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Some combinatorial properties of the Group $C_3 \oplus C_3 \oplus C_3$

by

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1. In this report the following property of the finite Abelian group $G = C_3 \oplus C_3 \oplus C_3$ is demonstrated:

<u>Theorem</u>:

Let S be a sequence of 14 elements (not necessarily distinct) of G, such that the sum of the elements is zero. Then S contains a non empty subsequence of length ≤ 3 which has sum zero.

This property was used in the proof of the equality $\lambda(G) = \Lambda(G)$ for groups G of the type $G = C_{3n} \oplus C_{3m} \oplus C_{3p}$ for suitable n, m and p. See [2]. For definitions and a description of the main problem in this field of research see [1].

2. Let G be a finite Abelian group of order ω and maximal element-order m. A zero-sequence of length $\leq m$ is called a short zero-sequence. As is shown in [1] there exists a finite number $\mu_B(G)$ such that any G-sequence of length $\geq \mu_B(G)$ contains a short zero-sequence. In general one has $m \leq \mu_B(G) \leq (\omega-1)(m-1) + 1$. For some groups the exact value of $\mu_B(G)$ is known.

So we have (see [1]):

$$\mu_B(G) = \omega = m \qquad \text{if } G \text{ is Cyclic } (G = C_m)$$

$$\mu_B(G) = n + 2m - 1 \qquad \text{if } G = C_n \oplus C_m \quad n \mid m$$

$$\mu_B(G) = 2^k - 1 \qquad \text{if } G = (C_2)^k$$

In particular we have $\mu_B(C_3) = 3$, $\mu_B(C_3 \oplus C_3) = 7$.

In this note we prove $\mu_B(C_3 \oplus C_3 \oplus C_3) = 17$.

The theorem formulated in (1) shows that the length 17 can be lowered to 14 if we restrict ourselves to sequences with sum zero.

3. For groups of the type $G = C_3 \oplus C_3 \oplus \ldots \oplus C_3$ we have the following property which simplifies the problem of determining $\mu_B(G)$:

<u>Lemma 1</u>:

Let n_k be the upper length of a $(C_3)^k$-sequence containing no short zero-sequences and consisting of <u>distinct</u> elements.
Then $\mu_B((C_3)^k) = 2\ n_k + 1$.

<u>proof</u>:

In $(C_3)^k$ any non zero element has order 3. Hence the only possible zero-sequences of length ≤ 3 are of one of the following types:

(i)	0	of length 1	
(ii)	a, - a	of length 2	$a \neq 0$
(iii)	a, a, a	of length 3	$a \neq 0$
(iv)	a, a+b, a-b	of length 3	$a \neq \pm b, 0$

The presence of a sequence of type (i), (ii) and (iii) can easily be excluded by excluding the element zero, the inverse of any element present and the appearance of any element more than twice.

The presence of a zero-sequence of type (i), (ii) or (iv) only depends on the set of elements contained in the sequence not regarding multiplicity.

Now let S be a sequence of n_k distinct elements containing no short zero-sequence. The S contains no sequence of the type (i), (ii), (iii) or (iv). The same however is true of the sequence $S \cup S$ which contains every element of S exactly twice, for the only type the presence of which depends on the multiplicity of the elements in S is type (iii) which is still not present.
Therefore we conclude $\mu_B((C_3)^k) > 2\ n_k$.

A sequence of $2\ n_k + 1$ elements not containing an element three times contains at least $n_k + 1$ distinct elements. Therefore it must contain a zero-sequence of the type (i), (ii) or (iv) and we conclude $\mu_B((C_3)^k) \leq 2\ n_k + 1$.

Hence $\mu_B((C_3)^k) = 2\ n_k + 1$.

We shall prove $\mu_B((C_3)^3) = 17$ by showing:

proposition:

$n_3 = 8$; all sequences of 8 distinct elements without short zero-subsequences are zero-sequences.

4. The proof of this proposition will be performed by enumeration of all possible cases. In this enumeration it will be seen that all sequences of length 8 containing no short zero-sequences and consisting of distinct elements are zero-sequences.
For the proof of the property described in (1) we need also the following property which will be proved seperately (see 12):

Lemma 2:

There exist no sequences of 7 distinct elements and sum zero which contains no short zero-sequence.

In 5 and 6 we describe some notations while in 7 some general remarks are made which simplify the enumeration starting in 8.

5. Denotation of the elements in $(C_3)^3$.

The group $(C_3)^3$ can be considered to be the additive group of the vector space $(\mathbb{F}_3)^3$. The elements of $(C_3)^3$ therefore may be denoted as vectors of length 3 consisting of elements which are one of the integers 0, 1 or 2.
In the enumeration we use the following denotation (written above the elements in vector notation)

O	A	B	C	D	E	F	G	H	I	J	K	L	M
$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\2\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\2\end{bmatrix}$	$\begin{bmatrix}2\\0\\1\end{bmatrix}$	$\begin{bmatrix}2\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\2\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\\2\end{bmatrix}$

$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{D}$	$\bar{E}$	$\bar{F}$	$\bar{G}$	$\bar{H}$	$\bar{I}$	$\bar{J}$	$\bar{K}$	$\bar{L}$	$\bar{M}$
$\begin{bmatrix}2\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\2\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\\0\end{bmatrix}$	$\begin{bmatrix}2\\0\\2\end{bmatrix}$	$\begin{bmatrix}0\\2\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\\2\end{bmatrix}$	$\begin{bmatrix}2\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\2\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\2\end{bmatrix}$	$\begin{bmatrix}1\\2\\2\end{bmatrix}$	$\begin{bmatrix}2\\1\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\\1\end{bmatrix}$

By this notation we have $X + \bar{X} = 0$ if $X = A, \ldots, M$.
We define $\bar{\bar{X}} = X$.

For the sequel we use the following parts of the addition table (in the above given notation)

+	A	B	C	G	$\bar{G}$	K	L	M	$\bar{K}$	$\bar{L}$	$\bar{M}$
G	K	L	M	$\bar{G}$	0	$\bar{F}$	$\bar{E}$	$\bar{D}$	$\bar{A}$	$\bar{B}$	$\bar{C}$
$\bar{G}$	$\bar{F}$	$\bar{E}$	$\bar{D}$	0	$\bar{G}$	A	B	C	F	E	D
K	F	$\bar{M}$	$\bar{L}$	$\bar{F}$	A	$\bar{K}$	$\bar{C}$	$\bar{B}$	0	H	$\bar{J}$
L	$\bar{M}$	E	$\bar{K}$	$\bar{E}$	B	$\bar{C}$	$\bar{L}$	$\bar{A}$	$\bar{H}$	0	I
M	$\bar{L}$	$\bar{K}$	D	$\bar{D}$	C	$\bar{B}$	$\bar{A}$	$\bar{M}$	J	$\bar{I}$	0
$\bar{K}$	$\bar{G}$	$\bar{J}$	H	$\bar{A}$	F	0	$\bar{H}$	J	K	C	B
$\bar{L}$	I	$\bar{G}$	$\bar{H}$	$\bar{B}$	E	H	0	$\bar{I}$	C	L	A
$\bar{M}$	$\bar{I}$	J	$\bar{G}$	$\bar{C}$	D	$\bar{J}$	I	0	B	A	M

Addition table part I.

+	D	E	F	H	I	J	$\bar{H}$	$\bar{I}$	$\bar{J}$
D	$\bar{D}$	K	L	$\bar{A}$	$\bar{K}$	F	$\bar{B}$	E	$\bar{L}$
E	K	$\bar{E}$	M	$\bar{M}$	D	$\bar{C}$	F	$\bar{K}$	$\bar{A}$
F	L	M	$\bar{F}$	E	$\bar{B}$	$\bar{L}$	$\bar{M}$	$\bar{C}$	D
H	$\bar{A}$	$\bar{M}$	E	$\bar{H}$	$\bar{J}$	$\bar{I}$	O	G	$\bar{G}$
I	$\bar{K}$	D	$\bar{B}$	$\bar{J}$	$\bar{I}$	$\bar{H}$	$\bar{G}$	O	G
J	F	$\bar{C}$	$\bar{L}$	$\bar{I}$	$\bar{H}$	$\bar{J}$	G	$\bar{G}$	O
$\bar{H}$	$\bar{B}$	F	$\bar{M}$	O	$\bar{G}$	G	H	J	I
$\bar{I}$	E	$\bar{K}$	$\bar{C}$	G	O	$\bar{G}$	J	I	H
$\bar{J}$	$\bar{L}$	$\bar{A}$	D	$\bar{G}$	G	O	I	H	J

Addition table part II.

6. <u>Further denotations</u>.

Let S be a sequence of distinct elements not containing a short zero-sequence. In order to know whether S can be extended to a larger sequence still having this property we describe all elements extension by which is impossible.
Extension by 0 is always impossible. Further if X is contained in S extension by $\bar{X}$ is forbidden. Finally if X and Y are two elements from S and Z = X + Y then extension by $\bar{Z}$ is not permitted.

We denote such a situation by a denotation

$$\begin{array}{|l} X_1 \;\dots\; X_r \\ \hline Y_1 \;\dots\; Y_s \end{array}$$

where $X_1, \dots, X_r$ are the elements of S and $Y_1, \dots, Y_s$ are the elements which are excluded as elements by which S can be extended. The list however will be not complete. The element 0 and the elements X_i extension by which trivially is impossible are not contained in the Y_i.

Example:

$$\begin{array}{|l} A\ B\ C \\ \hline 0\ \bar{A}\ \bar{B}\ \bar{C}\ \bar{D}\ \bar{E}\ \bar{F} \end{array}$$

is denoted by

$$\begin{array}{|l} A\ B\ C \\ \hline \bar{D}\ \bar{E}\ \bar{F} \end{array}$$

In the sequel the elements A, B and C will always be contained in the elements X_i. Therefore the elements $\bar{D}$, $\bar{E}$ and $\bar{F}$ will always be contained in the Y_i. In the denotation we will leave them out too.

The example given above is now denoted as

$$\begin{array}{|l} A\ B\ C \\ \hline \end{array}$$

7. Proof of the proposition; general remarks.

As $n_2 = 3$ we know that any subsequence of S of length ≥ 4 must contain three linearly independent elements. If not these 4 elements are contained in a subgroup $\cong C_3 \oplus C_3$ and therefore they contain a short zero-subsequence.

We may draw two conclusions:

1°) We may assume that S contains at least three linearly independent elements say A, B and C. From now on we assume therefore A, B and C $\in$ S.

2°) If we denote S as a matrix with 3 rows and $\ell(S)$ columns ($\ell(S)$ is the lentgth of S) no row in S contains the element zero more than three times.

As A, B, C $\in$ S we conclude that the remaining elements may introduce at most three times the element zero in the matrix. S contains at most three elements out of the set $W = \{D,E,F,H,I,J,\bar{H},\bar{I},\bar{J}\}$.

From the addition table part II one sees that $\{H,I,J\}$ and $\{\bar{H},\bar{I},\bar{J}\}$ are the only length three zero-subsequences formed out of W.

If S has length ≥ 8 S contains at least two elements out of the collection

$$Z = \{G,K,L,M,\bar{G},\bar{K},\bar{L},\bar{M}\}$$

In the sequel we consider extensions of {A,B,C} by two elements from Z and consider whether or not extension up to length 8 is possible. It will be seen that extension up to length 9 is impossible.

As {K,L,C}, {L,M,A} and {K,M,B} are zero-sequences only one of these at least two elements can be chosen out of the triple {K,L,M}.

We treat four possible cases:

CASE A : $G \in S$
CASE B : $\bar{G} \in S$
CASE C : $G, \bar{G} \notin S \qquad |S \cap Z| = 3$
CASE D : $G, \bar{G} \notin S \qquad |S \cap Z| = 2$

It is easily seen that one of these cases always is present if S is a sequence of length ≥ 8 containing no short zero-subsequence, consisting of distinct elements.

8. <u>Proof of the proposition; enumeration</u>.
Case A. $G \in S$. Situation:

$$\begin{array}{|l} A\ B\ C\ G \\ \hline \bar{K}\ \bar{L}\ \bar{M} \end{array}$$

The second element out of Z must belong to the triple {K,L,M}. By symmetry we may assume it to be K. Situation:

$$\begin{array}{|l} A\ B\ C\ G\ K \\ \hline \bar{K}\ \bar{L}\ \bar{M}\ M\ L\ F \end{array}$$

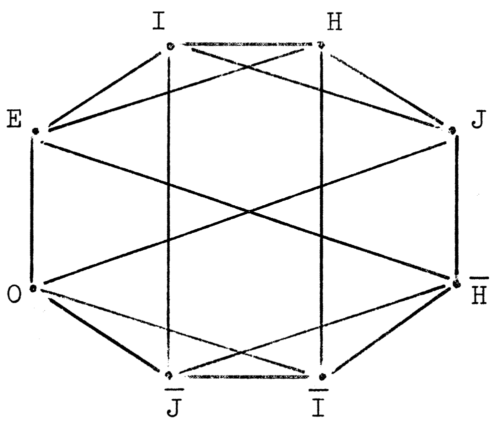
Further extension by elements of Z is not possible.
The remaining three elements have to be chosen from the collection $W_A = \{D,E,H,\bar{H},I,\bar{I},J,\bar{J}\}$.

We consider pairs in W_A having a sum equal zero or the inverse of an element already in S. It is not possible that S contains both members of such a pair. These pairs therefore are called forbidden pairs. The other pairs are called permissible pairs.
Elements extending S have to be chosen in such a way that all pairs formed out of them are permissible.
We construct a graph $G(W_A)$ the vertices of which are the elements from W_A. A pair of vertices is connected by an edge if the pair is permissible.
A maximal collection extending S is given by a maximal complete subgraph of $G(W_A)$.

In the table beneath we mark the permissible pairs by + and the forbidden pairs by - (use the addition table part II). Next to the table we design the graph $G(W_A)$

	D	E	H	I	J	$\bar{H}$	$\bar{I}$
E	+						
H	-	+					
I	-	+	+				
J	+	-	+	+			
$\bar{H}$	-	+	-	-	+		
$\bar{I}$	+	-	+	-	-	+	
$\bar{J}$	+	-	-	+	-	+	+

Permissible pairs in W_A



The graph $G(W_A)$

The only maximal complete subgraphs are triangles. Therefore we have only extensions by at most three elements.
The triangles I H J and $\bar{I}$ $\bar{H}$ $\bar{J}$ give no extensions as these sequences are zero-sequences themselves.

The remaining triagles E I H and D $\bar{J}$ $\bar{I}$ can be transformed into each other by interchanging the second and third row in S written as a matrix. Both triangles give essentially the same example of a sequence S of length 8 containing no short zero-subsequences. This example is given by the sequence:

$$\begin{matrix} \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{bmatrix}2\\1\\1\end{bmatrix} & \begin{bmatrix}1\\0\\1\end{bmatrix} & \begin{bmatrix}1\\2\\0\end{bmatrix} & \begin{bmatrix}0\\1\\2\end{bmatrix} \\ A & B & C & G & K & E & H & I \end{matrix} \qquad S_1$$

S_1 is easily seen to be a zero-sequence.

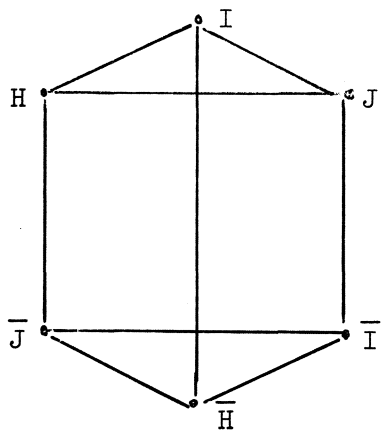
9. <u>Case B</u>. $G \in S$. Situation:

$$\left| \begin{array}{l} A\ B\ C\ \bar{G} \\ \hline D\ E\ F \end{array} \right.$$

The set W_B of elements in W extension by which is possible now contains only 6 elements. $W_B = \{H, I, J, \bar{H}, \bar{I}, \bar{J}\}$.
Again we denote the table of permissible pairs and the graph $G(W_B)$:

	H	I	J	$\bar{H}$	$\bar{I}$
I	+				
J	+	+			
$\bar{H}$	-	+	-		
$\bar{I}$	-	-	+	+	
$\bar{J}$	+	-	-	+	+

Permissible pairs in W_B



The graph $G(W_B)$

The only triangles in $G(W_B)$ are H I J and $\bar{H}$ $\bar{I}$ $\bar{J}$ which give no extensions of S as the corresponding sequences are zero-sequences.

We conclude that in case B, S cannot be extended by three elements from W. S must contain therefore three elements from Z.

The triple {K L M} contains at most one element in S. Hence at least one element from $\{\overline{K},\overline{L},\overline{M}\}$ has to be contained in S. By symmetry we may assume $\overline{K} \in S$. Situation:

$$\begin{array}{|l} \text{A B C } \overline{\text{G}} \ \overline{\text{K}} \\ \hline \text{D E F J } \overline{\text{H}} \end{array}$$

<u>CASE B 1)</u>:

S contains an element in {K,L,M}.

By symmetry we may assume $L \in S$. Situation:

$$\begin{array}{|l} \text{A B C } \overline{\text{G}} \ \overline{\text{K}} \text{ L} \\ \hline \text{D E F J } \overline{\text{H}} \text{ M K H} \end{array}$$

Extension is still possible by the elements $\{I,\overline{I},\overline{J},\overline{M}\}$. The pairs I, $\overline{I}$ and I, $\overline{J}$ are forbidden. Excluding $\overline{M}$ the only extension by two more elements is:

$$\underset{A}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \quad \underset{B}{\begin{bmatrix}0\\1\\0\end{bmatrix}} \quad \underset{C}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \quad \underset{\overline{G}}{\begin{bmatrix}2\\2\\2\end{bmatrix}} \quad \underset{\overline{K}}{\begin{bmatrix}1\\2\\2\end{bmatrix}} \quad \underset{L}{\begin{bmatrix}1\\2\\1\end{bmatrix}} \quad \underset{\overline{I}}{\begin{bmatrix}0\\2\\1\end{bmatrix}} \quad \underset{\overline{J}}{\begin{bmatrix}1\\0\\2\end{bmatrix}} \qquad S_2$$

S_2 has the demanded property but is a zero-sequence also.

Introducing $\overline{M}$ the situation becomes:

$$\begin{array}{|l} \text{A B C } \overline{\text{G}} \ \overline{\text{K}} \text{ L } \overline{\text{M}} \\ \hline \text{D E F J } \overline{\text{H}} \text{ M K H I } \overline{\text{J}} \ \overline{\text{I}} \end{array}$$

and further extension is seen to be impossible.

CASE B 2):

S contains no element in {K,L,M}.

For symmetry we may assume $\overline{L} \in S$. Situation:

$$\begin{array}{|ccccccc} A & B & C & \overline{G} & \overline{K} & \overline{L} & \\ \hline D & E & F & J & \overline{H} & \overline{I} & H \end{array}$$

Extension is still possible by $\{I,\overline{J},M,\overline{M}\}$ the pair I, $\overline{J}$ being forbidden. Extension by M leads us again to CASE B 1). Hence we have to introduce $\overline{M} \in S$. Situation:

$$\begin{array}{|ccccccccc} A & B & C & \overline{G} & \overline{K} & \overline{L} & \overline{M} & & \\ \hline D & E & F & J & \overline{H} & \overline{I} & H & I & \overline{J} \end{array}$$

Further extension is seen to be impossible.

10. CASE C.

S contains three elements in Z but G and $\overline{G}$ are not contained in S.

By symmetry it is easily seen that there are only two subcases namely $\overline{K}, \overline{L}, \overline{M} \in S$ or $K, \overline{L}, \overline{M} \in S$.

CASE C 1):

$\overline{K}, \overline{L}, \overline{M} \in S$. Situation:

$$\begin{array}{|ccccccc} A & B & C & & \overline{K} & \overline{L} & \overline{M} \\ \hline H & I & J & \overline{H} & \overline{I} & \overline{J} & G \end{array}$$

Extension is only possible by the elements $\{\overline{G},D,E,F\}$. Extension by $\overline{G}$ leads to CASE B.

As the pairs D,E, E,F and D,F are forbidden we conclude that extention by two more elements is not possible.

CASE C 2):

$K, \bar{L}, \bar{M} \in S$. Situation:

$$\left|\begin{array}{ccccccc} A & B & C & K & \bar{L} & \bar{M} & \\ \hline \bar{I} & H & \bar{H} & I & \bar{J} & J & G \end{array}\right.$$

Extension is only possible by the elements $\{\bar{G},D,E,F\}$. Extension by $\bar{G}$ leads to CASE B. The pairs D, F and E, F being forbidden the only extension up to length 8 is:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}2\\1\\1\end{bmatrix} \begin{bmatrix}2\\1\\2\end{bmatrix} \begin{bmatrix}2\\2\\1\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\0\\1\end{bmatrix} \qquad S_3$$

$$A \quad B \quad C \quad K \quad \bar{L} \quad \bar{M} \quad D \quad E$$

This is a sequence with the demanded property but also a zero-sequence.

11. CASE D.

S contains exactly two elements in Z but G and $\bar{G}$ are not contained in S.

By symmetry there are again two subcases: $\bar{K}, \bar{L} \in S$ or $K, \bar{L} \in S$.

CASE D 1):

$\bar{K}, \bar{L} \in S$. Situation:

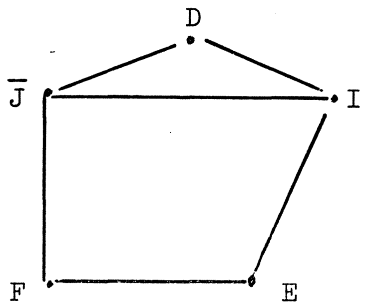
$$\left|\begin{array}{ccccc} A & B & C & \bar{K} & \bar{L} \\ \hline G & H & \bar{H} & \bar{I} & J \end{array}\right.$$

Extension is possible by $\bar{G}$, I, $\bar{J}$, D, E, F. Extension by $\bar{G}$ leads to CASE B. We put therefore $Z_{D_1} = \{I,\bar{J},D,E,F\}$.

Beneath we denote the permissible pairs and the grapg $G(Z_{D_1})$

	I	$\overline{J}$	D	E
$\overline{J}$	+			
D	+	+		
E	+	−	−	
F	−	+	−	+

Permissible pairs in W_{D_1}



The fraph $G(W_{D_1})$

The only triangle in $G(W_{D_1})$ is D, I, $\overline{J}$.

This leads to the sequence:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \begin{bmatrix}1\\2\\2\end{bmatrix} \begin{bmatrix}2\\1\\2\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}0\\1\\2\end{bmatrix} \begin{bmatrix}1\\0\\2\end{bmatrix} \qquad S_4$$

$$A \quad B \quad C \quad \overline{K} \quad \overline{L} \quad D \quad I \quad \overline{J}$$

S_4 has the demanded property but is also a zero-sequence.

<u>CASE D 2)</u>:

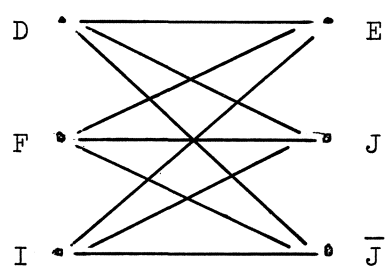
K, $\overline{L} \in$ S. Situation:

$$\begin{array}{|l} A\ B\ C\ K\ \overline{L} \\ \hline M\ L\ \overline{I}\ G\ H\ \overline{H} \end{array}$$

Extension is possible by elements from $\{\overline{G},\overline{M},D,E,F,I,J,\overline{J}\}$.
Extension by $\overline{G}$ (resp. $\overline{M}$) leads to CASE B (resp. CASE C). We put W_{D_2} = {D E F I J $\overline{J}$}. Again we give the permissible pairs and the graph $G(W_{D_2})$

	D	E	F	I	J
E	+				
F	–	+			
I	–	+	–		
J	+	–	+	+	
$\bar{J}$	+	–	+	+	–

Permissible pairs on W_{D_2}



The graph $G(W_{D_2})$

It is clear that this graph contains no triangle. Hence extension by three elements is impossible.

This completes the enumeration.

We have seen that all sequences S of length 8 distinct elements containing no short zero-subsequences are zero-sequences and that such sequences of length 9 are impossible. This proves our proposition.
Hence $n_3 = 8$ and $\mu_B(C_3 \oplus C_3 \oplus C_3) = 17$.

12. Proof of lemma 2.

To prove lemma 2 we show that all zero-sequences S of length 7 consisting of distinct elements are not irreducible; this means that any such sequence is the union of two proper zero-subsequences, one of which having length ≤ 3 as the total length is 7.

As the maximal length of an irreducible $C_3 \oplus C_3 \oplus C_3$ - sequence is 7 (see [1]) it follows that the distinctness of the elements in S is essential.

The proof uses an enumeration method which has been used to verify the equality $\lambda((C_3)^3) = \Lambda((C_3)^3)$ in [3, §9].

Lemma 2:

Any zero-sequence S in $C_3 \oplus C_3 \oplus C_3$ consisting of 7 distinct elements is not irreducible.

proof:

Like in 7) we may assume that S contains three linearly independent elements.

We choose a base in such a way that $\overline{A}$, $\overline{B}$ and $\overline{C}$ belong to S.

If we write S as a matrix S is denoted:

$$S = \begin{vmatrix} 2 & 0 & 0 & x_4 & x_5 & x_6 & x_7 \\ 0 & 2 & 0 & y_4 & y_5 & y_6 & y_7 \\ 0 & 0 & 2 & z_4 & z_5 & z_6 & z_7 \end{vmatrix}$$

We have $x_4 + x_5 + x_6 + x_7 \equiv 1 \bmod 3$ and similar for the y_i and the z_i.

We conclude that the C_3-sequences $(x_4\ x_5\ x_6\ x_7)$ etc. are permutations of one of the following C_3-sequences:

A	(1	0	0	0)
B	(2	2	0	0)
C	(1	1	1	1)
D	(2	2	2	1)
E	(1	1	2	0)

If a sequence A or B appears one of the rows in S contains ≥ 4 times the element zero and like in 7) we are done as 4 distinct elements in $C_3 \oplus C_3$ contain a short zero-sequence.

Suppose that the sequence $S' = \left(\begin{bmatrix} x_4 \\ y_4 \\ z_4 \end{bmatrix} \begin{bmatrix} x_5 \\ y_5 \\ z_5 \end{bmatrix} \begin{bmatrix} x_6 \\ y_6 \\ z_6 \end{bmatrix} \begin{bmatrix} x_7 \\ y_7 \\ z_7 \end{bmatrix} \right)$

contains a proper subsequence T' with sum $\begin{bmatrix} t \\ u \\ v \end{bmatrix}$

where t, u, v $\in \{0,1\}$. Then T' can be extended by some of the elements $\overline{A}$ $\overline{B}$ $\overline{C}$ to a proper zero-subsequence of S.
We prove that S' always contains such a subsequence T'.

As the rows A and B may be excluded the rows in S' can be supposed to be of the type C, D or E.
If S' contains an element X having all coordinated zero or one $\{X\}$ is a subsequence T' as described above, and we are done. Now D contains the element 2 three times, E contains 2 only once, and C does not contain 2. In order to have an element 2 in each column of S we must have at least 4 times the element 2 in S'. Therefore the row D must appear in S'.

Next consider all pairs of elements from S. The sum of each pair must have at least one coordinate 2 else we are done. Now C contains six pairs having sum 2, E contains only two pairs having sum 2 and D does not contain such a pair. We have six pairs in S. As D must appear in S' the two other rows must provide an element 2 in each pair. Hence the row C must appear also.

We may write S therefore as

$$S = \begin{vmatrix} 2 & 0 & 0 & 2 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & z_4 & z_5 & z_6 & z_7 \end{vmatrix}$$

As S is supposed to consist of distinct elements we must have $z_4 \neq z_5 \neq z_6 \neq z_4$. This means that the third row is of type E.
But now $z_7 = 1$ and the element $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is contained in S'. So we are done. This completes the proof of lemma 2.

13. Proof of the theorem.

Theorem:

Let S be a zero-sequence in $C_3 \oplus C_3 \oplus C_3$ of length 14. Then S contains a short zero-subsequence.

proof:

Suppose S contains no short zero-subsequence. Then no element of $C_3 \oplus C_3 \oplus C_3$ is contained three times in S and S contains therefore at least 7 distinct elements. Further there are at most 8 distinct elements in S as $n_3 = 8$.
There remain therefore two possibilities:

(i) S contains 7 distinct elements, each contained twice.
Now $S = \{x_1,x_2,\ldots,x_7,x_1,x_2,\ldots,x_7\}$. Put $T = \{x_1,\ldots,x_7\}$.
Then T is a sequence consisting of seven distinct elements containing no short zero-subsequences. Let t be the sum of t. As S is a zero-sequence we have $t + t = 0$. $C_3 \oplus C_3 \oplus C_3$ contains no element of order 2 so $t = 0$ and T is a zero-sequence. By lemma 2 this is a contradiction.

(ii) S contains 8 distinct elements.
Now $S = \{x_1,\ldots,x_8,x_1,\ldots,x_6\}$. Put $T = \{x_1,\ldots,x_8\}$.
$u = \{x_1,\ldots,x_6\}$. T is a sequence containing no short zero-subsequence consisting of 8 distinct elements. In the enumeration of all possible cases we have found that T must be a zero-sequence. As S is a zero-sequence also we have that U is a zero-sequence too.
But now $x_7 + x_8 = (x_1 + \ldots + x_8) - (x_1 + \ldots + x_6) = 0 - 0 = 0$ hence $\{x_7,x_8\}$ is a short zero-subsequence of T. This gives again a contradiction.

14. Final remarks.

Let G be a so called "homogeneous" Abelian group: $G = (C_n)^k$ n, k $\in \mathbb{Z}$. For these groups it is still an unsolved problem whether the conjective $\lambda(G) = \Lambda(G)$ generally is true or not. In this case $\Lambda(G) = k(n-1)$.

$\lambda(G)$ is as defined in [1] the maximal length of a G-sequence containing no zero-subsequences (primitive G-sequences).

For $G = C_3 \oplus C_3 \oplus C_3$ the proof of lemma 2 suggests us an example of a primitive G-sequence S of length $\lambda(G) = 6$ consisting of distinct elements.

It is easy to see that the sequence:

$$S' = \begin{bmatrix}2\\0\\0\end{bmatrix} \begin{bmatrix}0\\2\\0\end{bmatrix} \begin{bmatrix}0\\0\\2\end{bmatrix} \begin{bmatrix}1\\2\\1\end{bmatrix} \begin{bmatrix}1\\2\\1\end{bmatrix} \begin{bmatrix}1\\2\\0\end{bmatrix} \begin{bmatrix}1\\1\\2\end{bmatrix}$$

is a irreducible zero-sequence which contains only one element twice. Hence the subsequence

$$S = \begin{bmatrix}2\\0\\0\end{bmatrix} \begin{bmatrix}0\\2\\0\end{bmatrix} \begin{bmatrix}0\\0\\2\end{bmatrix} \begin{bmatrix}1\\2\\1\end{bmatrix} \begin{bmatrix}1\\2\\0\end{bmatrix} \begin{bmatrix}1\\1\\2\end{bmatrix}$$

is a primitive sequence consisting of distinct elements. This example suggests that there exists no general type of a maximal primitive sequence.

By lemma 2 it follows that an irreducible zero-sequence of length $\lambda(G) + 1$ contains at least one element twice. This leads to the following question:

Problem 1:

It is generally true that any irreducible zero-sequence of length $\lambda(G) + 1 = n(p-1) + 1$ in $G = (C_p)^n$ (p-prime) contains at least one element $p - 1$ times.

If this question is answered affirmatively it follows that all these sequences S can be written after a suitable choice of a base for $(C_p)^n$ as

$$S = \left|\begin{array}{cccc|cccc|cccc|cccc}
1 & 1 & \cdots & 1 & * & * & \cdots & * & * & & \cdots & * & * & & \cdots & * \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & * & & \cdots & * & * & & \cdots & * \\
\cdot & \cdot & & \cdot & 0 & 0 & & 0 & 1 & & & 1 & \cdot & & & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdots & \cdot & \cdot & & \cdots & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & & & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & & & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & & & \cdot & * & & & * \\
0 & 0 & & 0 & 0 & 0 & & 0 & 0 & & & 0 & 1 & & & 1
\end{array}\right|$$

$$\underbrace{\qquad\qquad}_{(p-1)} \quad \underbrace{\qquad\qquad}_{(p-1)} \quad \underbrace{\qquad\qquad}_{(p-1)} \quad \underbrace{\qquad\qquad}_{p}$$

were at the places marked $*$ integers can be chosen freely under the only restriction that the sum of all these integers in a row is equal one (mod p).
(It is easy to see that all sequences of this type are zero-sequences).

The problem is trivial for $p = 2$ or $n = 1$.
Lemma 2 solves it for $p = n = 3$.

Another question which is suggested by lemma 1 and by a conjecture formulated earlier by the author is the following:

<u>Problem 2</u>:

Is it generally true that any sequence in $G = (C_p)^n$ p-prime of length $\mu_B(G) - 1$ which does not contain a short zero-sequence consists of a collection of elements of G each taken (p-1) times?

In [1, §0] the author conjectures the answer to be positive for $n = 2$. The problem again is trivial for $p = 2$ or $n = 1$. For $p = 3$ the answer is positive by lemma 1. Further the only cases known are $n = 2$, $p = 5$ or 7 where the answer is positive too. See [1], [4].

References.

[1] P. van EMDE BOAS. A combinatorial problem on finite Abelian groups II. Math. Centre Report ZW - 1969 - 007.

[2] P.C. BAAYEN, P. van EMDE BOAS, D. KRUYSWIJK. A combinatorial problem on finite Abelian groups III. Math. Centre Report ZW - 1969 - 008.

[3] P. van EMDE BOAS, D. KRUYSWIJK. A combinatorial problem on finite Abelian groups. Math. Centre Report ZW - 1967 - 009.

[4] P. van EMDE BOAS. An ALGOL-60 Algorithm for the verification of a combinatorial conjecture on a finite Abelian group. Math. Centre Report ZW - 1969 - 014.