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Prospective Classification of Distance-Transitive Graphs

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ABSTRACT

The present state of the art in the classification of distance-transitive graphs is surveyed. A detailed treatment is given of the graphs on which a group with simple socle $PSL(n,q)$ $(n \geq 8)$ acts distance-transitively.

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1. INTRODUCTION

In this section we outline the programme of classifying all distance-transitive graphs. We mention some classification results, mainly concerning the case where the graph has an automorphism group with nonabelian simple socle. Section 2 provides some necessary conditions for a permutation group to act as a distance-transitive group of automorphisms on a graph and surveys recent general theory. Finally, in Section 3 a detailed treatment is given of a specific case where the simple socle is isomorphic to $PSL(n,q)$ with $n \geq 8$. We understand that this result (viz. Theorem 3.2) has also been established by INGLIS [14].

1.1. Definition. Let G be a group acting on a graph $\Gamma = (V\Gamma, E\Gamma)$ (i.e., we are given a morphism $G \rightarrow \mathrm{aut}\,\Gamma$). We say that G acts *distance-transitively* on Γ if its induced action on each of the sets

$$\{(\gamma,\delta) \mid \gamma,\delta \in V\Gamma,\ d(\gamma,\delta) = i\}$$

is transitive. Here, $d(\gamma,\delta)$ denotes the usual distance in Γ between its vertices γ and δ, and i runs through $\{0, \cdots, diam\Gamma\}$. A graph Γ is said to be *distance-transitive* if $\mathrm{aut}\,\Gamma$ acts distance-transitively on it.

Suppose G acts distance-transitively on the graph Γ. Then G is transitive on $V\Gamma$ and, for each i, $(1 \leq i \leq diam\Gamma =: d)$, the stabilizer G_γ of $\gamma \in V\Gamma$ acts transitively on the set $\Gamma_i(\gamma)$ of all vertices at distance i to γ. The most famous set of invariants of a distance-transitive graph is its *intersection array*

$$\{b_0, b_1, \cdots, b_{d-1}; c_1, \cdots, c_d\}$$

where, for $\gamma \in V\Gamma$ and $\delta \in \Gamma_i(\gamma)$, $b_i = |\Gamma_1(\delta) \cap \Gamma_{i+1}(\gamma)|$ and $c_i = |\Gamma_1(\delta) \cap \Gamma_{i-1}(\gamma)|$. (In fact, $c_0 = 1$, so there are only $2d-1$ relevant numbers involved.)

The notion of *distance-regularity* for a graph comes down to the existence of an intersection array for it. We shall not comment on the many different ways of looking at distance-regular graphs; we use the standard terminology and facts concerning this concept, as given in BANNAI & ITO [3], and BROUWER, COHEN & NEUMAIER [8].

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In reconstructing Γ from G, two data are crucial: for some $\gamma \in V\Gamma$, the subgroup $H := G_\gamma$ of G of all elements fixing γ, and an element g of G such that $\{g\gamma,\gamma\} \in E\Gamma$. For then, as a G-set, $V\Gamma$ can be identified with G/H, and, via this identification, we have $\{g_1H, g_2H\} \in E\Gamma$ if and only if $g_1^{-1}g_2 \in HgH$. In particular, for every distance-transitive graph Γ there is a group G, a subgroup H of G and an element $g \in G$ with $g \notin H$ and $g^{-1} \in HgH$ such that $\Gamma \cong \Gamma(G,H,g)$, where $\Gamma(G,H,g)$ stands for the graph Δ with $V\Delta = G/H$ and $E\Delta = \{\{g_1H, g_1gH\} \mid g_1 \in G\}$.

Suppose we are given G,H,g such that G acts distance-transitively on $\Gamma = \Gamma(G,H,g)$. Usually, it is not easy to find b_i, c_i from G and H, but the numbers $k_i := |\Gamma_i(\gamma)|$ can be found as H-orbit sizes: since H is transitive on $\Gamma_i(\gamma)$, and there is $t \in G$ with $t\gamma \in \Gamma_i(\gamma)$, we have $k_i = |H/H_{t\gamma}| = |H/(H \cap tHt^{-1})|$, whence $k_i = |HtH/H|$. There are a few problems left: supposing we have found the set of numbers $|HtH/H|$, how should it be ordered to give the k_i? (Lemma 2.7 below gives an indication.) Supposing the ordering of the k_i has been found. Although the b_i and c_i are related via $k_ib_i = k_{i+1}c_{i+1}$ the intersection array cannot yet be fully completed. Below we shall see why the question of determining Γ from G,H is of relevance.

1.2. Examples. (i). Set $V\Gamma = \binom{X}{d}$, where $X = \{1,\ldots,n\}$ for some n and $E\Gamma = \{\{x,y\} \mid |x \cap y| = d-1\}$. Then $\Gamma = (V\Gamma, E\Gamma)$ is the so-called *Johnson graph* $J(n,d)$. It has diameter d if $2d \le n$ and it is isomorphic to $J(n,n-d)$. The symmetric group Sym_n on n letters acts distance-transitively on $J(n,d)$.
(ii). Set $V\Gamma = \left[{V \atop d}\right]$, the set of all d-dimensional linear subspaces of a vector space V of dimension n over $\mathbf{F}_q$, and $E\Gamma = \{\{x,y\} \mid \dim x \cap y = d-1\}$. Then $\Gamma = (V\Gamma, E\Gamma)$ is the so-called *Grassmann graph* $G(n,d,q)$. Its diameter is d if $2d \le n$, and it is isomorphic to $G(n,n-d,q)$. The projective special linear group $PSL(n,q)$ acts distance-transitively on $G(n,d,q)$.

Imprimitivity of the action of G on $V\Gamma$ can be described completely in terms of the graph. For any j, denote by Γ_j the graph with vertex set $V\Gamma$ whose edges are the unordered pairs of vertices at distance j in Γ. Thus Γ_2 is disconnected if and only if Γ is bipartite. We say that Γ is *antipodal* if Γ_d, where $d = diam\Gamma$, is disconnected. Clearly, G is imprimitive on $V\Gamma$ if Γ is bipartite or antipodal. The following converse of this observation is due to SMITH [27].

1.3. Theorem. *An imprimitive distance-transitive graph is bipartite, antipodal, or of valency 2.*

In fact (cf. [8]) the theorem also holds if distance-transitivity is replaced by the weaker (combinatorial) condition of distance-regularity.

The theorem turns the classification of distance-transitive graphs into a two-stage problem. The first stage is to find all graphs whose automorphism groups act primitively on the vertex sets (i.e., the *primitive* distance-transitive graphs). The second stage is to find, for every primitive distance-transitive graph Γ, all related imprimitive distance transitive graphs $\tilde{\Gamma}$ (more precisely, assuming the valency is at least 3, all bipartite and all antipodal distance-transitive graphs $\tilde{\Gamma}$ for which a natural quotient exists that is isomorphic to Γ.) The number of such imprimitive distance-transitive graphs can be bounded in terms of $|G|$ (provided the valency of Γ is at least 3). Two techniques of finding imprimitive distance-transitive graphs related to a given primitive distance-transitive graph have proved to be of use: for finding bipartite graphs, a study of maximal cliques (cf. HEMMETER [12]), and for finding antipodal, a study of geodesics of the underlying primitive graph (cf. BROUWER & VAN BON [6]). By the two papers just cited, almost all imprimitive distance-transitive graphs related to a large class of primitive ones are determined.

The following result by PRAEGER, SAXL & YOKOYAMA [25] sheds light on the structure of aut Γ in the primitive case. The proof of this theorem depends on the classification of finite simple groups. Below, a group G is called *almost simple* if there is a nonabelian simple group X such that $X \le G \le \operatorname{aut} X$. (Then $X = \operatorname{soc} G$, the socle of G, i.e., the product of all minimal normal subgroups of G.)

1.4. Theorem. *Let* Γ *be a primitive distance-regular graph with a distance-transitive group* G *of automorphisms of* Γ. *Then one of*

(i) Γ *is a Hamming graph or, in case* $diam\Gamma = 2$, *its complement, and* G *is a wreath product;*

(ii) G *is almost simple;*

(iii) G *has an elementary abelian normal subgroup which is regular on* Γ.

1.5. The state of the art. In case (i) of Theorem 1.4, the graph is completely determined (although the group is not).

In case (ii), the classification of finite simple groups may be invoked so as to obtain an exhaustive list of possibilities for soc G. The following cases have already been treated: soc $G = \mathrm{Alt}_n$, the alternating group on n letters ($n \ge 5$), by SAXL [26] (for $n > 18$), IVANOV [18] (who also treats the imprimitive case), and LIEBECK, PRAEGER & SAXL [23]. For soc $G = PSL(n,q)$, $n \ge 8$, see Section 3. The case $n \le 7$ will be dealt with in a forthcoming paper. We understand that work on case (ii) for the classical groups of dimension at least 13 has been done in INGLIS [14]. At any rate, the distance-transitive groups of Lie type with a parabolic vertex stabilizer are all determined (cf. [8]). This leaves the exceptional groups of Lie type, where a classification is imminent (e.g., $E_8(q)$ has no distance-transitive permutation representation), and the sporadic groups. Some interesting examples of primitive distance-transitive graphs come from the sporadic groups M_{12}, M_{24}, J_1, J_2, and *Suz*. On the other hand, the Held group does not occur as soc G in case (ii) of the theorem (cf. VAN BON, COHEN & CUYPERS [7]).

Case (iii) is somewhat harder. Here, the first step is to use ASCHBACHER [2] to reduce to the case of an almost simple vertex stabilizer and the second to reduce the size of the regular normal subgroup, so that the study of an almost simple group on a relatively small module remains as the third step.

We end this section by mentioning two kinds of results that do not fit the partitioning induced by Theorem 1.4: If $diam\Gamma=1$, then G is doubly transitive on $V\Gamma$, and the pair G, Γ is known (see, e.g., LIEBECK [22] for further references). If $diam\Gamma = 2$, then G and Γ are determined by KANTOR & LIEBLER [20], LIEBECK [21,22], and LIEBECK & SAXL [24]. It seems that the methods employed can be extended to $diam\Gamma \le 4$ without too much difficulty. Finally, the case of small valency has been given some attention during the last few years, cf. A.A. IVANOV & A.V. IVANOV [19]. As a consequence, every distance-transitive graph (not necessarily primitive) of valency at most 13 is known. See [8] for details and further references.

2. General Theory

It may be transparent from the Praeger-Saxl-Yokoyama Theorem that the main problem in the classification of all distance-transitive graphs is: given a group G find all possible graphs Γ on which G acts primitively and distance-transitively. Thus, criteria are needed to determine the structure of such a graph Γ from knowledge of G. The lemma below is an example. It states that the stabilizer in G of a vertex in Γ is fairly large. The proof (cf. [8]) uses the fact that every orbit is self-paired and that hence every irreducible constituent of the permutation character of G on $V\Gamma$ occurs with multiplicity 1 and is real.

Notation. We shall, for any graph Γ, unless ambiguity arises, respectively denote by v, k, and d its number of vertices, its valency and its diameter.

2.1. Lemma. *If G is a distance-transitive group of automorphisms of a graph Γ of diameter d on v vertices, then*

(i) $v \le \sum_{\chi} \chi(1)$, *where the sum runs over all irreducible real characters χ of G;*

(ii) $v \le \min(\sqrt{(d+1)|G|},\ 1 + t + \sqrt{s|G|})$ *and $d \le r - 1$, where r (resp. s) stand for the number of irreducible real (resp. symplectic) characters of G, and t is the number of involutions in G.*

The following, trivial observation is quite useful: if $\pi = \sum_{\chi} a_\chi \chi$ is a permutation character of G, then $(\pi, 1_H^G) \le \sum_{\chi} a_\chi$, where 1_H^G denotes the permutation character of G on $V\Gamma \cong G/H$ (here H is the stabilizer in G of a vertex). This observation is usually applied with π a well-known permutation character. For instance, if G is a group of Lie type with Tits system (B,N,W,R) and $\pi = 1_B^G$, where B is a Borel subgroup of G, then

$$(\pi, 1_H^G) \le \sum_{\chi} \chi(1) = \tau + 1,$$

where χ runs over the irreducible characters of W and τ denotes the number of involutions in W. (Since, in any group, the number of irreducible symplectic character degrees plus the number of elements of order at most 2 equals the number of irreducible real character degrees, cf. Isaacs [16] p. 51, the above equality follows from the fact that all irreducible characters of W are real). In particular, H has at most τ+1 orbits on G/B. Inglis, Liebeck & Saxl determined the multiplicity-free permutation representations of $L(n,q)$, $n \ge 8$, (cf. Theorem 3.1 below) by making heavy use of this observation.

A useful diameter bound is given by Terwilliger (cf. [29]).

2.2. Theorem. *Let Γ be a distance-regular graph with intersection array $\{b_0, b_1, \cdots, b_{d-1};\ c_1, \cdots, c_d\}$. If Γ contains a quadrangle, then, for all i ($i = 1, \cdots, diam\Gamma$),*

$$c_i - b_i \ge c_{i-1} - b_{i-1} + a_1 + 2\,;$$

in particular, $diam\Gamma \le (k + c_d)/(a_1 + 2)$.

The criterium below, due to VAN BON [5], restricts the behavior of normal subgroups of a vertex stabilizer.

2.3. Proposition. *Let* Γ *be a graph of diameter d on which the group G acts distance-transitively as a group of automorphisms. For* $\gamma \in V\Gamma$, *denote by* G_γ^j *the kernel of the action of* G_γ *on* $\Gamma_j(\gamma)$. *If, for some* $i \ge 1$, *we have* $G_\gamma^i \ne 1$, *then*

$$G_\gamma^i \subset G_\gamma^{i-1} \subset \cdots \subset G_\gamma^1 \quad or \quad G_\gamma^i \subset G_\gamma^{i+1} \subset \cdots \subset G_\gamma^d .$$

Thus, the normal subgroups of G_γ whose actions vanish come in chains. The following corollary shows how the proposition can be used to derive information on Γ. We set $G_\gamma^{\le i}$ for the kernel of the action of G_γ on the union of all $\Gamma_j(\gamma)$ for $0 \le j \le i$, and, likewise $G_\gamma^{\ge i}$ for the kernel on the union of all $\Gamma_j(\gamma)$ for $i \le j \le d$.

2.4. Corollary. *Let* Γ, *G and* γ *be as in the above proposition. Assume, in addition, that G acts primitively on* $V\Gamma$.

(i) *If* $G_\gamma^{\le i} \ne 1$ *then* $|G_\gamma^{\le i}| > |G_\gamma^{\ge d-i}|$.

(ii) *Let* π *be a permutation of* $\{1,...,d\}$ *such that* K_i *is the kernel of the action of* $G_\gamma^{\pi(i)}$ *on* $\Gamma_{\pi(i)}(\gamma)$, *and* $|K_i| \ge |K_{i+1}|$ $(i = 0, \ldots, d)$, *where* $K_{d+1}=1$. *If* $|K_1| > |K_2| = |K_3| \ne 1$, *then* $\pi(1)=1$.

Another application (by VAN BON [5]) of this proposition concerns graphs defined on involutions. We recall that a class of *p*-transpositions in a group is a union of conjugacy classes of involutions with the property that the product of any two has order 1,2, or *p*.

2.5. Proposition. *Let* Γ *be a distance-transitive graph with distance-transitive group G. Suppose that the vertex set* $V\Gamma$ *of* Γ *is a conjugacy class of involutions in G, that G acts on* Γ *by conjugation and that there are elements in* $V\Gamma$ *which commute in G. Let* $x,y \in \Gamma$ *be such that x is adjacent to y. Then at least one of the following statements holds.*

(i) Γ *is a polygon or an antipodal 2-cover of a complete graph.*

(ii) *G is a 2-group.*

(iii) *The order of xy is an odd prime, and, if* $a,b \in V\Gamma$ *with ab of order 2, then a and b have maximal distance in* Γ; *moreover* $|wz| \ne 4$ *for all* $w,z \in V\Gamma$.

(iv) *The elements x and y commute, and if* $z \in \Gamma_2(x)$, *then the order of xz is 4 or a prime. In particular,* $O_2(C_G(x)) \ne \langle x \rangle$ *or there is an odd prime p such that* $C_G(x)$ *contains a normal subgroup generated by a class of p-transpositions.*

When starting with a group G having simple socle X, the situation where $H = C_G(\sigma)$, the centralizer in G of $\sigma \in \mathrm{aut}\, X$ can often be reduced to the case where $\sigma \in G$. The result is again due to VAN BON [5].

2.6. Lemma. *Let* Γ *be a graph on which G acts primitively distance-transitively, and denote by H the stabilizer in G of a vertex in* $V\Gamma$. *Suppose* σ *is an automorphism of G.*

(i) *If* σ *centralizes H and* $diam\Gamma \ge 3$, *then* $\sigma \in \mathrm{aut}\,\Gamma$.

(ii) *If* σ *normalizes H and* $diam\Gamma \ge 5$, *the same conclusion holds.*

The following result is one of the few tools that can be frequently employed. It goes back to TAYLOR & LEVINGSTON [28], except for the final statement, which follows from work of A.A. IVANOV [17], cf. [8].

2.7. Lemma. *Let* Γ *be a primitive distance-transitive graph with* $\mathrm{diam}\Gamma \geq 3$*, and, for* $\gamma \in V\Gamma$*, set* $k_i = \Gamma_i(\gamma)$.

(i) *There are* i, j *with* $1 \leq i \leq j \leq d$ *such that* $1 < k_1 < \cdots < k_i = \cdots = k_j > \cdots > k_d$.

(ii) *If* $i \leq j$ *and* $i+j \leq d$*, then* $k_i \leq k_j$.

(iii) *If* $k_i = k_j$ *for* i,j *with* $i < j$ *and* $i+j \leq d$*, then* $k_{i+1} = k_{j-1}$.

(iv) *If* $k_i = k_{i+1}$*, then* $k_i \geq k_j$ *for all* j.

(v) *If* $k_{j-1} = k$ *for some* j *with* $3 \leq j \leq d$*, then* $k = 2$.

(vi) *If* $k > 2$ *and* $k_{i+1} \leq k_i$*, then* $d \leq 3i$.

3. FINITE LINEAR GROUPS

The following theorem is due to N.F.J. INGLIS, M.W. LIEBECK & J. SAXL [15].

3.1. Theorem. *Let* G,H *be a pair consisting of a group* G *with* $PSL(n,q) \lhd G \leq \mathrm{aut}\, PSL(n,q)$, $n \geq 8$*, and a maximal subgroup* H *of* G *such that the permutation representation of* G *on the collection of cosets with respect to* H *is multiplicity free. Then one of the following statements holds (here,* $V = \mathbf{F}_q^n$ *and* ϕ *is the canonical projection* $\Gamma L(V) \to P\Gamma L(V)$*):*

(i) $G \leq P\Gamma L(V)$ *and* H *is the stabilizer of a proper subspace* U *of* V*;*

(ii) G *contains a graph automorphism and* H *is the stabilizer of a pair* $\{U,W\}$ *of subspaces of* V *with* $\dim U = 1$*, and* $\dim W = n-1$*;*

(iii) G *contains a graph automorphism,* $n = 2m$*, and* H *is the stabilizer of an* m*-dimensional subspace of* V*;*

(iv) G *contains a graph automorphism,* $n = 2m+1$*, and* H *is the stabilizer of a pair* $\{U,W\}$ *of subspaces of* V*, with* $U \subset W$, $\dim U = m$*, and* $\dim W = m+1$*;*

(v) $n=2m$ *and* $H = N_G(\phi K)$*, where* K *is a subgroup of* $SL(n,q)$ *isomorphic to* $SL(m,q^2)$*;*

(vi) $n=2m$ *and* $H = N_G(\phi K)$*, where* K *is a subgroup of* $SL(n,q)$ *isomorphic to* $Sp(2m,q)$*;*

(vii) q *is a square and* $H = N_G(\phi K)$*, where* K *is a subgroup of* $SL(n,q)$ *isomorphic to* $SL(n,q^{1/2})$*;*

(viii) q *is a square and* ϕK*, where* K *is a subgroup of* $SL(n,q)$ *isomorphic to* $SU(n,q^{1/2})$.

We use the above theorem to establish that all graphs on which G as above acts distance-transitively are known:

3.2. Theorem. *Let G be a group with $PSL(n,q) \lhd G \leq \mathrm{aut}\,PSL(n,q)$, $n \geq 8$. If Γ is a graph on which G acts primitively and distance-transitively, then Γ is a Grassmann graph (possibly a clique).*

Proof. As discussed in §1, we may put $\Gamma = \Gamma(G,H,g)$ where H is a subgroup and g is an element of G. Since the permutation representation of G on H is multiplicity free, we have one of the cases listed in Theorem 3.1. We shall treat them separately.

CASE (i). Let $n \geq 3$. Letting two cosets of H be adjacent if the unique subspaces of dimensions $d = \dim U$ they stabilize meet in a $d-1$-dimensional subspace, we find the Grassmann graphs $G(n,d,q)$. It is well known that this is the only graph structure on $V\Gamma$ turning it into a distance-transitive graph. Hence the theorem holds in this case.

CASE (ii). Let $n \geq 4$. We have a dichotomy according as U and W are incident or not. In the former case, the graph is of Lie type (rank 5), and is well known to be of rank 5 and not to possess a distance-transitive permutation action (cf. [8]).
Therefore, suppose H is the stabilizer of a pair U_0, W_0 of subspaces of V of dimension 1, $n-1$, respectively, with $U_0 \not\subseteq W_0$. This situation has been analyzed by DARAFSHEH [9]. A pair U, W can be in one of 5 positions relative to the pair U_0, W_0:
$\Gamma_0 = \{\{U_0,W_0\}\}$;
$\Gamma_\alpha = \{\{U,W\} \mid U = U_0 \text{ or } W = W_0\} \setminus \Gamma_0$;
$\Gamma_\beta = \{\{U,W\} \mid U \subset W_0 \text{ and } U_0 \subset W\} \setminus \Gamma_\alpha$;
$\Gamma_\gamma = \{\{U,W\} \mid U \subset W_0 \text{ and } U_0 \not\subseteq W, \text{ or } U_0 \subset W \text{ and } U \not\subseteq W_0\}$;
$\Gamma_\delta = \{\{U,W\} \mid U \not\subseteq W_0 \text{ and } U_0 \not\subseteq W\}$.

The sets Γ_α, Γ_β, and Γ_γ are H-orbits of respective sizes, $2(q^{n-1}-1)$, $q^{n-2}(q^{n-1}-1)/(q-1)$, and $2q^{n-2}(q^{n-1}-1)$. There is a single H-orbit in Γ_δ of size $(q^{n-1}-1)(q^{n-2}-1)$ consisting of all $U,W \in \Gamma_\delta$ with $(U+U_0) \cap W_0 \subset W$. The remainder of Γ_δ is partitioned into $q-2$ H-orbits, each of size $q^{n-2}(q^{n-1}-1)$, with $(U+U_0) \cap W_0 \cap W = \emptyset$. (Since H is transitive on the set of all points U distinct from U_0 and not in W_0, we can fix U and parametrize these orbits by $(U+U_0) \cap W_0$, ranging over the points on $U+U_0$ distinct from U, U_0, and $(U+U_0) \cap W_0$.)
First, observe that if $\Gamma_1(\{U_0,W_0\}) = \Gamma_\alpha$, then $\Gamma_2(\{U_0,W_0\})$ would contain all of Γ_β and Γ_γ, a contradiction. Thus, using that $|\Gamma_\alpha|$ is the minimal nontrivial H-orbit size, we must have $\Gamma_d(\{U_0,W_0\}) = \Gamma_\alpha$. (Recall that $n \geq 4$.) However, if $q>2$, it is readily seen that $\Gamma_\alpha \subset \Gamma_2(\{U_0,W_0\})$, by checking the various possibilities for $\Gamma_1(\{U_0,W_0\})$ among the H-orbits described above. This leads to the contradiction $4 \leq d=2$.
Finally, suppose $q=2$. Then $|\Gamma_\delta| \leq |\Gamma_\beta| \leq |\Gamma_\gamma|$, and so $\Gamma_1(\{U_0,W_0\}) = \Gamma_\delta$, and, taking W_1 on $(U+U_0) \cap W$ with W_1 not on any of $U+U_0$, W_0, W, we find $\{U,W\} \sim \{U_0,W_0\} \sim \{U,W_1\}$, whence $\Gamma_\alpha \subset \Gamma_2(\{U,W\})$, a contradiction as before.

CASE (iii) gives again rise to the Grassmann graphs, and can be dealt with as in (i).

CASE (iv). Let $n \geq 3$. The vertex set $V\Gamma$ can be described as follows. Let Δ be the doubled Grassmann graph on $\mathbf{F}_q^n$. (Thus, the vertices of Δ are the subspaces of dimension m or $m+1$ and adjacency is incidence.) Then H is the stabilizer of an edge in Δ, so $V\Gamma$ is the set of edges in Δ. Distance-transitive graphs which are line graphs have been determined, cf. BIGGS [4]. A direct check shows that if Γ is the line graph of Δ, it is distance-transitive if and only if $n = 3$ (in which case we obtain the generalized hexagon of order $(q, 1)$ associated with the projective plane of order q, on which $G \cap P\Gamma L(3,q)$

acts). Let $n \geq 4$, and assume that Γ is not the line graph of Δ. Fix $\gamma = \{U_0, W_0\} \in V\Gamma$, and put $H = G_\gamma$. Standard computations yield that the H-orbit containing $\{U,W\} \in V\Gamma$ has size as indicated in the table below.

orbit invariant	orbit size	restriction
$\dim U_0 \cap W = \dim W_0 \cap U = a$ and $\dim W_0 \cap W - \dim U_0 \cap U \in \{0,2\}$	$2q^{(m+1-a)^2} \left[{m \atop a}\right]\left[{m \atop m+1-a}\right]\left[{m+1-a \atop 1}\right]$	$1 \leq a \leq m$
$\dim U_0 \cap W - \dim W_0 \cap U \in \{1,-1\}$ and $\dim U_0 \cap U = a$	$2q^{(m+1-a)(m-a)} \left[{m \atop a}\right]\left[{m \atop m-a}\right]\left[{m-a \atop 1}\right]$	$0 \leq a \leq m-1$
$\dim U_0 \cap W = \dim W_0 \cap U = a$ and $\dim W_0 \cap W - \dim U_0 \cap U = 1$	$q^{(m+2-a)(m-a)} \left[{m \atop a}\right]\left[{m \atop m-a}\right]$	$0 \leq a \leq m$

Thus, the non-trivial H-orbit of minimal length is the one containing $\{U,W\} \in V\Gamma$ with $U = U_0$, i.e., such that $\{\{U,W\},\{U_0,W_0\}\}$ is an edge in the line graph of Δ. Consequently, our assumption and Lemma 2.7(ii) give that this orbit coincides with $\Gamma_d(\gamma)$. By Lemma 2.7(ii) and (iv), $\Gamma(\gamma)$ coincides with the unique nontrivial H-orbit of second smallest size. By the above table, this implies $q > 2$ and that $\Gamma(\gamma)$ is the H-orbit containing $\{U,W\} \in V\Gamma$ with $\dim U_0 \cap W - \dim W_0 \cap U = 1$ and $\dim U_0 \cap U = m-1$. A direct check shows that representatives from the smallest orbit occur in $\Gamma_2(\gamma)$, whence $d = 2$, a contradiction.

CASE (v). Let L be the unique subgroup of order $q+1$ in $SL(n,q)$ centralizing K. Then G-set of $\phi^{-1}G$-conjugates of L may be identified with $V\Gamma$, and we may write $H = N_G(L)$. In K, the elements of L have diagonal shape; a fixed generator l has diagonal $(\varepsilon, \varepsilon, ..., \varepsilon)$, where ε is a primitive $q+1$-st root of 1 in $\mathbf{F}_{q^2}$. For $i = 1, \cdots, m$, let l_i denote the element of K having diagonal shape, with its first i diagonal entries equal to ε, and the remaining $m-i$ equal to ε^{-1}. Then, each l_i with $i \neq 0, m$ generates a subgroup L_i which is G-conjugate but not H-conjugate to L. Since $L_i = L_{m-i}$, we may assume without loss of generality, $i \leq m/2$. The H-orbit of L_i has size $q^{2i(m-i)} \left[{m \atop i}\right]_{q^2}$ if $i \neq m/2$ and $q^{2i(m-i)} \left[{m \atop i}\right]_{q^2}/2$ otherwise.

We claim that, for $i = 1, ..., m/2$, the kernel $C_i = C_H(L_i^H)$ of the action of H on the H-conjugacy class of L_i coincides with $Scal_H$, the ϕ-image of the group of all scalar matrices in $\phi^{-1}(H) \cap GL(m,q^2)$. For, C_i stabilizes each subspace of dimension i as they are eigenspaces of certain L_i^h for $h \in H$ (if $i = m/2$, a little extra care is needed to exclude a possible interchange of the two eigenspaces, this is done by considering a second H-conjugate of L_i having exactly one eigenspace in common with the first), and so must be in the center of $GL(m,q^2)$. By Proposition 2.3, $m \leq 5$, since otherwise there would be more than 2 H-orbits with the same kernel.

Let $n \geq 8$. Then $m = 4$ or 5, and L_1^H and L_2^H are 2 orbits with the same kernel. Since $Scal_H$ is the biggest normal subgroup in H that can occur as the kernel of an orbit (note that K is never in a kernel) we must have $\Gamma_1(L) = L_i^H$, and $\Gamma_d(L) = L_j^H$ for $\{i,j\} = \{1,2\}$.

If i=1, then $L \sim L_1 \sim L_2$, so d=2, a contradiction. Suppose i=2. Then there are $n_1, n_2 \in l_2^H$ such that the two eigenspaces of n_1 both have trivial intersection with those of l_2, and those of n_2 both meet those of l_2 in a 1-dimensional subspace. Thus, $\langle n_1 \rangle$, $\langle n_2 \rangle \in \Gamma_1(L) \subseteq \Gamma_{\leq 2}(L_2)$, and, as $\langle n_1 \rangle$, $\langle n_2 \rangle \not\subseteq C_H(L_2)$, we get $\langle n_1 \rangle$, $\langle n_2 \rangle \in \Gamma_2(L_2)$. On the other hand, due to the above eigenspace intersections, $\{\langle n_s \rangle, \langle l_2 \rangle\}$, for s=1,2, represent two distinct G-orbitals in Γ_2, contradicting the distance transitivity assumption on G.

CASES (vi), (vii), and (viii). These cases are described by means of involutions, so 2.5 and 2.6 apply. Let x be an involution such that $H = C_G(x)$. Then $V\Gamma$ is the $PGL(n,q)$-conjugacy class of x. We claim that, in all three cases, an involution $y \in V\Gamma$ can be found with xy of order 4. Write σ for the Frobenius map $\xi \mapsto \xi^r$ where $r = q^{1/2}$, and τ for transpose inverse.
If $K \cong Sp(2m,q)$ $(m \geq 3)$, set

$$a = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}$$

and let $x = a\tau$, and $y = bxb^{-1}$, where R maps to an element of order 4 in $PSL(m,q)$.
If $K \cong SL(n,r)$ $(n \geq 3)$, take $x = \sigma$ and $y = a\sigma a^{-1}$, where a is given by the following matrix in which $\alpha \in \mathbf{F}_q \setminus \mathbf{F}_r$

$$\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}, \text{ respectively, } \begin{pmatrix} 1-\alpha(1-(\alpha^r-\alpha)^{-1}) & (\alpha^r-\alpha)^{-1}-1 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(extended by the identity matrix of dimension $n-3$ to a matrix of dimension n) if q is even, respectively, q is odd.
If $K \cong SU(n,r)$ $(n \geq 3)$, take $x = \sigma\tau$, and $y = axa^{-1}$, where a is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha^r & 1 & 0 \\ \alpha^r & 0 & 1 \end{pmatrix}$$

(extended by the identity matrix of dimension $n-3$ to a matrix of dimension n), where $\alpha\alpha^r = -1$.

Proposition 2.5 then yields that $\Gamma_1(x)$ is contained in $C_G(x)$ and $C_G(x) \cap x^G$ is a class of p-transpositions for some p. Now, let $n \geq 5$. Then, by ASCHBACHER [1] and FISCHER [10], case (vii) does not occur, while in case (vi) q=2 and $\Gamma(x)$ is the graph of transvections, i.e., $\Gamma(x)$ is the collinearity graph of the symplectic geometry of $Sp(2m,q)$; in particular, it contains a quadrangle, so Theorem 2.2 applies, showing that $diam\Gamma < 4$, a contradiction. (The relevant numbers k and a_1 follow from knowledge of the intersection array of the distance-transitive graph $\Gamma(x)$, see, for instance, HUBAUT [13]). In case (viii) for $n \geq 5$, we get r=2 and $\Gamma(x)$ is the graph of isotropic points of the hermitean variety, and again contains a quadrangle. Theorem 2.2 now gives $diam\Gamma \leq 8$, contradicting that the number of H-orbits equals the number of G-classes in $PSL(n,r)$ (cf. GOW [11]), the final contradiction. □

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