrarily chosen indices $k_i > 1$.

The multivariate weak Gaussian stochastic realization problem with $k_1 = k_2 = \ldots = k_m = 1$ is known as the factor analysis problem in psychology and statistics and as the confluence analysis problem or representation by errors-in-variables model in econometrics. For references to the psychological and statistics literature see [2,21,23,29,35,49,50,51,52], and for the econometrics literature [1,14,16,20,28,31,32,33,34,38,40,43,44]. A survey of the theory up to 1956 is given by T.W. Anderson and H. Rubin [2]. For a review of the history of factor analysis in psychology see [21,1.1;29,p.74-76;51,Preface].

Despite the age of problem 2.7 for $k_i = 1$, $i \in Z_m$, it is apparently still unsolved. The existence question has been discussed. The minimality question is hardly ever mentioned, [2,p.114 I] being an exception. O. Reiersøl [46,p.127-128] touches on this question in his study of the identifiability of the factor analysis problem. For the psychological literature see [21,Ch.5]. The classification problem has been posed under the name of identification question, see [2,section 5] and the references quoted there. Again, O. Reiersøl [46] treats this question in the context of the identifiability problem. This question is also unsolved. In factor analysis the equivalence class is often reduced by imposing adhoc conditions on the matrix of factor loadings. Approximations of factor models have been proposed in which the variance of the error term is minimized [21,Ch.9]. Most of the literature on factor analysis deals with statistical tests and not with the system theoretic questions of characterization of minimality and classification.

Why have econometricians given up on confluence analysis or factor analysis? The problem has been introduced into econometrics by R. Frisch [14]. T.C. Koopmans has treated it in [31]. O. Reiersøl has partly developed the subject [43,44,45,46]. Apparently econometricians have turned away from this analysis because of the "identifiability" problem mentioned below 2.8 and because of the presence of unobservable random variables in the model. See [16,p.992;20,p.977] for discussions of this point. R.E. Kalman's [24,25,26,27] suggestion of a system theoretic approach to econometric modeling should be taken seriously by econometricians.

Problem 2.10. *The* strong multivariate Gaussian stochastic realization problem. *Given a probability space* (Ω,F,P), $m \in Z_+$, $k_1, ..., k_m \in Z_+$, *for* $i \in Z_m$, $y_i:\Omega \rightarrow R^{k_i}$, $y=(y_1, ..., k_m) \in G(0,Q_y)$.
a. Does there exist a Gaussian factor model with $n \in Z_+$ *and a random variable* $x:\Omega \rightarrow R^n$, *such that*

$$(F^{y_1},...,F^{y_m} \mid F^x) \in CIG,$$

$$F^x \subset (F^{y_1} \vee \cdots \vee F^{y_m}).$$

If so, call this Gaussian factor model a strong Gaussian stochastic realization *of y.*
b. Determine the $n \in Z_+$ *for which in a.*

$$(F^{y_1}, ..., F^{y_m} \mid F^x) \in CIG_{\min}$$

Call a strong Gaussian stochastic realization of y minimal *if n satisfies this condition.*
c. Classify all minimal strong Gaussian stochastic realizations of y.

Problem 2.10 has first been formulated in a Hilbert space framework in [12].

Problem 2.11. *The* weak multivariate finite stochastic realization problem. *Given* k, $m \in Z_+$, *and a probability measure* p_0 *on* $(Z_k)^m$.
a. Does there exist a finite factor model with $n \in Z_+$ *and a probability measure* p_1 *on* $(Z_k)^m \times Z_n$ *such that, if for* $i \in Z_m$ $y_i:\Omega \rightarrow Z_k$, $x:\Omega \rightarrow Z_n$ *are random variables with the distribution of* $(y_1, \ldots ,y_m,x)$ *equal to* p_1 *and*

$$(F^{y_1}, \ldots, F^{y_m} \mid F^x) \in CI$$

then the probability distribution of $(y_1, \ldots, y_m)$ *equals the given probability distribution* p_0*?*
b. What is the minimal $n \in Z_+$ *for which question a. has a solution?*
c. Classify all minimal stochastic realizations p_1.

The motivation for problem 2.11 comes from summarizing data of psychological tests. This problem has been formulated by the sociologist P.F. Lazersfeld in 1950 [36], and subsequently developed by him [37] and others [4,39]. See the last two references for literature on this subject. A discussion of this problem is given by T.W. Anderson [4]. One may also pose a multivariate strong finite stochastic realization problem.
A generalization of the factor analysis problem has been posed by D.J. Bartholomew [6,7].
The strong Gaussian stochastic realization problem may be generalized as follows.

Problem 2.12. *The* multivariate σ-algebraic stochastic realization problem. *Given* $F_1, \ldots ,F_m \in \underline{F}$. *Determine all σ-algebras* $G \in \underline{F}$ *such that*

$$(F_1,...,F_m \mid G) \in CI_{\min},$$

$$G \subset (F_1 \vee F_2 \vee \cdots \vee F_m).$$

Preliminary results on problem 2.12 will not be stated here because of space limitations. The two-variate version of problem 2.12 is discussed in [53,54].

3. The multivariate Gaussian stochastic realization problem

A special case of problem 2.8 will be discussed.

A technical result

Proposition 3.1. *Let $m \in Z_+$, and for $i \in Z_m$ $y_i:\Omega \rightarrow R^{k_i}$, $x:\Omega \rightarrow R^n$, be zero mean random variables. Then a., b., and c. below are equivalent:*

a.

$$(F^{y_1},...,F^{y_m} \mid F^x) \in CIG;$$

b.1.

$$(y_1, ..., y_m, x) \in G;$$

b.2. *with respect to some basis, for all $i, j \in Z_m$, $i \neq j$,*

$$Q_{ij} = Q_{ix} Q_x^{-1} Q_{xj},$$

where $x \in G(0,Q_x)$, $Q_x > 0$, $Q_{ij} = E[y_i y_j^T]$, and $Q_{ix} = E[y_i x^T]$;

c. *there exist for $i \in Z_m$, $w_i:\Omega \rightarrow R^{k_i}$, $w_i \in G(0,Q_{w_i})$, and $H_i \in R^{k_i \times n}$, such that $x \in G(0,Q_x)$, $(x, w_1, ..., w_m)$ independent, and*

$$y_i = H_i x + w_i,\ i \in Z_m.$$

The weak and strong two-variate Gaussian stochastic realization problem have been solved, see [52] and for another representation [12]. It is of interest to know that the solution has been obtained by exploiting the canonical variable representation [3, Ch.12], by using a suitable representation and by solving a Riccati-like inequality.

Comments on the errors-in-variables model

The errors-in-variables model 2.2 leads to a difficulty that is due to non-minimality. Assume given this model with

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\quad k_1 < k_2 \text{ say},$$

$$z = Gu,\ y_1 = u + w_1,\ y_2 = z + w_2,$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I \\ G \end{bmatrix} u + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

If one accepts the partition of y into (y_1, y_2) one may ask for the minimal $x:\Omega \rightarrow R^n$ such that

$$(F^{y_1}, F^{y_2} \mid F^x) \in CIG_{\min},$$

or, equivalently,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

with (x, w_1, w_2) independent and Gaussian. Now it may be the case that $n = dim(x) < k_1 < k_2$ [52]. Then the errors-in-variables model defined above, in which

$$(F^{y_1}, F^{y_2} \mid F^u) \in CIG,$$

with $k_1 = dim(u) > n$, is necessary non-minimal. This non-minimality will lead to a rather large class of equivalent realizations. This in turn will lead to difficulties if one tries to estimate parameters of this model.

However the problem is not as bad as it seems. Generically the dimension of a minimal x is $rank(E[y_1 y_2^T]) = k_1$, if $k_1 < k_2$, and then there is no difficulty with non-minimality [52].

The three variate Gaussian stochastic realization problem

The multivariate weak Gaussian stochastic realization problem will probably have to be solved by induction on the number of components of the observation vector. Since the case of two variables has been solved in [52], the next case in line is the three-variate case.

Assume one is given for $i \in Z_3$ $y_i : \Omega \rightarrow R^{k_i}$, $(y_1, y_2, y_3) \in G(0, Q_y)$. In the two-variate case referred to above, a key role is played by the canonical variable decomposition developed by H. Hotelling [22]. For the three-variate case a canonical form seems also necessary. One can speak here about a canonical variable form because the problem is specified in terms of spaces rather than in terms of the particular variables.

PROBLEM 3.2. [22, p.375]. *Consider* $Q \in R^{k \times k}$ *with* $Q = Q^T \geqslant 0$, $k = k_1 + k_2 + k_3$. *Define an* internal transformation *as* $Q \rightarrow SQS^T$ *with*

$$S = blockdiagonal(S_1, S_2, S_3),$$

where for i=1,2,3, $S_i \in R^{k_i \times k_i}$. *Call* Q_1, $Q_2 \in R^{k \times k}$ equivalent *if there exists an internal transformation* S *such that* $Q_1 = SQ_2S^T$. *Determine a canonical form for Q under this equivalence relation, thus for the set*

$$\underline{T}(Q) = \left\{ \begin{array}{c} Q_1 \in R^{k \times k} \mid Q_1 = Q_1^T \geqslant 0 \text{ and there exists an internal} \\ \text{transformation } S \text{ such that } Q_1 = SQS^T \end{array} \right\}.$$

Problem 3.2 is posed by Hotelling at the conclusion of the paper [22] in which he solves the analogous problem for two sets of random variables. There are some references on this problem, see [30] and the references quoted there.

A conjecture has been formulated for the canonical form requested in problem 3.2. In mathematical terms it is somewhat elaborate, hence it will be described in words. Each of the three sets of variables may be decomposed into several parts consisting of:

1. components that are independent;
2. components that are correlated with those of one other variable but independent of the components of the third;
3. components that are dependent.

For the third set of components a further decomposition should be given but how is not yet clear.

The weak multivariate Gaussian stochastic realization problem 2.9 with $k_i = 1$ for all $i \in Z_m$, can be shown to be equivalent to the following problem.

PROBLEM 3.3. *Given* $Q \in R^{m \times m}$, $Q = Q^T \geqslant 0$. *Determine* (n, H, Q_w) *with* $n \in Z_+$, $H \in R^{m \times n}$, $Q_w \in R^{m \times m}$, $Q_w = Q_w^T \geqslant 0$ *and diagonal, such that*

$$Q = HH^T + Q_w.$$

In particular, determine the minimal $n \in Z_+$ *for which this is possible and classify all solutions* (n, H, Q_w) *for the minimal n.*

A characterization for the existence of just one factor has already been given by C. Spearman [49]. See also [2,section 4] for results of this problem. R.E. Kalman has presented solutions to specific cases of this problem [24,p.10;27]. It seems that algebraic and geometric methods must be used to attack problem 3.3.

In the following attention is restricted to the three-variate strong Gaussian stochastic realization problem 2.8. It can be shown that this problem has a solution iff there exists a $T \in R^{n \times k}$, with $k = \sum_{i=1}^{i=3} k_i$, such that,

1. $x = Ty$, $Q_x > 0$,
2. $Q_{ij} = Q_{ix} Q_x^{-1} Q_{xj}$, *for all* $i \in Z_3$, $i \neq j$, *where* Q_{ij}, Q_{ix} *are as defined in 3.1.*

In the following attention is further restricted.

CASE 3.4. *Consider the three-variate strong Gaussian stochastic realization problem with* $k_1 = k_2 = k_3 = 1$, $y_1, y_2, y_3 : \Omega \to R$, $(y_1, y_2, y_3) \in G(0, Q_y)$, $rank(Q_y) = 3$. *Assume, without loss of generality, that*

$$Q_y = \begin{bmatrix} 1 & q_{12} & q_{13} \\ q_{12} & 1 & q_{23} \\ q_{13} & q_{23} & 1 \end{bmatrix}.$$

If $rank(Q_y) < 3$ then there are other special cases.

It is first shown that the existence question of problem 2.10 is trivial. For example, either x_1, x_2, or x_3 defined by

$$x_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ x_2 = \begin{bmatrix} y_1 \\ y_3 \end{bmatrix}, \ x_3 = \begin{bmatrix} y_2 \\ y_3 \end{bmatrix},$$

satisfies the conditions of 2.9.a.

The next question is the minimality of the dimension of the random variable x. In general the minimal n can be 0, 1, or 2. One may distinguish special cases based on the conjectured canonical form for problem 3.2. Thus if $q_{12} = q_{13} = q_{23} = 0$, which implies that (y_1, y_2, y_3) are independent, then the dimension of x is zero. Of all the other special cases, that of q_{12}, q_{13}, q_{23} nonzero is the most interesting one.

PROPOSITION 3.5. *Given case 3.4 of the three-variate strong Gaussian stochastic realization problem. Assume that for* $i, j \in Z_3$, $i \neq j$, *one has* $q_{ij} \neq 0$. *Then there exists a solution* $x : \Omega \to R$ *of this problem of dimension n=1 iff one of the following conditions is satisfied:*

1. $q_{12} = q_{13} q_{23}$;
2. $q_{13} = q_{12} q_{23}$;
3. $q_{23} = q_{12} q_{13}$.

Moreover, if $q_{12} = q_{13} q_{23}$ *then* $x = y_3$ *and* $(F^{y_1}, F^{y_2} \mid F^{y_3}) \in CIG$. *The condition* $(F^{y_1}, F^{y_2} \mid F^{y_3}) \in CIG$ *implies directly that* $(F^{y_1}, F^{y_2}, F^{y_3} \mid F^{y_3}) \in CIG$.

Consider again case 3.4 of the three-variate Gaussian stochastic realization problem. In the cases not discussed above the dimension of the minimal factor is 2. What is the classification of all such x's? This question leads to a set of quadratic equations. Thus one has to determine all $T \in R^{2 \times 3}$ such that

$$TQ_y T^T = I,$$

$$q_{12} = q_1 T^T T q_2, \quad q_{13} = q_1 T^T T q_3, \quad q_{23} = q_2 T^T T q_3,$$

where q_1, q_2, q_3 are the columns of Q_y. As of yet no transparent way of representing the solution is known. The solution will have to be based on the theory of quadratic equations.

4. Realization of stochastic processes

The problem formulation

The problem to be considered here is the modeling of a process or time series. Specifically, this involves the selection of a class of stochastic dynamic systems that may be used to model the process, and the formulation and solution of a stochastic realization problem. This modeling problem arises in econometrics, psychology, control and communication and in other areas of science and engineering.

This modeling problem becomes more interesting if one demands that the inherent causality relation or dependency between the components of the observed process is exhibited. This problem should be considered in the light of the discussion of section 2. Thus below one will see the analogue for stochastic processes of the regression model, the errors-in-variables model and the factor analysis model.

A question is then what concept of causality or dependence relation to use in this context. As mentioned before definitions of a causality relation between a pair of random variables or stochastic processes have been proposed in the econometric literature. The two concepts most often mentioned are Granger causality [17,18,19] and Sims causality [47,48]; see also the references [8,13,42,58] and the references quoted there. The notion of a pair of feedback free processes has also been introduced. Considering the discussion of [8] it seems that the only interesting system theoretic question is the decomposition of a stochastic system into subsystems that are interconnected serially and/or in parallel without feedback connections. The usefulness of such a general decomposition is not clear. Another topic must be mentioned in this context namely the feedback decomposition of Gaussian processes [15].

Below attention is concentrated on two methods to represent the causality relation between the components of a stochastic process. The first method concerns a rather strong causality concept, in which one set of components is the input of a stochastic system and another set of components the output of the same system. The second method is factor analysis for stochastic processes.

For a mathematical problem formulation assume given of an observed Gausian stochastic process the family of finite dimensional distributions. In the rest of this section some attempts to model such a process are reviewed.

Modeling of time series in psychometrics

In the past psychologists have applied factor analysis to time series in a rather naieve way. Below the criticism of this approach by T.W. Anderson [5] is summarized. Psychologists may have data from a patient over time. The application of factor analysis then proceeds by averaging these data over time and applying a factor analysis to the averaged data. T.W. Anderson's criticism is that in this approach,

> "... there is a danger of missing important and interesting characteristics which are significant because of their development in time..." [5, p. 9].

See also the criticism in [5,section 6]. Finally he suggests psychologists apply time series analysis, in particular ARMA-represention, to psychological time series. In system theoretic terms his advice is to use stochastic dynamic systems and stochastic realization theory.

Stochastic realization without causality relations

Econometric practice for the modeling of time series is to apply the regression model where the regressor may contain observations with lagged time indices. One may interpret this representation as one in which some variables are caused or explained by other, possibly delayed, variables. The resulting equations may be combined into an autoregressive representation possibly with moving average part.

The criticism from system theory on this procedure is that the selection of regressors is arbitrary, see the comment on the regression model, and that the concept of state of a dynamical system is not made explicit.

The system theoretic approach to the modeling problem posed above is then to define dynamic systems in which the state is made explicit and to use realization theory. In the problem under discussion one would use stochastic realization theory for which only the case of Gaussian processes has been developed in some detail [10,11,12].

Stochastic realization in the class of stochastic systems with inputs

DEFINITION 4.1. *A* Gaussian system with input *is an object described by*

$$x_{t+1} = Ax_t + Bu_t + Mv_t,$$
$$y_t = Cx_t + Du_t + Nv_t,$$

where $v:\Omega \times T \to R^p$, $v_t \in G(0, Q_v)$ *is a Gaussian white noise process,* $u:\Omega \times T \to R^m$ *a stochastic process in some class* U, $x:\Omega \times T \to R^n$, $y:\Omega \times T \to R^k$ *are stochastic processes defined by the above relations,* $A \in R^{n \times n}$, $B \in R^{n \times m}$, $M \in R^{n \times p}$, $C \in R^{k \times n}$, $D \in R^{k \times m}$, $N \in R^{k \times p}$, (v,u) *are independent, and* $T=Z$. *The observations consist of the processes* (u,y).

For the class of inputs U one may take white noise processes or zero mean stationary Gaussian processes possibly with a rational spectral density.

PROBLEM 4.2. *Given a zero mean stationary Gaussian process with values in* R^r, *the class of Gaussian systems with input and a class of input processes* U. *Determine* $k,m \in N$, *a permutation matrix* $P \in R^{r \times r}$ *and a Gaussian system with inputs with* k,m *being respectively the dimensions of the output space and the input space, such that*

$$r = k + m,$$
$$P \begin{pmatrix} u \\ y \end{pmatrix}, \text{ equals the given process in distribution.}$$

Such a system will be called a weak Gaussian stochastic realization *of the given observed process. It will be called* minimal *if the dimension of the system is minimal. Moreover, classify all minimal such realizations.*

Note the analogy of definition 4.1 with the regression model of 2.1. Problem 4.2 has been inspired by work of J.C. Willems [56,57] on deterministic realization problems.

As a criterion for the usefulness of definition 4.1 one must pose that the dimension m of the input space and the dimension k of the output space can be uniquely determined from the family of finite dimensional distributions of the given process. Apparently this cannot be done without imposing additional conditions. If the class of inputs U consists only of white noise processes then one can show that the dimension of the input space is not unique although its maximal value is unique. If the class of inputs consists of zero mean stationary Gaussian processes then the example of a series connection of two Gaussian systems with inputs shows that the dimension of the input space cannot be uniquely determined. Possibly the condition that the dimension of the input space is minimal may be necessary to achieve uniqueness. Therefore this approach remains to be investigated in more detail.

The dynamic errors-in-variables system

The opportunity is taken to criticize a stochastic system suggested in [12,p.449]. The stochastic realization problem proposed there is to represent an observed process as an element of the following class of dynamical systems.

DEFINITION 4.3. *A* dynamic errors-in-variables system *is an object described by*

$$x_{t+1} = Ax_t + Bu_t,$$
$$y_t = Cx_t,$$
$$z_{1t} = u_t + w_{1t},$$
$$z_{2t} = y_t + w_{2t},$$

where $w_1, z_1 : \Omega \times T \rightarrow R^{k_1}$, $w_2, z_2 : \Omega \times T \rightarrow R^{k_2}$, *are stochastic processes with* w_1, w_2 *independent Gaussian white noise processes,* $u : \Omega \times T \rightarrow R^{k_1}$ *is a stochastic process from some class* $\underline{U}$, $x : \Omega \times T \rightarrow R^n$, $y : \Omega \times T \rightarrow R^{k_2}$ *are stochastic processes,* $A \in R^{n \times n}$, $B \in R^{n \times k_1}$ $C \in R^{k_2 \times n}$ *and the observed process is* (z_1, z_2).

Apparently an assumption is imposed on the data, namely the partition of the observation in the components z_1, z_2. If one disregards this aspect, then the class of these stochastic systems is a proper subset of the class defined in 4.1 above.

Stochastic realization and factor analysis
Can factor analysis be used in the modeling of time series? Confluence analysis as proposed by R. Frisch and developed by O. Reiersøl has also been formulated for time series, see [14,43,44]. Specifically they have considered roughly the following representation.

DEFINITION 4.4. *The* dynamic factor analysis model. *The observed process* $z : \Omega \times T \rightarrow R^r$ *is specified by*

$$z_t = r_t + w_t,$$
$$Sr_t = 0,$$

where $w : \Omega \times T \rightarrow R^r$ *is a stationary Gaussian white noise process with for all* $t \in T$ $w_t \in G(0, Q_w)$ *and* Q_w *diagonal,* $r : \Omega \times T \rightarrow R^r$ *is a zero mean stationary Gaussian process independent of w and* $S \in R^{p \times r}$.
Equivalently, one can say that

$$z_t = Hs_t + w_t,$$

where $s : \Omega \times T \rightarrow R^q$ *is a zero mean stationary Gaussian process independent of w and* $H \in R^{r \times q}$.

The above definition differs from [43,44] in that the assumption of uncorrelatedness of random variables has been replaced by the assumption of Gaussian distributions and independence of random variables.

The dynamic factor analysis model has apparently not been investigated since the work of O. Reiersøl [43,44]. The difficulties with the "identifiability" issue as discussed in section 2 has halted progress on this approach. The question whether factor analysis can be used in the modeling of time series remains therefore unanswered.

The criticism from a system theorist of the dynamic factor analysis model 4.4 is that it does not specify the dynamic behavior of the process r or the process s. In system theory one uses state space representations in which the state is made explicit. One could extend the dynamic factor analysis model by specifying that the stochastic process s is the output of a Gaussian stochastic system [12,54]. This approach remains to be investigated.

Conclusion
The analysis of the stochastic realization of a stochastic process while exhibiting the dependencies between the components is incomplete. The approach of searching for a realization in the class of Gaussian stochastic systems where the observed process has to be partitioned into an input and output process is unsatisfactorily without further conditions. The criticism of the regression model should be taken seriously also for this approach. The combination of factor analysis and Gaussian stochastic dynamic systems may be useful. For the moment a definition of a stochastic dynamic

system that incorporates both the factor and the stochastic dynamic system properties is still missing. More research is wanted here.

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