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ON A CLASS OF NEARLY SINGULAR OPTIMAL CONTROL PROBLEMS

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On a class of nearly singular optimal control problems

by

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ABSTRACT

For a class of linear singular optimal control problems with a non-unique singular arc, the solution of the corresponding nearly singular problem is analyzed and a limit solution based on formal singular perturbations is derived. The result is verified by using an asymptotic power series expansion satisfying the Riccati equation of the nearly singular problem.

KEY WORDS & PHRASES: *Cheap control, singular perturbation*

1. INTRODUCTION

We consider the class of linear, time-invariant, n-dimensional dynamical systems

$$\dot{x} = Ax + Bv, \quad x(0) = x_0 \qquad (1.1ab)$$

with performance index

$$J = \int_0^\infty x'Qx + \varepsilon^2 v'Rv \, dt, \quad 0 < \varepsilon << 1, \qquad (1.1c)$$

where Q is a symmetric positive semi-definite matrix and R is symmetric and positive definite. We denote the n-dimensional state space by $\mathcal{X}$. The control vector takes its values in the linear n-dimensional space $\mathcal{U}$ and $v(\cdot): \mathbb{R}^+ \to \mathcal{U}$ is assumed to be a piece-wise continuous mapping. In this paper we analyze the problem of perfect regulation for a class of cheap optimal control problems of the type (1.1). For $\varepsilon = 0$ (1.1) reduces to a singular optimal control problem, which, as it is shown in [3], may have a family of solutions. As $\varepsilon \to 0$ the solution of (1.1) will tend to one of these solutions. In order to formulate such a class of singular problems in terms of A,B and Q we introduce some concepts of geometric system theory in section 2. For a more extensive exposition we refer to WONHAM [10]. In section 3 we specify the class of problems (1.1) to which our investigations apply and carry out some transformations in order to bring the system in its most suitable form. In section 4, a formal method for selecting the appropriate singular solution is presented, while in the sections 5 and 6 we prove the correctness of the result by perturbing the solution of (1.1) with respect to ε. It is remarked that the convergence of x satisfying (1.1) for $\varepsilon \to 0$ can also be proved by analyzing its Laplace transform see FRANCIS [1,2].

2. SOME CONCEPTS OF GEOMETRIC SYSTEM THEORY

Before giving a definition of controllability subspaces we introduce the concept of (A,B)-invariant subspaces.

DEFINITION 2.1. A subspace $\mathcal{V} \subset \mathcal{X}$ is called (A,B)-invariant if for any $x_0 \in \mathcal{V}$ there exists a control $u(\cdot): \mathbb{R}^+ \to \mathcal{U}$ such that $x(t)$ satisfying (1.1ab) remains in $\mathcal{V}$ for $t > 0$.

Let $\mathcal{B} = \operatorname{Im}B$. It can be proved that (A,B)-invariant subspaces may be characterized by the property $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$, or, equivalently, by the existence of a family of feedbacks

$$(2.1) \qquad \underline{F}(\mathcal{V}) = \{F: \mathcal{X} \to \mathcal{U} \mid (A+BF)\,\mathcal{V} \subset \mathcal{V}\},$$

so that the closed loop system that starts $\mathcal{V}$ remains in $\mathcal{V}$ for $t > 0$. The class of (A,B)-invariant subspaces contained in some subspace of $\mathcal{X}$ is closed under addition and, thus, has a supremal element, see [10]. In the sequel we denote the supremal (A,B)-invariant subspace contained in $\mathcal{K} = \operatorname{Ker}Q$ by $\mathcal{V}^*_{\mathcal{K}}$.

DEFINITION 2.2. A subspace $\mathcal{R} \subset \mathcal{X}$ is called a controllability subspace if for any $x_0, x_1 \in \mathcal{R}$ there exists a $T > 0$ and a $u(\cdot): \mathbb{R}^+ \to \mathcal{U}$ such that $x(t)$ given by (1.1ab) satisfies $x(T) = x_1$ and $x(t) \in \mathcal{R}$ for $0 < t < T$.

Clearly, a controllability subspace is also (A,B)-invariant. Given a subspace $\mathcal{B}_0 \subset \mathcal{X}$ and a mapping $A_F: \mathcal{X} \to \mathcal{X}$, we define the subspace $\mathcal{R}_0 \subset \mathcal{X}$ by

$$(2.2) \qquad \mathcal{R}_0 = \langle A_F \mid \mathcal{B}_0 \rangle \equiv \mathcal{B}_0 + A_F\mathcal{B}_0 + \ldots + A_F^{n-1}\mathcal{B}_0.$$

It can be shown that $\mathcal{R}$ is a controllability subspace if and only if

$$(2.3) \qquad \mathcal{R} = \langle A+BF \mid \mathcal{B} \cap \mathcal{R} \rangle \quad \text{for } F \in \underline{F}(\mathcal{R}).$$

Furthermore, the class of controllability subspaces contained in some subspace of $\mathcal{X}$ is closed under addition and, thus, has a supremal element. The supremal controllability subspace contained in $\mathcal{K} = \operatorname{Ker}Q$ we denote by $\mathcal{R}^*_{\mathcal{K}}$. It can be proved that

(2.4) $\mathcal{R}^*_K = \langle A+BF|\mathcal{B} \cap \mathcal{V}^*_K\rangle$ for $F \in \underline{F}(\mathcal{V}^*_K)$.

3. THE NEARLY SINGULAR OPTIMAL CONTROL PROBLEM

For the class of problems (1.1) we assume that

(3.1) $\mathcal{X} = \mathcal{K} + \mathcal{B}$.

Furthermore, it is supposed that $\mathcal{R}^*_K \neq 0$ as this property characterizes the class of problems we are aiming at, while condition (3.1) is meant as a restriction to focus our attention to a representive subclass for which the limit problem has a non-unique solution. The present study can be seen as the counterpart of the work by O'Malley and Jameson [8,9], where implicitly $\mathcal{R}^*_K = 0$. Since $A\mathcal{K} \subset \mathcal{X} = \mathcal{K} + \mathcal{B}$, we have that $\mathcal{V}^*_K = \mathcal{K}$ (see section 2). Let $\kappa = \dim \mathcal{K}$. We assume that the state space $\mathcal{X}$ is the span of n basis vectors $e_1,\ldots,e_n$ chosen in such a way that $\mathcal{K}$ is the span of last κ of them. The control space $\mathcal{U}$ is the span of m basis vectors $d_1,\ldots,d_m$ chosen in such way that $B^{-1}e_1,\ldots,B^{-1}e_{n-\kappa}$ has the same span as the first $n-\kappa$ basis vectors d_i, so

(3.2a) $\mathcal{K} = \{x|x \in \mathcal{X},\ x_1 =\ldots= x_{n-\kappa} = 0\}$

and

(3.2b) $B^{-1}\mathcal{K} = \{v|v \in \mathcal{U},\ v_1 =\ldots= v_{n-\kappa} = 0\}$,

where B^{-1} denotes the functional inverse of B, see [10, p.6]. By regular mappings H: $\mathcal{X} \to \mathcal{X}$ and G: $\mathcal{U} \to \mathcal{U}$ any system (A,B,Q,R) can be transformed into a system ($H^{-1}AH$, $H^{-1}BG$, H'QH, G'RG) of the required form. Note that H'QH and G'RG are symmetric and positive (semi-)definite.

Consequently, we may restrict ourselves to systems (1.1) of the form

(3.3) $$\begin{pmatrix}\dot{x}_s\\ \dot{x}_k\end{pmatrix} = \begin{pmatrix}A_s & A_{sk}\\ A_{ks} & A_k\end{pmatrix}\begin{pmatrix}x_s\\ x_k\end{pmatrix} + \begin{pmatrix}B_s & 0\\ 0 & B_k\end{pmatrix}\begin{pmatrix}v_s\\ v_k\end{pmatrix}$$

satisfying (3.2). It is noted that because of (3.1) B_s is one to one.

For the control vector we write

$$(3.4)\qquad \begin{pmatrix} v_s \\ v_k \end{pmatrix} = \begin{pmatrix} 0 & -B_s^{-1}A_{sk} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_s \\ x_k \end{pmatrix} + \begin{pmatrix} u_s \\ u_k \end{pmatrix},$$

so that (1.1) becomes

$$(3.5ab)\qquad \begin{pmatrix} \dot{x}_s \\ \dot{x}_k \end{pmatrix} + \begin{pmatrix} A_s & 0 \\ A_{ks} & A_k \end{pmatrix} \begin{pmatrix} x_s \\ x_k \end{pmatrix} + \begin{pmatrix} B_s & 0 \\ 0 & B_k \end{pmatrix} \begin{pmatrix} u_s \\ u_k \end{pmatrix}, \begin{pmatrix} x_s(0) \\ x_k(0) \end{pmatrix} = \begin{pmatrix} x_{s0} \\ x_{k0} \end{pmatrix}$$

with performance index

$$(3.5c)\qquad J = \int_0^\infty (x_s',x_k') \begin{pmatrix} Q_s & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_s \\ x_k \end{pmatrix} + \varepsilon^2 \Bigg[(x_s',x_k') \begin{pmatrix} 0 & 0 \\ 0 & M_x \end{pmatrix} \begin{pmatrix} x_s \\ x_k \end{pmatrix},$$

$$2(x_s',s_k') \begin{pmatrix} 0 & 0 \\ N_0 & N_k \end{pmatrix} \begin{pmatrix} u_s \\ u_k \end{pmatrix} + (u_s',u_k') \begin{pmatrix} R_s & R_{us}' \\ R_{ks} & R_k \end{pmatrix} \begin{pmatrix} u_s \\ u_k \end{pmatrix} \Bigg] dt,$$

where $M_k = (B_s^{-1}A_{sk})'R_sB_s^{-1}A_{sk}$, $N_k = -(B_s^{-1}A_{sk})'R_{ks}'$ and $N_{ks} = (B_s^{-1}A_{sk})'R_s$. In the sequel we denote by A,B,Q,M,N and R the mappings of (3.5). About these mappings we make the following hypotheses. Let G, C_k and D_k be such that $GG' = Q$, $C_kC_k' = R_k^{-1}$ and $D_kD_k' = M_k$. Then

(H3.1) the pair (A,B) is stabilizable

(H3.2) the pair (G,A) is detectable

(H3.3) the pair $(A_k-B_kR_k^{-1}N_k', B_kC_k)$ is stabilizable

(H3.4) the pair $(D_k'-C_k'N_k', A_k-B_kR_k^{-1}N_k')$ is detectable.

It is known that under the assumptions (H3.1) and (H3.2), (3.5) has an optimal solution with

$$(3.6a)\qquad u = -\varepsilon^{-2}R^{-1}(B'P_\varepsilon+\varepsilon^2N')x,$$

where P_ε is the unique positive semi-definite symmetric solution of the algebraic Riccati equation

(3.6b) $$P_\varepsilon(A-BR^{-1}N')+(A-BR^{-1}N')'P_\varepsilon-\varepsilon^{-2}P_\varepsilon BR^{-1}B'P_\varepsilon+Q+\varepsilon^2(M-NR^{-1}N') = 0.$$

4. THE FORMAL LIMIT SOLUTION

Since the cost of control is small, it is expected that by some appropriately chosen initial pulse the solution will tend rapidly to the subspace K. In order to analyze this behaviour we carry out the following transformations

(4.1) $$u = \hat{u}/\varepsilon, \quad t = \tau\varepsilon \quad \text{and } J = \hat{J}\varepsilon.$$

Substituting (4.1) into (3.5) and formally letting $\varepsilon \to 0$ we obtain

(4.2a) $$d\hat{x}/d\tau = B\hat{u}$$

(4.2b) $$\hat{J} = \int_0^\infty \hat{x}'Q\hat{x} + \hat{u}'R\hat{u}\, d\tau.$$

We consider the feedback

(4.3a) $$\hat{u} = R^{-1}B'\hat{P}\hat{x}$$

with $\hat{P}$ satisfying

(4.3b) $$\hat{P}BR^{-1}B'\hat{P} = Q.$$

Partitioning the inverse of R as

(4.4) $$R^{-1} = T = \begin{pmatrix} T_s & T'_{ks} \\ T_{ks} & T_k \end{pmatrix},$$

we write the unique positive semidefinite solution of (4.3b) as

$$\text{(4.5a)} \qquad P = \begin{pmatrix} P_{s0} & 0 \\ 0 & 0 \end{pmatrix}$$

with $P_{s0} > 0$ satisfying

$$\text{(4.5b)} \qquad P_{s0}B_sT_sB_s'P_{s0} = Q_s.$$

The corresponding closed loop system reads

$$\text{(4.6)} \qquad \begin{pmatrix} d\hat{x}_s/d\tau \\ d\hat{x}_k/d\tau \end{pmatrix} = \begin{pmatrix} -B_sT_sB_s'P_{s0} & 0 \\ -B_kT_{ks}B_s'P_{s0} & 0 \end{pmatrix}\begin{pmatrix} \hat{x}_s \\ \hat{x}_k \end{pmatrix}.$$

Integration yields

$$\text{(4.7a)} \qquad \hat{x}_s(\tau) = e^{-B_sT_sB_s'P_{s0}\tau}x_{s0},$$

$$\text{(4.7b)} \qquad \hat{x}_k(\tau) = x_{k0} - \int_0^{\tau} B_kT_{ks}B_s'P_{s0}\hat{x}_s(\bar{\tau})d\bar{\tau}.$$

It is noted that $B_sT_sB_s'P_{s0} = P_{s0}^{-1}Q_s$ is positive definite. Consequently, as $\tau \to \infty$ $\hat{x}_s \to 0$ and $\hat{x}_k \to x_{k0} - \xi_{k0}$ with

$$\text{(4.8)} \qquad \xi_{k0} = B_kT_{ks}T_s^{-1}B_s^{-1}x_{s0}.$$

Letting $\varepsilon \to 0$, we observe that at the initial point the solution jumps from (x_{s0},x_{k0}) to $(0,x_{k0}-\xi_{k0})$. Once the solution is in the subspace K it remains there as K is A invariant for (3.5). The performance index will be zero as $\varepsilon \to 0$ for any feedback $u_k = F_kx_k$. For the purpose of selecting the appropriate feedback we consider the optimal control problem for x_k given by (3.5ac) with $x_s = 0$ for $t > 0$:

$$\text{(4.9ab)} \qquad \dot{\bar{x}}_k = A_k\bar{x}_k + B_k\bar{u}_k, \quad \bar{x}_k(0) = x_{k0} - \xi_{k0}$$

$$\text{(4.9c)} \qquad \bar{J} = \int_0^{\infty} \bar{x}_k'M_k\bar{x}_k + 2\bar{x}_k'N_k\bar{u}_k + \bar{u}_k'R_k\bar{u}_k dt.$$

From (H3.2) and (H3.3) it follows that an optimal solution exists with

$$(4.10) \qquad \bar{u}_k = -R_k^{-1}(B_k'\bar{P}_k + N_k')\bar{x}_k,$$

where $\bar{P}_k$ is the unique positive semi-definite solution of the algebraic Riccati equation

$$(4.10b) \qquad \bar{P}_k(A_k - B_kR_k^{-1}N_k') + (A_k - B_kR_k^{-1}N_k')'\bar{P}_k - \bar{P}_kB_kR_k^{-1}B_k'\bar{P}_k + (M_k - N_kR_k^{-1}N_k') = 0,$$

see KUČERA [4]. Thus, the optimal solution reads

$$(4.11) \qquad \bar{x}_k(t) = e^{(A_k - B_kR_k^{-1}B_k'\bar{P}_k - B_kR_k^{-1}N_k')t}(x_{k0} - \xi_{k0}).$$

<u>REMARK</u>. It is not obvious that $x_k(t,\varepsilon) \to \bar{x}_k(t)$ for $t \geq \delta > 0$ and $\varepsilon \to 0$, as $\bar{x}_k$ follows from the order $0(\varepsilon^2)$ terms of the performance index. Since $x_s = 0(\varepsilon)$, x_s is also present with terms of order $0(\varepsilon^2)$, so before hand it is not clear that the system can be decomposed in the above way.

5. ASYMPTOTIC SOLUTION OF THE RICCATI EQUATION

Let us assume that the positive semi-definite solution of the algebraic Riccati equation (3.6b) can be expanded as

$$(5.1) \qquad P_\varepsilon = \varepsilon \sum_{j=0}^{\infty} P^{(j)}\varepsilon^j, \quad P^{(j)} = \begin{pmatrix} P_{sj} & P'_{ksj} \\ P_{ksj} & P_{kj} \end{pmatrix}.$$

Substitution of (5.1) into (3.6b) yields, by setting $\varepsilon = 0$, $P^{(0)} = \hat{P}$ with $\hat{P}$ given by (4.5). Equating the coefficients of the terms to ε we obtain

$$(5.2) \qquad P_{s0}A_s + A_s'P_{s0} - P_{s1}B_sT_sB_s'P_{s0} - P_{s0}B_sT_sB_s'P_{s1} = 0$$

and

$$(5.3) \qquad -P_{s0}B_sT_sN_{ks}' - P_{s0}B_sT_{sk}N_k' - P_{s0}B_sT_sB_s'P_{ks1}' - P_{s0}B_sT_{sk}B_k'P_{k1} = 0.$$

Since $P_{s0}B_s > 0$ (5.3) is equivalent to

$$(5.4)\qquad T_s N'_{ks} + T_{sk} N'_k + T_s B'_s P'_{ks1} + T_{sk} B'_k P_{k1} = 0.$$

Equating the terms of $0(\varepsilon^2)$ we obtain the equation

$$(5.5)\qquad \begin{aligned} &-P_{ks1} B_s T_s N'_{ks} - P_{ks1} B_s T_{sk} N'_k + P_{k1} A_k - P_{k1} B_k T_{ks} N'_{ks} - P_{k1} B_k T_k N'_k \\ &-N_{ks} T_s B'_s P_{ks1} - N_k T'_{ks} B'_s P'_{ks1} + A'_k P_{k1} - N_{ks} T'_{ks} B'_k P_{k1} - N_k T_k B'_k P_{k1} \\ &-P_{ks1} B_s T_s B'_s P'_{ks1} - P_{ks1} B_s T_{sk} B_k P_{k1} - P_{k1} B_k T_{ks} B'_s P'_{ks1} - P_{k1} B_k T_k B'_k P_{k1} \\ &+M_k - N_{ks} T_s N'_{ks} - N_{ks} T_{sk} N'_k - N_k T_{ks} N'_{ks} - N_k T_k N'_k = 0. \end{aligned}$$

From (4.4) we derive that

$$(5.6)\qquad R_k^{-1} = T_k - T_{ks} T_s^{-1} T'_{ks}.$$

Using (5.4) and (5.6) we reduce equation (5.5) to

$$(5.7)\qquad \begin{aligned} &P_{k1}[A_k - B_k R_k^{-1} N'_k] + [A_k - B_k R_k^{-1} N'_k]' P_{k1} + \\ &\qquad - P_{k1} B_k R_k^{-1} B'_k P_{k1} + M_k - N_k R_k^{-1} N'_k = 0, \end{aligned}$$

which has a unique positive semi-definite solution $P_{k1} = \bar{P}_k$ see (4.10b). This iteration process can be continued to yield uniquely determined coefficients $P^{(j)}$, $j = 2,3,\ldots$.

6. THE SINGULARLY PERTURBED CLOSED LOOP SYSTEM

Substitution of (3.6a) and (5.1) into (3.5ab) gives the closed loop system

$$(6.1ab)\qquad \begin{pmatrix} \dot{x}_s \\ \dot{x}_k \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1} C_{ss}(\varepsilon) & C_{sk}(\varepsilon) \\ \varepsilon^{-1} C_{ks}(\varepsilon) & C_{kk}(\varepsilon) \end{pmatrix} \begin{pmatrix} x_s \\ x_k \end{pmatrix}, \quad \begin{pmatrix} x_s(0) \\ x_k(0) \end{pmatrix} = \begin{pmatrix} x_{s0} \\ x_{k0} \end{pmatrix}$$

with

$$\varepsilon^{-1}C_{ss}(\varepsilon) = A_s - \varepsilon^{-2}B_sT_sB_s'P_{s\varepsilon} - \varepsilon^{-2}B_sT_{sk}B_k'P_{ks\varepsilon},$$

$$\varepsilon^{-1}C_{ks}(\varepsilon) = A_{ks} - \varepsilon^{-2}B_kT_{ks}B_s'P_{s\varepsilon} - \varepsilon^{-2}B_kT_kB_k'P_{ks\varepsilon},$$

$$C_{sk}(\varepsilon) = -\varepsilon^{-2}B_sT_sB_s'P_{ks\varepsilon} - B_sT_sN_{ks}' - \varepsilon^{-2}B_sT_{sk}B_k'P_{k\varepsilon} - B_sT_{sk}N_k',$$

$$C_{kk}(\varepsilon) = A_k - \varepsilon^{-2}B_kT_{ks}B_s'P_{ks\varepsilon}' - B_kT_{ks}N_{ks}' - \varepsilon^{-2}B_kT_kB_k'P_{k\varepsilon} - B_kT_kN_k'.$$

THEOREM 6.1. *Let* $(x_s(t), x_k(t))$ *be the solution of* (6.1ab), *then*

$$\left|\begin{pmatrix} x_s(t) \\ x_k(t)\end{pmatrix} - \begin{pmatrix} \hat{x}_s(t/\varepsilon) \\ \hat{x}_k(t/\varepsilon)\end{pmatrix} - \begin{pmatrix} 0 \\ \bar{x}_k(t)\end{pmatrix} + \begin{pmatrix} 0 \\ x_{k0}-\xi_{k0}\end{pmatrix}\right| = O(\varepsilon) \tag{6.2}$$

for $t \geq 0$ *with* $\hat{x}_s$, $\hat{x}_k$, $\bar{x}_k$ *and* ξ_{k0} *given by* (4.7)-(4.11).

PROOF. Since all eigenvalues of (6.1a) have negative real parts, see KWAKERNAAK and SIVAN [5, p.233], $|x_s|$ and $|x_k|$ have upperbounds of order $O(1)$. Integration of the equation for x_s yields

$$x_s(t) = e^{\varepsilon^{-1}C_{ss}(\varepsilon)t}x_{s0} + \int_0^t e^{\varepsilon^{-1}C_{ss}(\varepsilon)(t-\tau)}C_{sk}(\varepsilon)x_k(\tau)d\tau \tag{6.3}$$

or

$$x_s(t) = e^{\varepsilon^{-1}C_{ss}(0)t}x_{s0} + O(\varepsilon). \tag{6.4}$$

We now introduce the dependent variable

$$x_r = x_k - C_{ks}(\varepsilon)C_{ss}^{-1}(\varepsilon)x_s. \tag{6.5}$$

From (6.1a) we derive the corresponding differential equation

$$\dot{x}_r = [C_{kk}(\varepsilon) - C_{ks}(\varepsilon) - C_{ss}^{-1}(\varepsilon)C_{sk}(\varepsilon)]\{x_r + C_{ks}(\varepsilon)C_{ss}^{-1}(\varepsilon)x_s\}.$$

From (6.3) it follows that x_s is of the order $0(\varepsilon)$ in the L_1 norm, so that

$$(6.6)\qquad x_r(t) = e^{[C_{kk}(\varepsilon)-C_{ks}(\varepsilon)C_{ss}^{-1}(\varepsilon)C_{sk}(\varepsilon)]t}\{x_{k0}-C_{ks}(\varepsilon)C_{ss}^{-1}(\varepsilon)x_{s0}\}+0(\varepsilon).$$

Substitution of (6.3) and (6.6) into (6.5) yields

$$(6.7)\qquad x_k(t) = e^{[C_{kk}(0)-C_{ks}(0)C_{ss}^{-1}(0)C_{sk}(0)]t}\{x_{k0}-C_{ks}(0)C_{ss}^{-1}(0)x_{s0}\} + \\ - C_{ks}(0)C_{ss}^{-1}(0)e^{\varepsilon^{-1}C_{ss}(0)t}x_{s0} + 0(\varepsilon).$$

It is noted that

$$(6.8)\qquad C_{ks}(0)C_{ss}^{-1}(0) = B_kT_{ks}T_s^{-1}B_s^{-1},$$

so that

$$(6.9)\qquad C_{kk}(0) - C_{ks}(0)C_{ss}^{-1}(0)C_{sk}(0) = A_k - B_kR_k^{-1}B_k'\bar{P}_k - B_kR_k^{-1}N_k'.$$

According to (6.4) and (6.7) x_s and x_k satisfy (6.2), which completes the proof. □

7. AN EXAMPLE

As an illustration of the method of approximating the solution of a nearly singular system we present the following example

$$(7.1a)\qquad \dot{x} = Ax + Bv, \quad x(0) = x_0,$$

$$(7.1b)\qquad J = \int_0^\infty x'Qx + \varepsilon^2 v'Rv\ dt$$

with

$$(7.1c)\qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Putting (7.1) in the required form (3.5) we obtain

(7.2a) $$\dot{x}_1 = u_1, \qquad x_1(0) = x_{10},$$

(7.2b) $$\dot{x}_2 = x_1+u_2, \quad x_2(0) = x_{20},$$

(7.3c) $$J = \int_0^\infty x_1^2 + \varepsilon^2(x_2^2-2x_2u_1+u_1^2+u_2^2)\,dt.$$

In the limit $\varepsilon \to 0$ the system jumps initially from (x_{10},x_{20}) to $(0,x_{20})$, see (4.7) and (4.8). In order to analyze the limit solution in the subspace $x_1 = 0$ for $t > 0$, we consider the optimal control problem (4.9) for the system (7.2), so

(7.4a) $$\dot{\bar{x}}_2 = \bar{u}_0, \quad \bar{x}_2(0) = x_{20},$$

(7.4b) $$J = \int_0^\infty \bar{x}_2^2 + \bar{u}_2^2\,dt.$$

The optimal solution satisfies $\bar{u}_2 = -\bar{x}_2$, see (4.10). For the problem (7.1) the algebraic Riccati equation reads

(7.5) $$Q + P_\varepsilon A' + A'P_\varepsilon - \varepsilon^2 P_\varepsilon BR^{-1}B'P_\varepsilon = 0,$$

which has the positive definite solution

(7.6) $$P_\varepsilon = \begin{pmatrix} \varepsilon\sqrt{1+\varepsilon^2} & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 \end{pmatrix}.$$

Since $u_\varepsilon = -\varepsilon^{-2}R^{-1}B'P_\varepsilon x_\varepsilon$, the closed loop system reads

(7.7a) $$\dot{x}_{\varepsilon 1} = -\varepsilon^{-1}\sqrt{1+\varepsilon^2}\, x_{\varepsilon 1},$$

(7.7b) $$\dot{x}_{\varepsilon 2} = -x_{\varepsilon 2},$$

Consequently, the solution converges to the given limit solution as $\varepsilon \to 0$.

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