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# Berry-Esseen bounds for L-statistics with unbounded weight functions

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Berry-Esseen bounds of order $n^{-\frac{1}{2}}$ are established for linear combinations of order statistics with unbounded weight functions. The weight functions are allowed to tend to infinity in neighbourhoods of zero and one at a logarithmic rate. A finite number of discontinuity points in the weight function is also permitted, provided a local smoothness condition is imposed on the inverse of the underlying distribution of the observations. The present report supplements HELMERS and HUŠKOVÁ (1984), where (part of) Theorem 1, which deals with the case of a continuous unbounded weight function, was presented, together with an outline of its proof; Theorem 2 is a new result covering the case of a discontinuous unbounded weight function. The relation with recent work of VAN ZWET (1984) and FRIEDRICH (1985) is briefly pointed out.

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*Note:* This paper is an extended version of R. Helmers and M. Hušková: A Berry-Esseen bound for *L*-statistics with unbounded weight functions, Proceedings of the Third Prague Symposium on Asymptotic Statistics (1984), 93-101, (ed. P. Mandl, M. Hušková).

## 1. INTRODUCTION AND RESULTS

Let $X_1, X_2, \ldots, X_n$ be independent random variables (r.v.) with common distribution function (df) $F$ and let $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$ be the corresponding order statistics. Let $J$ be a fixed real-valued weight function on $(0,1)$. We consider $L$-statistics (or linear combinations of order statistics)

$$T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{i:n} \tag{1.1}$$

where the weights $c_{in}$ are of either one of the following forms:

$$c_{in} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds, \quad i=1, \ldots, n \tag{1.2}$$

or

$$c_{in} = J\left(\frac{i}{n+1}\right), \quad i=1, \ldots, n. \tag{1.3}$$

Let

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$$F_n^*(x) = P(T_n^* \leqslant x) \quad \text{for} \quad -\infty < x < \infty \tag{1.4}$$

where

$$T_n^* = (T_n - E(T_n))/\sigma(T_n). \tag{1.5}$$

In the past decade there has been considerable interest into the asymptotic distribution theory for $L$-statistics. It is well-known that $T_n^*$ is asymptotically normally distributed under quite general conditions. A survey of such results was given by SERFLING (1980). We also refer to a recent paper of MASON (1981), which contains the best result so far obtained in this area.

More recently attention has been paid to the problem of establishing Berry-Esseen bounds for $L$-statistics. We mention the work of BJERVE (1977), HELMERS (1977, 1981, 1982), SERFLING (1980) and VAN ZWET (1984). These authors obtained Berry-Esseen bounds for $L$-statistics for the case of bounded weights. The purpose of this paper is to derive Berry-Esseen bounds for $L$-statistics with unbounded weight functions. Let $\Phi$ denote the standard normal df and define $F^{-1}$ by

$$F^{-1}(s) = \inf\{x\colon F(x) \geqslant s\} \quad \text{for} \quad 0 < s < 1.$$

In our results- stated in the form of two theorems- we establish Berry-Esseen bounds of order $n^{-\frac{1}{2}}$ for statistics of the form (1.1). Our first result reads as follows

THEOREM 1. *Suppose these exist numbers $\delta > 0$, $\epsilon > 0$ and $K > 0$ such that*
(I) *the function $J$ satisfies a Lipschitz condition of order 1 on $[\epsilon, 1-\epsilon]$, whereas on neighbourhoods $(0,\epsilon)$ and $(1-\epsilon, 1)$ of zero and one $J$ is twice differentiable with second derivative $J''$, satisfying*

$$|J''(s)| \leqslant K[s(1-s)]^{-2} \tag{1.6}$$

(II) *the inverse $F^{-1}$ satisfies*

$$|F^{-1}(s)| \leqslant K[s(1-s)]^{-\frac{1}{4}+\delta} \quad \text{for} \quad 0 < s < 1 \tag{1.7}$$

$$|F^{-1}(s_1) - F^{-1}(s_2)| \leqslant K|s_1 - s_2| \cdot [(s_1(1-s_1))^{-\frac{5}{4}+\delta} + (s_2(1-s_2))^{-\frac{5}{4}+\delta}] \tag{1.8}$$

*for $0 < s_1, s_2 < \epsilon$ and $1-\epsilon < s_1, s_2 < 1$. Then $\sigma^2(J, F) > 0$ where*

$$\sigma^2(J, F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(F(x))J(F(y))(F(\min(x,y)) - F(x)F(y))dxdy \tag{1.9}$$

*implies that*

$$\sup_x |F_n^*(x) - \Phi(x)| = O(n^{-\frac{1}{2}}) \quad \text{as} \quad n \to \infty \tag{1.10}$$

*whenever either (1.2) or (1.3) is satisfied.*

Theorem 1 allows weight functions $J$ tending to infinity in the neighbourhood of 0 and 1 at a logarithmic rate. An example is provided by the weight function $\Phi^{-1}$, the normal quantile function. Then $T_n$ is an asymptotically efficient $L$-estimator of normal scale.

We should perhaps also note that it is easily checked that in fact (1.8) implies (1.7). However, the stronger assumption (1.8) is only needed in the treatment of certain terms appearing in the proof of the Lemma's 2.2 and 2.4, whereas assumption (1.7) seems to be a crucial requirement to make whole proof of Theorem 1 work. For these reasons we preferred to state Theorem 1 in its present form. Note that assumption (1.7) is satisfied if $E|X_1|^r < \infty$, for some $r > 4$.

Our second theorem is a modification of Theorem 1 in which we allow points of discontinuity in the weight function $J$. The price for this is a local smoothness condition on $F^{-1}$.

THEOREM 2. *Suppose that there exist numbers $\delta > 0$, $\epsilon > 0$, $K > 0$, $\eta > 0$ and a positive integer $\kappa$ such that*

(iii) *the function J possesses a finite number of jumps at* $s_1, \ldots, s_k \in (0,1)$ *and otherwise satisfies assumption (I)*

(iv) *the inverse* $F^{-1}$ *satisfies assumption (II) and, in addition,*

$$|F^{-1}(u)-F^{-1}(v)| \leq K|u-v| \tag{1.11}$$

*for all* $u,v \in (s_i-\eta, s_i+\eta)$ *for* $i=1,2,\ldots,k$. *Then* $\sigma^2(J,F)>0$, *with* $\sigma^2(J,F)$ *as in (1.9), implies that*

$$\sup_x |F_n^*(x)-\Phi(x)| = O(n^{-\frac{1}{2}}), \quad as \quad n\to\infty \tag{1.12}$$

*whenever either (1.2) on (1.3) is satisfied.*

Our method of proof resembles those of VAN ZWET (1977) and DOES (1982) as these authors also combine smoothing techniques with appropriate conditioning arguments. We note that Theorem 1 for the case that the weights are of the form (1.2) also occurs in HELMERS and HUŠKOVÁ (1984), together with an outline of its proof. The omitted details are to be found in section 2 of the present report.

After Theorem 1 was obtained the Ph.D. thesis of K.O. FRIEDRICH (Freiburg) appeared. In his thesis FRIEDRICH obtained a slightly better result, then the one given in Theorem 1. On the other hand, Theorem 2 cannot be deduced from Friedrich's result, as he does not allow discontinuity points in the score function generating the weights.

We conclude this section by remarking that a different possible way of arriving at our results would have been the verification of the assumptions of Theorem 1.1 of VAN ZWET (1984) for our case; more specifically, any set of assumptions implying the two requirements mentioned on page 438 of VAN ZWET (1984) would entail a Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for *L*-statistics. In fact, FRIEDRICH's (1985) approach resembles this latter method, as he verifies the assumptions of his Berry-Esseen theorem for arbitrary statistics, which is an extension of VAN ZWET's (1984) Theorem 1.1.

## 2. PROOF OF THEOREM 1

We begin by collecting a few preliminary results which we shall need in our proofs. Also we introduce some more notation which will be used throughout this paper.

Define a function $r$ on $(0,1)$ by

$$r(u) = [u(1-u)]^{-1} \quad \text{for} \quad 0<u<1. \tag{2.1}$$

Application of Lemma A2.3 of ALBERS, BICKEL and VAN ZWET (1976) easily yields for any integers $1 \leq m \leq n$

$$E|U_{m:n}-\frac{m}{n}|^{\alpha} = O(n^{-\frac{\alpha}{2}}(r(\frac{m}{n+1}))^{-\frac{\alpha}{2}}) \tag{2.2}$$

for any $\alpha>0$. In addition one can directly generalize (2.2) to the following bound:

$$E|U_{m:n}-\frac{m}{n}|^{\alpha}U_{m:n}^{\beta} = O(n^{-\frac{\alpha}{2}}(r(\frac{m}{n+1}))^{-\frac{\alpha}{2}}(\frac{m}{n})^{\beta}) \tag{2.3}$$

for any $\alpha>0$, $\beta \in \mathbb{R}$, and any $-\beta \leq m \leq n$.

The quantity $\sigma^2(J,F)$ (cf. (1.9)) given by

$$\sigma^2(J,F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))J(F(y))\,(F(\min(x,y))-F(x)F(y))dxdy \tag{2.4}$$

is the asymptotic variance of $n^{\frac{1}{2}}T_n$; it follows directly from Theorem 1 of MASON (1981) that

$$\lim_{n\to\infty} n\sigma^2(T_n) = \sigma^2(J,F) \tag{2.5}$$

with $T_n$ as in (1.1), under the present set of assumptions, in the case that (1.3) is satisfied. The same result for the case that (1.2) holds then follows directly from the relation (2.80) given in the final part of this section.

Let, for $n\geq 1$, $(U_{1:n}, \ldots, U_{n:n})$ denote the order statistics corresponding to a sample of size $n$ from the uniform distribution on (0,1). For any integer $1\leq m\leq[\frac{1}{4}\epsilon n]$, let $V=(V_{1:m-1}, \ldots, V_{m-1:m-1})$, $Z=(Z_{1:n-2m}, \ldots, Z_{n-2m:n-2m})$ and $W=(W_{1:m-1}, \ldots, W_{m-1:m-1})$ be vectors of order statistics corresponding to samples of sizes $m-1$, $n-2m$, and $m-1$ from the uniform distribution on (0,1) and let $V$, $Z$, and $W$, $U_{m:n}$ and $U_{n-m+1::n}$ be independent. Then the joint distribution of $(U_{1:n}, \ldots, U_{n:n})$ is the same as that of

$$\begin{aligned} &U_{m:n}V_{1:m-1}, \ldots, U_{m:n}V_{m-1:m-1}, U_{m:n},(U_{n-m+1:n}-U_{m:n})Z_{1:n-2m}+ \\ &U_{m:n}, \ldots, (U_{n-m+1:n}-U_{m:n})Z_{n-2m:n-2m}+U_{m:n}, U_{n-m+1:n}, \\ &(1-U_{n-m+1:n})W_{1:m-1}+U_{n-m+1:n}, \ldots, (1-U_{n-m+1:n})W_{m-1:m-1}+U_{n-m+1:n}. \end{aligned} \tag{2.6}$$

Since the joint distribution of $X_{i:n}$, $i=1,\ldots,n$ is the same as that of $F^{-1}(U_{i:n})$, $i=1,\ldots,n$ it follows directly from (2.1) that the distribution of $T_n$ (cf. (1.1)) can be identified with that of

$$\begin{aligned} &T_{1n}(U_{m:n})+c_{mn}F^{-1}(U_{m:n})+T_{2n}(U_{m:n}, U_{n-m+1:n})+c_{n-m+1n}F^{-1}(U_{n-m+1:n})+ \\ &T_{3n}(U_{n-m+1:n}) \end{aligned} \tag{2.7}$$

where

$$T_{1n}(U_{m:n}) = \sum_{i=1}^{m-1} c_{in}F^{-1}(V_{i:m-1}U_{m:n}) \tag{2.8}$$

$$T_{2n}(U_{m:n}, U_{n-m+1:n}) = \sum_{i=1}^{n-2m} c_{in}F^{-1}(Z_{i:n-2m}(U_{n-m+1:n}-U_{m:n})+U_{m:n}) \tag{2.9}$$

and

$$T_{3n}(U_{n-m+1:n}) = \sum_{i=1}^{m-1} c_{in}F^{-1}(W_{i:m-1}(1-U_{n-m+1:n})+U_{n-m+1:n}). \tag{2.10}$$

Clearly, the r.v.'s $T_{1n}(U_{m:n})$, $T_{2n}(U_{m:n}, U_{n-m+1:n})$ and $T_{3n}(U_{n-m+1:n})$ are conditionally independent, conditionally given $U_{m:n}=u$ and $U_{n-m+1:n}=v$ for any $0<u<v<1$. This fact will be crucial in what follows.

Define, for $\frac{m}{n}\leq s\leq\frac{n-m}{n}$, the function $\psi_n$ by

$$\psi_n(s) = \int_s^{\frac{n-m}{n}} J(y)dy - \frac{\left[\frac{n-m}{n}-s\right]}{\frac{n-2m}{n}} \int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y)dy \tag{2.11}$$

and note that

$$\psi_n\left[\frac{m}{n}\right]=\psi_n\left[\frac{n-m}{n}\right]=0.$$

Let $\Gamma_{n-2m}$ denote the empirical df based on $Z_1, \ldots, Z_{n-2m}$; i.e. $\Gamma_{n-2m}(s)=(n-2m)^{-1}\sum_{i=1}^{n-2m} I_{(0,s)}(Z_i)$ for $0<s<1$, where $Z_1, \ldots, Z_{n-2m}$ are independent uniform (0,1) r.v.'s corresponding to the order statistics $Z_{1:n-2m}, \ldots, Z_{n-2m:n-2m}$. Here and elsewhere $I_A(\cdot)$ denotes the indicator of a set $A$. For any r.v. $X$, with $0<\sigma(X)<\infty$, we write $\tilde{X}$ for $X-EX$ and $X^*$ for $(X-EX)/\sigma(X)$.

*We shall now first prove (1.10) for the case that the weights are of the form (1.2).* Similarly as in HELMERS (1981, 1982) we begin by writing

$$T_{2n}(U_{m:n},\ U_{n-m+1:n}) = \int_0^1 \psi_n \left[\frac{m}{n} + \frac{n-2m}{n}\,\Gamma_{n-2m}(s)\right] dF^{-1}(U_{m:n}) \tag{2.12}$$

$$+(n-2m)^{-1}\sum_{i=1}^{n-2m} F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})Z_i)\int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y)dy.$$

To proceed we note that, as $J$ is Lipschitz of order 1 on $[\epsilon, 1-\epsilon]$ (cf. assumption (I)), we can approximate $T_{2n}$ from above and below for sufficiently large $n$ by r.v.'s $T_{2n+}$ and $T_{2n-}$ defined by

$$T_{2n\pm}(U_{m:n},\ U_{n-m+1:n}) = \int_0^1 \{\psi_n \left[\frac{m}{n} + \frac{n-2m}{n} s\right] + \tag{2.13}$$

$$\frac{n-2m}{n}(\Gamma_{n-2m}(s)-s)\psi_n' \left[\frac{m}{n} + \frac{n-2m}{n} s\right] \pm 2^{-1} L \left[\frac{n-2m}{n}\right]^2 (\Gamma_{n-2m}(s)-s)^2 I_{[\epsilon,1-\epsilon]}(s) +$$

$$2^{-1}\left[\frac{n-2m}{n}\right]^2 (\Gamma_{n-2m}(s)-s)^2 \psi_n'' \left[\frac{m}{n} + \frac{n-2m}{n} s\right] I_{(0,\epsilon)\cup(1-\epsilon,1)}(s) +$$

$$6^{-1}\left[\frac{n-2m}{n}\right]^3 (\Gamma_{n-2m}(s)-s)^3 \psi_n''' \left[\frac{m}{n} + \frac{n-2m}{n}(\lambda s + (1-\lambda)\Gamma_{n-2m}(s))\right] \cdot$$

$$\cdot\, I_{(0,\frac{\epsilon}{2})\cup(1-\frac{\epsilon}{2},1)}(s)\} dF^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s) +$$

$$(n-2m)^{-1}\sum_{i=1}^{n-2m} F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})Z_i)\int_{\frac{m}{n}}^{\frac{n-m}{n}} J(y)dy$$

where $L$ is the Lipschitz constant and $\lambda$ a random point in $[0,1]$; i.e.

$$T_{2n-}(U_{m:n},\ U_{n-m+1:n}) \leqslant T_{2n}(U_{m:n},\ U_{n-m+1:n}) \leqslant T_{2n+}(U_{m:n},\ U_{n-m+1:n}). \tag{2.14}$$

Define (cf. (2.7))

$$T_{n\pm} = T_n + T_{2n\pm}(U_{m:n},\ U_{n-m+1:n}) - T_{2n}(U_{m:n},\ U_{n-m+1:n}). \tag{2.15}$$

In the following lemma we relate $T_n^*$ with $T_{n+}^*$ and $T_{n-}^*$ (cf. HELMERS (1981); (1982) for a similar approach).

LEMMA 2.1. *If the assumptions of Theorem 1 are satisfied, then*

$$P(T_n^* \leqslant x) \leqslant P(T_{n-}^* \leqslant x_{n+}) \tag{2.16}$$

*and*

$$P(T_n^* \leqslant x) \geqslant P(T_{n+}^* \leqslant x_{n-}) \tag{2.17}$$

*for appropriate sequences* $x_{n+}$, $n=1,2,...$ *and* $x_{n-}$, $n=1,2,...$ *satisfying*

$$x_{n\pm} = x(1+O(n^{-\frac{1}{2}})) + O(n^{-\frac{1}{2}}) \tag{2.18}$$

*uniformly in* $x$.

PROOF. The relation (2.7) and (2.15) together imply

$$P(T_n^* \leqslant x) \leqslant P(T_{n-}^* \frac{\sigma(T_{n-})}{\sigma(T_n)} + \frac{E(T_{n-}-T_n)}{\sigma(T_n)} \leqslant x) \tag{2.19}$$

and, similarly,

$$P(T_n^* \leq x) \geq P(T_{n+}^* \frac{\sigma(T_{n+})}{\sigma(T_n)} + \frac{E(T_{n+}-T_n)}{\sigma(T_n)} \leq x) \tag{2.20}$$

for $-\infty < x < \infty$. It is immediate from (2.9) and (2.15) and assumption (I) that

$$|T_{n\pm}-T_n| = |T_{2n\pm}-T_{2n}| = O(\int_{\epsilon}^{1-\epsilon} (\Gamma_{n-2m}(s)-s)^2 dF^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)) \tag{2.21}$$

and a simple calculation shows that

$$E|T_{n\pm}-T_n| = O(n^{-1}) \tag{2.22}$$

and

$$\sigma^2(T_{n\pm}-T_n) \leq E(T_{n\pm}-T_n)^2 = O(n^{-2}). \tag{2.23}$$

Application of (2.23) and the elementary inequality

$$|\sigma(T_{n\pm})-\sigma(T_n)| = \sigma(T_{n\pm}-T_n)$$

directly yields

$$\frac{\sigma(T_{n\pm})}{\sigma(T_n)} = 1+O(n^{-\frac{1}{2}}), \quad \frac{E(T_{n\pm}-T_n)}{\sigma(T_n)} = O(n^{-\frac{1}{2}}). \tag{2.24}$$

Together all these results implies the desired statements. □

In view of Lemma 2.1 it obviously suffices to show

$$\sup_x |P(T_{n\pm}^* \leq x)-\Phi(x)| = O(n^{-\frac{1}{2}}) \tag{2.25}$$

in stead of (1.10). To prove (2.25) we show that for some sufficiently small $\gamma>0$

$$\int_{|t|\leq n^{\gamma}} |t|^{-1}|\rho_{n\pm}^*(t)-e^{-\frac{1}{2}t^2}|dt = O(n^{-\frac{1}{2}}) \tag{2.26}$$

and

$$\int_{n^{\gamma}\leq|t|\leq\gamma n^{\frac{1}{2}}} |t|^{-1}|\rho_{n\pm}^*(t)|dt = O(n^{-\frac{1}{2}}) \tag{2.27}$$

where $\rho_{n\pm}^*$ denotes the characteristic function (ch.f.) of $T_{n\pm}^*$. An application of Esseen's smoothing lemma (see, e.g., FELLER (1971), p. 538) will then complete the proof of (2.25).

We first prove (2.26). To start with we note that (2.7)-(2.10) and the remark following (2.10) directly yields

$$\rho_{n\pm}^*(t) = E[\phi^*_{T_{1n}(U_{m:n})}(t)\cdot\phi^*_{T_{2n\pm}(U_{m:n},\, U_{n-m+1:n})}(t)\cdot\phi^*_{T_{3n}(U_{n-m+1:n})}(t) \tag{2.28}$$
$$\cdot\exp(it\sigma_{n\pm}^{-1}(ET_{n\pm}|\,U_{m:n},\, U_{n-m+1:n})-ET_{n\pm}))]$$

where $\sigma_{n\pm}^2=\sigma^2(T_{n\pm})$, and, for any r.v. $X$ with $E|X|<\infty$,

$$\phi_X^*(t) = E(\exp(it\sigma_{n\pm}^{-1}(X-E(X|\,U_{m:n},\, U_{n-m+1:n})))|U_{m:n},\, U_{n-m+1:n}). \tag{2.29}$$

Note that the expression within square brackets in (2.28) is precisely equal to the conditional ch.f. of $T_{n\pm}^*$, where the conditioning is on $U_{m:n}$, and $U_{n-m+1:n}$. The expectation operator $E$ in (2.28) refers to the expected value taken w.r.t. $(U_{m:n},\, U_{n-m+1:n})$.

We continue with the analysis of $\rho_{n\pm}^*(t)$. In the next lemma we derive asymptotic approximations for the first and third factor within square brackets in (2.28); i.e. for $\phi^*_{T_{1n}(u)}(t)$ and $\phi^*_{T_{3n}(v)}(t)$ for

$0<u<\epsilon$ and $1-\epsilon<v<1$. The choice of $m$ will be specified later, but is any case $m=m(n)\to\infty$ as $n\to\infty$.

LEMMA 2.2. *If the assumptions of Theorem 1 are satisfied, then for any real t and* $0<u<\epsilon$

$$|\phi^*_{T_{1n}(u)}(t)-1+\frac{1}{2}t^2\sigma_{n\pm}^{-2}\sigma^2(T_{1n}(u))| = O(n^{-\frac{1}{2}}(\log n)^3|t|^3u^{-\frac{3}{4}+3\delta}m^{\frac{3}{2}}) \tag{2.30}$$

*and*

$$\sigma^2(T_{1n}(u)) = O(n^{-2}(\log n)^2u^{-\frac{1}{2}+2\delta}m) \tag{2.31}$$

*as* $m,\ n\to\infty$. *The relation (2.30) and (2.31) remain valid if we replace* $T_{1n}(u)$ *by* $T_{3n}(v)$ *and* $u$ *by* $1-v$.

PROOF. It suffices to deal with

$$E\,|T_{1n}(u)-ET_{1n}(u)|^3. \tag{2.32}$$

By Jensen's inequality we obtain

$$E|T_{1n}(u)-ET_{1n}(u)|^3 \leq (\sum_{j=1}^{m-1}|\int_{\frac{j-1}{n}}^{\frac{j}{n}} J(s)ds\,|(E\,|F^{-1}(V_{j:m-1}u)-EF^{-1}(V_{j:m-1}u))^3)^{\frac{1}{3}})^3. \tag{2.33}$$

In view of assumption (1.8) there exist constants $B_1$ and $B_1^*$ such that for $0<u<\epsilon$

$$E(F^{-1}(V_{j:m-1}u)-EF^{-1}(V_{j:m-1}u))^4 \leq \tag{2.34}$$

$$B_1u^4E(V_{j:m-1}-\frac{j}{m})^4(r^{+5-4\delta}(V_{j:m-1}u)+r^{+5-4\delta}(\frac{j}{m}u)) \leq B_1^*u^{-1+4\delta}m^{-2}(\frac{j}{m})^{-3+4\delta}$$

for $5\leq j\leq m$. For $j\leq 5$ we easily obtain the following bound:

$$E(F^{-1}(V_{j:m-1}u)-EF^{-1}(V_{j:m-1}u))^4 \leq \tag{2.35}$$

$$B_2u^{-1+4\delta}\int_0^1 y^{-1+4\delta}\frac{m!}{(j-1)!(m-1-j)!}y^{j-1}(1-y)^{m-j-1}dy \leq B_2^*u^{-1+4\delta}m^{1-4\delta}$$

for some constants $B_2$ and $B_2^*$. Together these three inequalities yield

$$E|T_{1n}(u)-ET_{1n}(u)|^3 = O(n^{-3}\log^3 nu^{-\frac{3}{4}+3\delta}(m^{\frac{3}{4}-3\delta}+ \tag{2.36}$$

$$m^{\frac{3}{4}-3\delta}(\sum_{j=5}^{m-1}j^{-\frac{3}{4}+\delta})^3) = O((\log n)^3u^{-\frac{3}{4}+\delta}m^{\frac{3}{2}}n^{-3}).$$

The assertion (2.30) now follows from (2.36), (2.5), and an appropriate three term Taylor expansion for $\phi^*_{T_{1n}(u)}(t)$. The second statement of the lemma (2.31) follows directly from (2.30) and a simple moment inequality. □

We also need an asymptotic approximation for $\phi^*_{T_{2n\pm}(u,v)}$ for $0<u<\epsilon$, $1-\epsilon<v<1$. Note that the r.v. $S_n(u,v)$ appearing is the following lemma corresponds to the leading term in the stochastic expansion (2.13), conditional on $U_{m:n}=u$ and $U_{n-m+1:n}=v$.

LEMMA 2.3. *If the assumptions of Theorem 1 are satisfied, then for any* $|t|\leq\gamma n^{\frac{1}{2}}$ *and* $0<u<\epsilon$, $1-\epsilon<v<1$

$$|\phi^*_{T_{2n\pm}(u,v)}(t)-\exp(-\frac{1}{2}t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(u,v))| = O(n^{-\frac{1}{2}}(t^2((F^{-1}(u))^2+(F^{-1}(v))^2)n^{-\frac{3}{2}}\cdot(\frac{m}{n})^{\delta}+ \tag{2.37}$$

$$|t|^3)\exp(-\tfrac{1}{2}t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(u,v)))+$$
$$n^{-1}t^2((F^{-1}(u))^2+(F^{-1}(v))^2)+n^{-\frac{1}{2}}m^{-\frac{1}{2}}|t|(|F^{-1}(u)|+F^{-1}(v)|)$$

*where*

$$S_n(u,v) = -\left[\frac{n-2m}{n}\right]\int_0^1 J\left[\frac{m}{n}+\frac{n-2m}{n}s\right](\Gamma_{n-2m}(s)-s)dF^{-1}(u+(v-u)s). \tag{2.38}$$

**Proof.** The proof of this lemma is a highly technical matter. In view of (2.13) and (2.29) we may write

$$\phi^*_{T_{2n\pm}(u,v)}(t) = Ee^{it\sigma_{n\pm}^{-1}\tilde{T}_{2n\pm}(u,v)} = Ee^{it\sigma_{n\pm}^{-1}\{S_n(u,v)+\tilde{Q}_n(u,v)+\tilde{R}_n(u,v)\}} \tag{2.39}$$

where (cf. (2.38))

$$S_n(u,v) = -\left[\frac{n-2m}{n}\right]\int_0^1 J\left[\frac{m}{n}+\frac{n-2m}{n}s\right](\Gamma_{n-2m}(s)-s)dF^{-1}(u+(v-u)s) \tag{2.40}$$

and

$$Q_n(u,v) = \pm 2^{-1}L\left[\frac{n-2m}{n}\right]^2\int_0^1(\Gamma_{n-2m}(s)-s)^2 I_{[\frac{\epsilon}{2},1-\frac{\epsilon}{2}]}dF^{-1}(u+(v-u)s)+ \tag{2.41}$$
$$-2^{-1}\left[\frac{n-2m}{n}\right]^2\int_0^1(\Gamma_{n-2m}(s)-s)^2 J'\left[\frac{m}{n}+\frac{n-2m}{n}s\right]I_{(0,\frac{\epsilon}{2})\cup(1-\frac{\epsilon}{2},1)}(s)dF^{-1}(u+(v-u)s)$$

and

$$R_n(u,v) = -6^{-1}\left[\frac{n-2m}{n}\right]^3. \tag{2.42}$$
$$\cdot\int_0^1(\Gamma_{n-2m}(s)-s)^3 J''\left(\frac{m}{n}+\frac{n-2m}{n}(\lambda s+(1-\lambda)\Gamma_{n-2m}(s)\right)I_{(0,\frac{\epsilon}{2})\cup(1-\frac{\epsilon}{2},1)}(s)dF^{-1}(u+(v-u)s)$$

for $0<u<\epsilon$ and $1-\epsilon<v<1$.

A simple Taylor expansion argument yields

$$\phi^*_{T_{2n\pm}(u,v)}(t) = Ee^{it\sigma_{n\pm}^{-1}S_n(u,v)}\{1+it\sigma_{n\pm}^{-1}\tilde{Q}_n(u,v)\}+ \tag{2.43}$$
$$+O(t^2\sigma_{n\pm}^{-2}\sigma^2(Q_n(u,v))+|t|\sigma_{n\pm}^{-1}E|R_n(u,v)|)$$

for any real $t$. Since $S_n(u,v)$ is a sum of i.i.d. r.v.'s with zero means and $Q_n(u,v)$ is a von-Mises functional of degree 2 one easily obtains:

$$\rho^*_{T_{2n\pm}(u,v)}(t) = \rho_n^{n-2m}\left[\frac{t}{(n-2m)\sigma_{n\pm}}\right]+ \tag{2.44}$$
$$it\sigma_{n\pm}^{-1}(n-2m)\rho_n^{n-2m-1}\left[\frac{t}{(n-2m)\sigma_{n\pm}}\right]Ee^{it\sigma_{n\pm}^{-1}h_n(Z_1)(n-2m)^{-1}}\tilde{g}_n(z_1,z_1)+$$
$$it\sigma_{n\pm}^{1}(n-2m)(n-2m-1)\rho_n^{n-2m-2}\left[\frac{t}{(n-2m)\sigma_{n\pm}}\right]E^{it\sigma_{n\pm}^{-1}(n-2m)^{-1}\{h_n(z_1)+h_n(z_2)\}}$$
$$g_n(Z_1,Z_2)+O(t^2\sigma_{n\pm}^{-2}\sigma^2(Q_n(u,v))+|t|\sigma_{n\pm}^{-1}E|R_n(u,v)|).$$

Here $\rho_n$ denotes the ch.f. of $h_n(Z_1)$, i.e. of

$$h_n(Z_1) = -\left[\frac{n-2m}{n}\right]\int_0^1 J\left[\frac{m}{n}+\frac{n-2m}{n}s\right](\chi_{(0,s)}(Z_1)-s)dF^{-1}(u+(v-u)s) \tag{2.45}$$

whereas the function $g_n$, appearing in (2.44), is given by

$$g_n(Z_i,Z_j) = \pm Ln^{-2}\int_0^1(\chi_{(0,s)}(Z_i)-s)(\chi_{(o,s)}(Z_j)-s)I_{[\epsilon,1-\epsilon]}(s)dF^{-1}(u+(v-u)s)+ \tag{2.46}$$

$$-2^{-1}\left[\frac{n-2m}{n}\right]^2\frac{1}{(n-2m)^2}\int_0^1 J\left[\frac{m}{n}+\frac{n-2m}{n}s\right](\chi_{(0,s)}(Z_i)-s)$$

$$(\chi_{(0,s)}(Z_j)-s)I_{(0,\epsilon)\cup(1-\epsilon,1)}(s)dF^{-1}(u+(v-u)s),\ \text{for}\ 1\leq i,j\leq 2.$$

To proceed we remark that Lemma 1 of PETROV (1975), page 109, directly yields for any $|t|\leq\gamma n^{\frac{1}{2}}$

$$|\rho_n^{n-2m}\left[\frac{1}{(n-2m)\sigma_{n\pm}}\right]-\exp(-\tfrac{1}{2}t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(u,v)))|= \tag{2.47}$$

$$= O(n^{-\frac{1}{2}}|t|^3\exp(-\tfrac{1}{4}t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(u,v))).$$

Here we have used the fact that $E|h_n(Z_1)|^3$ is bounded, and $Eh^2(Z_1)$ is bounded away from zero, uniformly for $0<u,v<1$; by the assumptions of Theorem 1. Secondly we note that

$$|it\sigma_{n\pm}^{-1}(n-2m)\rho_n^{n-2m-1}\left[\frac{t}{(n-2m)\sigma_{n\pm}}\right]Ee^{it\sigma_{n\pm}^{-1}(n-2m)^{-1}h_n(Z_1)}\tilde{g}_n(Z_1,Z_1)|= \tag{2.48}$$

$$O|t^2\sigma_{n\pm}^{-2}E|h_n(Z_1)g_n(Z_1,Z_1)|\exp(-\tfrac{1}{4}t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(u,v)))$$

and, similarly, also that

$$|it\sigma_{n\pm}^{-1}(n-2m)(n-2m-1)\rho_n^{n-2m-2}\left[\frac{t}{(n-2m)\sigma_{n\pm}}\right]Ee^{it\sigma_{n\pm}^{-1}(n-2m)^{-1}\{h_n(Z_1)+h_n(Z_2)\}} \tag{2.49}$$

$$g_n(Z_1,Z_2)| = O(|t|^3\sigma_{n\pm}^{-3}E|h_n(Z_1)h_n(Z_2)g_n(Z_1,Z_2)|\exp(-\tfrac{1}{4}t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(u,v)))(|t|^3n^{-\frac{1}{2}}+1)$$

A simple computation using the assumptions of Theorem 1 and Hölder's inequality yields

$$E|h_n(Z_1)g_n(Z_1,Z_1)| = O(n^{-2}\int_0^1\int_0^1 r^\delta\left[\frac{m}{n}+\frac{n-2m}{n}s\right]r\left[\frac{m}{n}+\frac{n-2m}{n}y\right]\cdot \tag{2.50}$$

$$\cdot E|\chi_{(0,s)}(Z_1)-s|\,|\chi_{(0,y)}(Z_1)-y|^2dF^{-1}(u+(v-u)s)dF^{-1}(u+(v-u)y))=$$

$$O(n^{-2}(\frac{m}{n})^\delta((F^{-1}(u))^2+(F^{-1}(v))^2),\ \text{for some}\ \delta>o$$

and uniformly in $0<u,v<1$, and, quite similarly,

$$E|h_n(Z_1)h_n(Z_2)g_n(Z_1,Z_2)| = O(n^{-2}\int_0^1\int_0^1\int_0^1 r^\delta\left[\frac{m}{n}+\frac{n-2m}{n}s\right] \tag{2.51}$$

$$r^\delta\left[\frac{m}{n}+\frac{n-2m}{n}y\right]r\left[\frac{m}{n}+\frac{n-2m}{n}w\right]E|\chi_{(0,s)}(Z_1)-s|$$

$$|\chi_{(0,y)}(Z_2)-y|\,|\chi_{(0,w)}(Z_1)-w|\,|\chi_{(0,w)}(Z_2)-w|\cdot$$

$$\cdot dF^{-1}(u+(v-u)s)dF^{-1}(u+(v-u)y)dF^{-1}(u+(v-u)w)) = O(n^{-2}),$$

uniformly in $0<u,v<1$. We can also easily deduce from (2.5) and (2.23) that

$$\sigma_{n\pm}^{-1} = O(n^{+\frac{1}{2}})$$

which together with the relations (2.44), (2.47)-(2.51) yields the first order bound on the r.h.s. of (2.37).
It remains to consider the terms involving $\sigma^2(Q_n(u,v))$ on $E|R_n(u,v)|$ on the r.h.s. of (2.43), to establish the corresponding order bounds for these quantities. Because of (2.41) we directly see that

$$\sigma^2(Q_n(u,v)) = O(\sigma^2\{\int_\epsilon^{1-\epsilon} (\Gamma_{n-2m}(s)-s)^2 dF^{-1}(u+(v-u)s)\}+ \tag{2.52}$$

$$\sigma^2\{(\int_0^\epsilon + \int_{1-\epsilon}^1)(\Gamma_{n-2m}(s)-s)^2 J'\left[\frac{m}{n}+\frac{n-2m}{n}s\right] dF^{-1}(u+(v-u)s)\}).$$

It is easily checked that

$$\sigma^2(\int_\epsilon^{1-\epsilon} (\Gamma_{n-2m}(s)-s)^2 dF^{-1}(u+(v-u)s)) = \tag{2.53}$$

$$O(n^{-2}(\int_\epsilon^{1-\epsilon} s(1-s)dF^{-1}(u+(v-u)s))^2) = O(n^{-2})$$

uniformly for $0<u,v<1$, whereas

$$\sigma^2(\int_0^\epsilon(\Gamma_{n-2m}(s)-s)^2 J'\left[\frac{m}{n}+\frac{n-2m}{n}s\right] dF^{-1}(u+(v-u)s)) =$$

$$O((\int_0^\epsilon n^{-1}s(1-s)r\left[\frac{m}{n}+\frac{n-2m}{n}s\right] dF^{-1}(u+(v-u)s))^2 = O(n^{-2}((F^{-1}(u))^2+(F^{-1}(v))^2).$$

Together these last three results yields that

$$\sigma^2(Q_n(u,v)) = O(n^{-2}((F^{-1}(u))^2+(F^{-1}(v))^2),$$

the desired result. To complete our proof of Lemma 2.3 we have to show that

$$E|R_n(u,v)| = O(n^{-1}m^{-\frac{1}{2}}(|F^{-1}(u)|+|F^{-1}(v)|). \tag{2.54}$$

To establish (2.54) we combine the following inequalities; for some constant $B>0$

$$E|\Gamma_{n-2m}(s)-s|^3 \le B[(n-2m)^{\frac{3}{4}}(s(1-s))^{\frac{3}{4}}+(n-2m)^{\frac{3}{2}}(s(1-s))^{\frac{3}{2}}](n-2m)^{-3} \tag{2.55}$$

$$|J''\left[\frac{m}{n}+\frac{n-2m}{n}(\lambda s+(1-\lambda)\Gamma_{n-2m}(s))\right]| \le B\{r^2\left[\frac{m}{n}+\frac{n-2m}{n}s\right]+ \tag{2.56}$$

$$r^2\left[\frac{m}{n}+\frac{n-2m}{n}\Gamma_{n-2m}(s)\right]\}$$

together with a probability bound of LAI (1975), page 827: for every $\alpha\in(0,1)$ and $a>0$ there exists $\lambda>0$ such that

$$P(\sup_{s\in(n^{-\alpha},1-n^{-\alpha})}\Gamma_{n-2m}(s)/s\le a) = O(\exp(-\lambda n^{\frac{(1-\alpha)}{2}})). \tag{2.57}$$

After some computations, using (2.55)-(2.57) we find that

$$E|R_n(u,v)| = O(E\int_0^1|\Gamma_{n-2m}(s)-s|^3\{r^2\left[\frac{m}{n}+\frac{n-2m}{n}s\right]+ \tag{2.58}$$

$$r^2\left[\frac{m}{n}+\frac{n-2m}{n}\Gamma_{n-2m}(s)\right]\}dF^{-1}(u+(v-u))=$$

$$O(E(\int_0^{n^{-\alpha}}+\int_{1-n^{-\alpha}}^{1})|\Gamma_{n-2m}(s)-s|^3\left[\frac{m}{n}\right]^{-2}dF^{-1}(u+(v-u)s)+$$

$$E(\int_{n^{-\alpha}}^{1-n^{-\alpha}}|\Gamma_{n-2m}(s)-s|^3\{r^2\left[\frac{m}{n}+\frac{n-2m}{n}s\right]+r^2\left[\frac{m}{n}+\frac{n-2m}{n}\Gamma_{n-2m}(s)\right]\}\cdot$$

$$\cdot\{I\{\Gamma_{n-2m}(s)/s>a\}+I\{\Gamma_{n-2m}(s)/s\leqslant a\}\}dF^{-1}(u+(v-u)s))=$$

$$O(n^{-1}m^{-\frac{1}{2}}(|F^{-1}(u)|+|F^{-1}(v)|).$$

which proves (2.54). This completes the proof of the lemma. □

To deal with the fourth factor within square brackets in (2.28) it will be convenient to have

LEMMA 2.4. *If the assumptions of Theorem 1 are satisfied, then*

$$E|E(T_{n\pm}|U_{m:n},U_{n-m+1:n})-ET_{n\pm}|^3\cdot I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n})= \tag{2.59}$$

$$O(n^{-\frac{3}{2}}\left[\frac{m}{n}\right]^{\frac{3}{4}+3\delta}(\log n)^3).$$

PROOF. In view of (2.7) - (2.10) and (2.15), and the assumptions (I) and (II) it suffices clearly to estimate the following four quantities:

$$M_{n1}=E|\sum_{j=5}^{m-1}\int_{\frac{j-1}{n}}^{\frac{j}{n}}J(y)dy|(E(F^{-1}(U_{j:n})|U_{m:n})-EF^{-1}(U_{j:n})|^3I_{(0,\epsilon)}(U_{m:n}) \tag{2.60}$$

$$M_{n2}=E|\int_{\epsilon}^{1-\epsilon}(s(1-s))(n-2m)^{-1}dF^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)- \tag{2.61}$$

$$EF^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)|^3I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n})$$

$$M_{n3}=E|\int_0^{\epsilon}s(1-s)(n-2m)^{-1}J'\left[\frac{m}{n}+\frac{n-2m}{n}s\right]\cdot \tag{2.62}$$

$$\cdot dF^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)-EF^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)|^3I_{(0,\epsilon)}(U_{m:n})$$

and

$$M_{n4}=E|\int_0^1(F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)- \tag{2.63}$$

$$EF^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s))ds|^3I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n}).$$

To begin with the treatment of $M_{n1}$ we note that

$$|E(F^{-1}(U_{j:n})|U_{m:n})-F^{-1}\left[\frac{j}{m}U_{m:n}\right]|I_{(0,\epsilon)}(U_{m:n})\leqslant \tag{2.64}$$

$$KE|V_{j:m-1}-\frac{j}{m}|U_{m:n}r^{\frac{5}{4}-\delta}(U_{m:n})(r^{\frac{5}{4}-\delta}(V_{j:m-1})+$$

$$r^{\frac{5}{4}-\delta}\left[\frac{j}{m}\right])=O(m^{-\frac{1}{2}}r^{\frac{3}{4}-\delta}\left[\frac{j}{m}\right]U_{m:n}r^{\frac{5}{4}-\delta}(U_{m:n})$$

uniformly for $5 \leqslant j \leqslant m-1$.

Quite similarly, we obtain also that

$$|EF^{-1}(\frac{j}{m}U_{m:n}) - F^{-1}\left[\frac{j}{n+1}\right]| = O\left[\frac{j}{m}r^{\frac{5}{4}-\delta}(\frac{j}{m})r^{\frac{3}{4}-\delta}\left[\frac{m}{n+1}\right]\right] \tag{2.65}$$

and

$$|F^{-1}\left[\frac{j}{n+1}\right] - EF^{-1}(U_{j:n})| = O(n^{-\frac{1}{2}}r^{\frac{3}{4}-\delta}\left[\frac{j}{n+1}\right])$$

uniformly for $5 \leqslant j \leqslant m-1$. Combining these results, with the fact that uniformly for $1 \leqslant j \leqslant m$

$$EF^{-1}(U_{j:n})I(U_{j:n} > \frac{\epsilon}{2}) = O(n^{-1}) \tag{2.66}$$

we arrive at

$$M_{n1} = O(n^{-\frac{3}{2}}\left[\frac{m}{n}\right]^{\frac{3}{4}+3\delta}(\log n)^3). \tag{2.67}$$

We note that (2.66) is easily inferred from Lemma A.2.1 of ALBERS, BICKEL and VAN ZWET (1976), together with an application of assumption (II) and Hölder's inequality.

We next turn to $M_{n3}$. An elementary calculation shows that

$$M_{n3} = O(\frac{1}{(n-2m)^3}E|\int_0^{\epsilon}|F^{-1}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s) - \tag{2.68}$$

$$F^{-1}\left[\frac{m}{n}+\frac{n-2m}{n}s\right]|r\left[\frac{m}{n}+\frac{n-2m}{n}s\right]ds|^3 I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n}).$$

Since, for $0<s<\epsilon$, $r(s) \leqslant Cs^{-1}$ for some constant $C>0$ and by applying assumption (II) we arrive at

$$M_{n3} = O(\frac{1}{(n-2m)^3}E|(|U_{m:n}-\frac{m}{n}|+|U_{n-m+1:n}-\frac{n-m}{n}|)\int_0^{\epsilon}(r^{\frac{5}{4}-\delta}\left[\frac{m}{n}+\frac{n-2m}{n}s\right]+ \tag{2.69}$$

$$r^{\frac{5}{4}-\delta}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)r\left[\frac{m}{n}+\frac{n-2m}{n}s\right]ds|^3 I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n})) =$$

$$O(E|(|U_{m:n}-\frac{m}{n}|+|U_{n-m+1:n}-\frac{n-m}{n}|)(\left[\frac{m}{n}\right]^{-\frac{5}{4}+\delta}+$$

$$U_{m:n}^{-\frac{1}{4}+\delta}\left[\frac{m}{n}\right]^{-1})|^3 I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n})) = O(\left[\frac{m}{n}\right]^{-\frac{9}{4}+3\delta}n^{-\frac{3}{2}}).$$

The term $M_{n2}$ can be treated in a completely similar way. Finally we consider $M_{n4}$. A simple computation now yields

$$M_{n4} = O(E(|U_{m:n}-\frac{m}{n+1}|+|U_{n-m+1:n}-\frac{n-m+1}{n+1}|)^3 I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)} \tag{2.70}$$

$$(U_{n-m+1:n})(\int_0^1(r^{\frac{5}{4}-\delta}(U_{m:n}+(U_{n-m+1:n}-U_{m:n})s)+$$

$$r^{\frac{5}{4}-\delta}\left[\frac{m}{n}+\frac{n-2m}{n}s\right])ds)^3) = O(\left[\frac{m}{n}\right]^{\frac{3}{4}+3\delta}n^{-\frac{3}{2}}).$$

Together all these results directly imply (2.59) and the lemma is proved. □

We are now in a position to complete the proof of (2.26). Take $m=[n^{\frac{1}{3}}]$. Application of an exponential bound for uniform order statistics (see, e.g., Lemma A2.1 of ALBERS, BICKEL and VAN ZWET (1976)) yields

$$\int_{|t|\leq n^{\gamma}} |t|^{-1}|\rho^{n\pm^*}(t)-Ee^{itT^*_{n\pm}}I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n})|dt = O(n^{-\frac{1}{2}}). \tag{2.71}$$

Using (2.28), (2.54) and the Lemma's 2.2, 2.3 and 2.4 we find after some elementary computations for all $|t|\leq n^{\gamma}$ and for some sufficiently small $\gamma>0$

$$\begin{aligned}&|Ee^{itT^*_{n\pm}}I_{(0,\epsilon)}(U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n})-e^{-\frac{1}{2}t^2}| \leq \\ &|E[(1-\tfrac{1}{2}t^2\sigma_{n\pm}^{-2}\sigma^2(T_{1n}(U_{m:n})|U_{m:n}))(1-\tfrac{1}{2}t^2\sigma_{n\pm}^{-2}(T_{3n}(U_{n-m+1:n})|U_{n-m+1:n})) \\ &(\exp(-\tfrac{1}{2}t^2\sigma_{n\pm}^{-2}\sigma^2(S_n(U_{m:n},\ U_{n-m+1:n})))|U_{m:n},\ U_{n-m+1:n}) \\ &(1+it(E(T^*_{n\pm}|U_{m:n},\ U_{n-m+1:n})-\tfrac{1}{2}t^2(E(T^*_{n\pm}|U_{m:n},\ U_{n-m+1:n}))^2)I_{(0,\epsilon)}( \\ &U_{m:n})I_{(1-\epsilon,1)}(U_{n-m+1:n})]-e^{-\frac{1}{2}t^2}|+O(n^{-\frac{1}{2}}(t^2+|t|^3)e^{-\frac{1}{5}t^2}+O(n^{-\frac{1}{2}-\frac{1}{2}\delta}).\end{aligned} \tag{2.72}$$

Combining now (2.54), (2.51), (2.72) we arrive at (2.26) after some calculations involving conditional moments. We note that we have used here the well-known fact that

$$\begin{aligned}\sigma_{n\pm}^2 &= E\{\sigma^2(T_{1n}(U_{m:n})|U_{m:n})+\sigma^2(T_{2n\pm}(U_{m:n},\ U_{n-m+1:n})|U_{m:n},\ U_{n-m+1:n})+ \\ &\sigma^2(T_{3n}(U_{n-m+1:n})|U_{n-m+1:n})\}+E(E(T_{n\pm}|U_{m:n},\ U_{n-m+1:n})-ET_{n\pm})^2.\end{aligned} \tag{2.73}$$

Also we employ the easily verified inequality

$$\begin{aligned}&|\sigma(T_{2n\pm}(U_{m:n},\ U_{n-m+1:n})|U_{m:n},\ U_{n-m+1:n})- \\ &\sigma(S_n(U_{m:n},\ U_{n-m+1:n})|U_{m:n},\ U_{n-m+1:n})|^2 \leq \sigma^2((T_{2n\pm}(U_{m:n},\ U_{n-m+1:n})- \\ &S_n(U_{m:n},\ U_{n-m+1:n}))|U_{m:n},\ U_{n-m+1:n}) \leq 2\sigma^2(Q_n(U_{m:n},\ U_{n-m+1:n})|U_{n-m+1:n})+ \\ &2\sigma^2(R_n(U_{m:n},\ U_{n-m+1:n})\,U_{m:n},\ U_{n-m+1:n})\end{aligned} \tag{2.74}$$

with $Q_n$ and $R_n$ as in (2.41) and (2.4.2). The first term on the r.h.s. of (2.74) is estimated in (2.52) and (2.53). A similar bound for the second term on the r.h.s. of (2.74), i.e., for $\sigma^2(R_n(u,v))$, is easily obtained by an argument like (2.58). Here we write

$$\begin{aligned}&\sigma^2(R_n(u,v)) \leq ER_n^2(u,v) = \\ &O(E(\int_0^1|\Gamma_{n-2m}(s)-s|^3|J''\left[\frac{m}{n}+\frac{n-2m}{n}(\lambda s+(1-\lambda)\Gamma_{n-2m}(s)\right]|dF^{-1}(u+(v-u)s))^2) = \\ &O(E((\int_0^{n^{-\alpha}}+\int_{1-n^{-\alpha}}^1)|\Gamma_{n-2m}(s)-s|^3\left[\frac{m}{n}\right]^{-2}dF^{-1}(u+(v-u)s))^2+ \\ &E(\int_{n^{-\alpha}}^{1-n^{-\alpha}}|\Gamma_{n-2m}(s)-s|^3r^2\left[\frac{m}{n}+\frac{n-2m}{n}s\right]dF^{-1}(u+(v-u)s))^2) = \\ &O(n^{-3-2\alpha}\left[\frac{m}{n}\right]^{-2}((F^{-1}(u))^2+(F^{-1}(v))^2)+n^{-3}\left[\frac{m}{n}\right]^{-1}(F^{-1}(u))^2+(F^{-1}(v))^2) = \\ &O(n^{-2}m^{-1}((F^{-1}(u))^2+(F^{-1}(v))^2)\end{aligned} \tag{2.75}$$

provided we take $\alpha\in(0,1)$ (cf. (2.57)) sufficiently large.

Next we prove (2.27). Take $m=[\frac{1}{4}\epsilon n]$. Using (2.28) once more we find for all $|t|\leq\gamma n^{\frac{1}{2}}$

$$|\rho^*_{n\pm}(t)| \leq E|\phi^*_{T_{2n\pm}(U_{m:n},\ U_{n-m+1:n})}(t)|. \tag{2.76}$$

Clearly $T_{2n\pm}(u,v)$ (cf. (2.13)) is the sum of a non-degenerate $U$-statistic of degree 2-which is precisely equal to $S_n(u,v)+Q_n(u,v)$, with a kernel, which is bounded by $C(|F^{-1}(v)|+|F^{-1}(v)|)$ for some constant $C>0$, and a remainder term - which is the third order term in (2.13) - satisfying

$$E|R_n(u,v)| = O(n^{-\frac{3}{2}}(|F^{-1}(u)|+|F^{-1}(v)|). \tag{2.77}$$

This latter order bound is immediate from (2.58), this time with the choice $m=[\frac{1}{4}\epsilon n]$.

We now follow the argument given on page 505 of HELMERS and VAN ZWET (1982), (cf their relation (3.10)), together with the elementary estimate

$$|\phi^*_{T_{2n\pm}(u,v)}| = |Ee^{it\sigma_{n\pm}^{-1}T_{2n\pm}(u,v)}| \leq |Ee^{it\sigma_{n\pm}^{-1}(S_n(u,v)+Q_n(u,v))}|+|t|\sigma_{n\pm}^{-1}E\,|R_n(u,v)| \tag{2.78}$$

to find that for some sufficiently small $\gamma>0$

$$\int_{n^\gamma<|t|\leq\gamma n^{\frac{1}{2}}} |t|^{-1}|\rho^*_{n\pm}(t)|dt \leq \int_{n^\gamma\leq|t|\leq\gamma n^{\frac{1}{2}}} |t|^{-1}E\,|\phi_{T_{2n\pm}(U_{m:n},\ U_{n-m+1:n})}(t)|dt \leq \tag{2.79}$$

$$\int_{n^\gamma\leq|t|\leq\gamma n^{\frac{1}{2}}} |t|^{-1}(E\,|e^{it\sigma_{n\pm}^{-1}(S_n(u_{m:n},\ U_{n-m+1:n})+Q_n(U_{m:n},\ U_{n-m+1:n}))} +$$

$$n^{\frac{1}{2}}\sigma_{n\pm}^{-1}E\,|R_n(U_{m:n},\ U_{n-m+1:n})|=O(n^{-\frac{1}{2}}E[|F^{-1}(U_{m:n})|^3+|F^{-1}(U_{n-m+1:n})|^3+$$

$$|F^{-1}(U_{m:n})|^p+|F^{-1}(U_{n-m+1:n})|^p+|F^{-1}(U_{m:n})|+|F^{-1}(U_{n-m+1:n})|]$$

for some constant $p>\frac{5}{3}$. Since $E|F^{-1}(U_{n:m})|^r$, for $r=1,p$ and 3, are $O(1)$, with $m=[\frac{1}{4}\epsilon n]$, we have proved (2.27). This completes the proof of Theorem 1 for the case that the weights are of the form (1.2).

It remains to establish (1.10) for the case that the weights of the form (1.3), i.e., the weights are given by $c_{in}=J(\frac{i}{n+1})$ for $i=1,2,...,n$, $n\geq1$. The basic new result we shall need is the following order bound:

$$\sigma^2(n^{-1}\sum_{i=1}^{n}(n\int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds-J\left[\frac{i}{n+1}\right])X_{i:n}) = O(n^{-\frac{3}{2}-2\delta}(\log n)^2). \tag{2.80}$$

To prove this is suffices to estimate

$$Q_{n1} = \sigma^2(\sum_{\nu=2}^{[n\epsilon]}(\int_{\frac{\nu-1}{n}}^{\frac{\nu}{n}} J(s)ds-\frac{1}{n}J\left[\frac{\nu}{n+1}\right])F^{-1}(U_{\nu:n})) \tag{2.81}$$

$$Q_{n2} = \sigma^2(\sum_{\nu=[n\epsilon]}^{[n(1-\epsilon)]}(\int_{\frac{\nu-1}{n}}^{\frac{\nu}{n}} J(s)ds-\frac{1}{n}J\left[\frac{\nu}{n+1}\right])F^{-1}(U_{\nu:n})) \tag{2.82}$$

and

$$Q_{n3} = \sigma^2(\sum_{\nu=n-[n\epsilon]}^{n}(\int_{\frac{\nu-1}{n}}^{\frac{\nu}{n}} J(s)ds-\frac{1}{n}J\left[\frac{\nu}{n+1}\right])F^{-1}(U_{\nu:n})) \tag{2.83}$$

where $\epsilon > o$ as in Theorem 1

It follows directly from assumption (II) that

$$\sigma^2(F^{-1}(U_{\nu:n})) \leq E(F^{-1}(U_{\nu:n}) - F^{-1}\left(\frac{\nu}{n+1}\right))^2 = O(n^{-1} r^{\frac{3}{2}-2\delta}\left(\frac{\nu}{n+1}\right)). \tag{2.84}$$

Moreover, with the aid of assumption (I), we obtain

$$\left| \int_{\frac{\nu-1}{n}}^{\frac{\nu}{u}} J(s)ds - \frac{1}{n} J\left(\frac{\nu}{n+1}\right) \right| = O(n^{-2} r\left(\frac{\nu}{n+1}\right) + n^{-3} r^2\left(\frac{\nu}{n+1}\right)) \tag{2.85}$$

for $\nu \leq n\epsilon$ or $\nu \geq n(1-\epsilon)$, whereas the r.h.s. of (2.85) becomes $O(n^{-2})$ for $n\epsilon < \nu < n(1-\epsilon)$. These last two bounds directly imply that

$$\sigma^2(Q_{n1}) = O(n^{-\frac{3}{2}-2\delta}), \quad \sigma^2(Q_{n3}) = O(n^{-\frac{3}{2}-2\delta}).$$

It follows directly from the argument given on page 35 of HELMERS (1982) that $\sigma^2(Q_{n2}) = O(n^{-3})$, as $n \to \infty$. Together these order bounds for $\sigma^2(Q_{ni})$ for $i = 1,2,3$, imply (2.80) and the proof of Theorem 1 is complete. □

## 3. PROOF OF THEOREM 2.

The proof of Theorem 2 is a slight modification of the proof of Theorem 1. Without loss of generality we shall assume that there exists only one discontinuity, i.e., the function $J$ possesses exactly one jump at the point $s_1$ in the interval (0,1). Also we shall only give the proof of Theorem 2 for the case that the weights are of the form (1.2): i.e.,

$$c_{in} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds$$

for $i = 1,2,...,n$, $n \geq 1$. The other case, when the weights are generated by (1.3),can be treated quite similarly to the argument given at the end of Section 2.

We give the modifications which are needed to carry the proof of Theorem 1 over to our present more general situation. Our new proof will require an additional conditioning argument. To begin with we consider the basic decomposition (2.7) and rewrite $T_{2n}(U_{m:n}, U_{n-m+1:n})$ as follows:

$$T_{2n}(U_{m:n}, U_{n-m+1:n}) = S_{1n}(U_{m:n}, U_{m_1:n}) + \int_{\frac{m_1-n}{n}}^{\frac{m_1}{n}} J(s)dsF^{-1}(U_{m_1:n}) + \tag{3.1}$$

$$S_{2n}(U_{m_1:n}, U_{m_2:n}) + \int_{\frac{m_{2-1}}{n}}^{\frac{m_2}{n}} J(s)dsF^{-1}(U_{m_{2-1}:n}) + S_{3n}(U_{m_2:n}, U_{n-m+1:n})$$

where

$$m_1 = [ns_1 - cn^{\frac{1}{2}} \log n], \quad m_2 = [ns_1 + cn^{\frac{1}{2}} \log n] \tag{3.2}$$

for $c > 0$, and with

$$S_{1n}(U_{m:n}, U_{m_1:n}) = \sum_{i=1}^{m_1-m-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)dsF^{-1}(U_{m:n} + (U_{m_1:n} - U_{m:n})Z'_{i:m_1-m-1}) \tag{3.3}$$

$$S_{2n}(U_{m_1:n}, U_{m_2:n}) = \sum_{i=1}^{m_2-m_1-1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)dsF^{-1}(U_{m_1:n}+(U_{m_2:n}-U_{m_1:n})Z''_{i:m_2-m_1-1}) \tag{3.4}$$

and

$$S_{3n}(U_{m_2:n}, U_{n-m+1:n}) = \sum_{i=1}^{n-m-m_2} \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)dsF^{-1}(U_{m_2:n}+(U_{n-m+1:n}-U_{m_2:n})\cdot \tag{3.5}$$

$$\cdot Z'''_{i:n-m-m_2}).$$

The random vectors $V=(V_{1:m-1},\ldots,V_{m-1,m-1})$, $Z'=(Z_{1:m_1-m-1},\ldots,Z_{m_1-m-1:m_1-m-1})$, $Z''=(Z_{1:m_2-m_1-1},\ldots,Z_{m_2-m_1-1:m_2-m_1-1})$, $Z'''=(Z_{1:n-m-m_2},\ldots,Z_{n-m-m_2:n-m-m_2})$ and $W=(W_{1:m-1},\ldots,W_{m-1:m-1})$ are the vectors of order statistics corresponding to the samples of sizes $m-1$, $m_1-m-1$, $m_2-m_1-1$, $n-m-m_2$ and $m-1$ from the uniform distribution on (0,1); the vectors $V$ and $W$ are defined in the paragraph after (2.5) in Section 2. We let the vectors $V$, $Z'$, $Z''$, $Z'''$, $W$ and $(U_{m:n}, U_{m_1:n}, U_{m_2:n}, U_{n-m+1:n})$ be independent. The r.v's $S_{1n}(U_{m:n}, U_{m_1:n})$ and $S_{3n}(U_{m_2:n}, U_{n-m+1:n})$ can now be treated similarly as $T_{2n}(U_{m:n}, U_{n-m+1:n})$ in Section 2, whereas the r.v. $S_{2n}(U_{m_1:n}, U_{m_2:n})$ can be analysed in a way similar to that of $T_{1n}(U_{m:n})$ and $T_{3n}(U_{n-m+1:n})$. The analysis of the r.v's $S_{1n}(U_{m:n}, U_{m_1:n})$ and $S_{3n}(U_{m_2:n}, U_{n-m+1:n})$ resembles closely the one given for $T_{2n}(U_{m:n}, U_{n-m+1:n})$ in Section 2 (cf. Lemma 2.3). We only have to replace the function $\psi_n$ (cf. (2.11)) by functions

$$\psi_{1n}(u) = \int_u^{\frac{m_1}{n}} J(y)dy - \frac{\frac{m_1}{n}-u}{\frac{m_1-m}{n}} \int_{\frac{m}{n}}^{\frac{m_1}{n}} J(y)dy \tag{3.6}$$

and

$$\psi_{2n}(u) = \int_u^{\frac{n-m_2}{n}} J(y)dy - \frac{\frac{n-m_2}{n}-u}{\frac{n-m_2-m}{n}} \int_{\frac{m_2}{n}}^{\frac{n-m}{n}} J(y)dy \tag{3.7}$$

respectively.

The r.v. $S_{2n}(U_{m_1:n}, U_{m_2:n})$ causes the only new difficulty in the proof, as compared with the proof of Theorem 1. In view of assumption (IV) and (2.2) we easily check that

$$E\,|F^{-1}(u+(v-u)Z''_{i-m_2-m_1-1})-(F^{-1}(u+(v-u)\frac{i}{m_2-m_1})|^k = \tag{3.8}$$

$$O(((m_2-m_1)r(\frac{i}{(m_2-m_1)}))^{-\frac{k}{2}}),$$

whenever $u,v\in(s_1-\eta, s_1+\eta)$, $k>0$, for some $\eta>0$. It follows directly from (3.8) that

$$E\,|S_{2n}(u,v)-ES_{2n}(u,v)|^2 = O((n^{-1}\sum_{i=1}^{m_2-m_1-1} E((F^{-1}(u+(v-u)Z''_{i:m_2-m_1-1})- \tag{3.9}$$

$$F^{-1}(u+(v-u)\frac{i}{(m_2-m_1)}))^2 = O(n^{-\frac{3}{2}}\log^2 n)$$

$$E\,|S_{2n}(u,v)-ES_{2n}(u,v)|^3 = O(n^{-\frac{3}{2}}n^{-\frac{3}{4}}\log^3 n) \tag{3.10}$$

whenever $u,v \in (s_1-\eta, s_1+\eta)$ which further implies

$$|\phi^*_{S_{2n}(u,v)}(t)-1+\frac{t^2}{2}\sigma_n^{-2}\sigma^2(S_{2n}(u,v))| = O(|t|^3 n^{-\frac{3}{4}}\log^3 n) \tag{3.11}$$

for $u,v \in (s_1-\eta, s_1+\eta)$.

Finally, to obtain the assertion corresponding to that of Lemma 2.4 it remains to show that

$$E|\sum_{i=1}^{m_2-m_1-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}} J(y)dy\,|EF^{-1}(U_{m_1:n}+(U_{m_2:n}-U_{m_1:n})Z''_{i:m_2-m_1-1})|U_{m_1:n},\,U_{m_2:n})- \tag{3.12}$$

$$EF^{-1}(U_{m_1:n}+(U_{m_2:n}-U_{m_1:n})Z''_{i:m_2-m_1-1})|^3 = O(n^{-2}n^{-\frac{1}{4}}\log^3 n).$$

Proceeding quite similarly as above we obtain

$$E|\sum_{i=1}^{m_2-m_1-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}} J(y)dy\,|EF^{-1}(U_{m_1:n}+(U_{m_2:n}-U_{m_1:n})Z''_{i:m_2-m_1-1}|U_{m_1:n},\,U_{m_2:n})- \tag{3.13}$$

$$EF^{-1}(U_{m_1:n}+(U_{m_2:n}-U_{m_1:n})Z''_{i:m_2-m_1-1})|^3 I_{(s_1-\eta,s_1+\eta)}(U_{m_2:n})I_{(s_1-\eta,s_1+\eta)}(U_{m_1:n}) =$$

$$O(n^{-3}(m_2-m_1)^{\frac{3}{2}}) = O(n^{-2-\frac{1}{4}}\log^3 n).$$

The remaining part follows easily from the fact that

$$P(U_{m_i:n}\in(s_i-\epsilon,s_i+\epsilon)) = O(n^{-q}),\quad i=1,2 \tag{3.14}$$

and

$$E|\sum_{i=m_1+1}^{m_2-1}\int_{\frac{i-1}{n}}^{\frac{i}{n}} J(y)dyF^{-1}(U_{i:n})|^3 = O(1) \tag{3.15}$$

for $\epsilon>0$ and $q>0$ arbitrary. This completes the proof of Theorem 2.

REFERENCES


[1] W. ALBERS, P.J. BICKEL, and W.R. VAN ZWET (1976), Asymptotic expansions for the power of distribution free tests in the one sample problem. *Ann. Statist. 4,* 108-156.

[2] S. BJERVE (1977). Error bounds for linear combinations of order statistics. *Ann. Statist. 5,* 237-255.

[3] R.J.M.M. DOES (1982). Berry-Esseen theorems for simple linear rank statistics under the null-hypothesis. *Ann. Probability 10,* 982-991.

[4] W. FELLER (1971). *An Introduction to Probability Theory and Its Applications,* Vol. 2, 2nd edition, Wiley, New York.

[5] K.O. FRIEDRICH (1985). *Berry-Esseen Abschätzungen für Nichtparametrische Statistiken,* Ph.D. Thesis, Albert-Ludwig-Universität, Freiburg in Breisgau, 66 p.

[6] R. HELMERS (1977). The order of the normal approximation for linear combinations of order statistics with smooth weight functions, *Ann. Probability 5,* 940-953.

[7] R. HELMERS (1981). A Berry-Esseen theorem for linear combinations of order statistics. *Ann. Probability 9,* 342-347.

[8] R. HELMERS (1982). *Edgeworth expansions for linear combinations of order statistics,* Mathematical Centre Tracts 105, Centre for Mathematics and Computer Science, Amsterdam.

[9] R. HELMERS and W.R. VAN ZWET (1982). The Berry-Esseen bound for *U*-statistics. *Proc.*

*Symp. on Statistical Decision Theory and Related Topics, III,* 497-512 (ed. J.O. Berger, S.S. Gupta).

[10] R. Helmers and M. Hušková (1984). A Berry-Esseen bound for *L*-statistics with unbounded weight functions. *Proc. Third Prague Symp. on Asymptotic Statistics,* 93-101 (ed. P. Mandl, M. Hušková)

[11] T.L. Lai (1975). On Chernoff-Savage statistics and sequential rank tests. *Ann. Statist., 3,* 825-845.

[12] D.M. Mason (1981). Asymptotic normality of linear combinations of order statistics with a smooth score function. *Ann. Statist. 9,* 899-904.

[13] V.V. Petrov (1975). *Sums of Independent Random Variables,* Springer, Berlin.

[14] R.J. Serfling (1980). *Approximation Theorems of Mathematical Statistics,* Wiley, New York.

[15] W.R. van Zwet (1977). Asymptotic expansions for the distribution functions of linear combinations of order statistics. *Proc. Symp. on Statistical Decision Theory and Related Topics II,* 421-437, (ed. D.S. Moore, S.S. Gupta).

[16] W.R. van Zwet (1984). A Berry-Esseen bound for symmetric statistics. *Z. Wahrsch. Verw. Gebiete, 66,* 425-440.