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PSEUDOCOMPACTNESS AND THE STONE-ČECH COMPACTIFICATION FOR TOPOLOGICAL GROUPS

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1. Introduction

In [6] we introduced the concept of an S-group. A topological group G is said to be an S-group of it is a non-compact topological Hausdorff-group such that βG is a compact topological group in which G is (canonically) embedded as a dense subgroup. Such groups have been considered earlier by Glicksberg [1] and Kister [5].
In [1], Glicksberg remarked that it is easy to see that an S-group is pseudocompact, but he left open the converse question. We shall prove that a topological group G is an S-group if and only if G is pseudocompact and non-compact. This will be a direct consequence of the fact that an arbitrary product of pseudocompact topological groups is pseudocompact and the simplest case of Theorem 1 of [1]. The proof that a product of pseudocompact topological groups is pseudocompact is surprisingly simple if one characterizes pseudocompactness of a topological group in terms of its Bohr compactification. We conclude with some remarks about 0-dimensional S-groups.

2. Preliminaries.

In the sequel all topological spaces are Hausdorff spaces. If X is a topological space, C(X) will denote the Banach algebra of all complex-valued bounded continuous functions on X (pointwise defined operations and supremumnorm). A subset A of X is said to be C^*-embedded in X if every $f \in C(A)$ can be extended to an element of C(X). A topological space X is called pseudocompact if every continuous real-valued function on X is bounded. There are several characterizations of pseudocompactness. We mention:

2.1 THEOREM. Let X be a completely regular topological space.
The following conditions are equivalent:
(i) X is pseudocompact.
(ii) For any decreasing sequence $\{V_n\}_{n\in\mathbb{N}}$ of non-empty open sets in X, $\bigcap_{n\in\mathbb{N}} \overline{V}_n \neq \emptyset$.

(iii) Any sequence $\{W_n\}_{n\in \mathbb{N}}$ of non-empty, pairwise disjoint, open sets in X has a cluster point)[1].

(iv) Any non-void closed G_δ-set in βX meets X.

PROOF. (i) $\Longleftrightarrow$ (ii) : C.f. [1]; (i) $\Longleftrightarrow$ (iii) : c.f.[2], 9.13, and for (i) $\Longleftrightarrow$ (iv), c.f. [1] (compare also [2], 6 I 1).

Recall that βX (X a completely regular space) denotes the Stone-Čech compactification of X, that is, a compact topological space in which X is C^*-embedded as a dense subspace; βX is unique up to a homeomorphism which leaves X pointwise invariant. βX is characterized by the property of being a compact space in which X is densely embedded such that any continuous function from X into a compact topological space has a continuous extension to all of βX. There is a close relationship between pseudocompactness and Stone-Čech compactifications: c.f. 2.1 above. Moreover, a well known theorem of Glicksberg [1] states that $\beta(\prod_{\alpha\in A} X_\alpha) = \prod_{\alpha\in A} \beta X_\alpha$ if and only if $\prod_{\alpha\in A} X_\alpha$ is pseudocompact (here, for every $\alpha \in A$, X_α is a completely regular space). We need only the simplest case of this theorem, the proof of which is quite straightforward:

2.2 THEOREM. If X and Y are completely regular spaces and if $X \times Y$ is pseudo-compact, then the canonical injection of $X \times Y$ into $\beta X \times \beta Y$ extends to a homeomorphism of $\beta(X\times Y)$ onto $\beta X \times \beta Y$.

PROOF. C.f. [1].

If G is a topological group, we denote the Bohr compactification of G by (α, G_c). This means that G_c is a compact topological group and that $\alpha : G \to G_c$ is a continuous homomorphism of G onto a dense subgroup of G_c such that for any continuous homomorphism ψ of G into a compact topological group H there exists a continuous homomorphism $\tilde{\psi} : G_c \to H$ such that $\psi = \tilde{\psi} \circ \alpha$. This is analoguous to the characterization of a βX mentioned above; the analogon of the C^*-embedding of X into βX is : if $f : G \to \mathbb{C}$ is an almost periodic function (c.f. [3],18.2 for a definition),

)[1] That is: any such a sequence is not locally finite.

then an $\tilde{f} \in C(G_c)$ exists such that $f = \tilde{f} \circ \alpha$. A Bohr compactification (α, G_c) of G always exists (c.f. [4] for references) and is unique up to a topological isomorphism which leaves $\alpha(G)$ pointwise invariant. Hence we may speak about "the" Bohr compactification of G. In general, $\alpha : G \to G_c$ is not injective, and even if α is injective (for instance, if G is a locally compact abelian group) α need not be a homeomorphism. However, we have:

2.3 THEOREM. Let G be a topological group. The following are equivalent:

(i) α is a topological embedding of G into G_c.

(ii) G is totally bounded.

(iii) G is a subgroup of a compact topological group K.

In this case, G_c may be identified with the closure of G in any compact topological group K_0 in which G is embedded as a subgroup, and α may be identified with the inclusion mapping of G in its closure.

PROOF. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are well-known and (iii) $\Rightarrow$ (i) is easy. For proofs we refer to [6], Proposition 3.7 and Lemma 3.2.

Every topological group G is completely regular ([3], 8.4), hence topologically embedded in its Stone-Čech compactification βG.The embedding mapping will always be denoted by γ. An S-group is a non-compact topological group G such that βG is a compact topological group in which G is embedded by γ as a dense subgroup. In Section 4 of [6] we exhibited a large variety of non-S-groups. Moreover, we have (c.f. [6], Theorems 3.3 and 3.8):

2.4 THEOREM. Let G be a topological group. The following are equivalent:

(i) G is an S-group.

(ii) C(G) is exactly the space of all almost periodic functions on G.

(iii) There exists a homeomorphism $\bar{\alpha}$ of βG onto G_c such that $\alpha = \bar{\alpha} \circ \gamma$.

(iv) G is totally bounded and every $f \in C(G)$ is right uniformly continuous.

PROOF. The proofs are rather trivial, using the various characterizations of (α, G_c) and $(\gamma, \beta G)$, Theorem 2.3 and the fact that all almost periodic functions are (left and right) uniformly continuous.

3. The existence of S-groups.

3.1 PROPOSITION. If G is a non-compact group, and G × G is pseudocompact, then G is an S-group.

PROOF. Since G × G is pseudocompact we may identify the Stone-Čech compactification of G × G with $(\gamma\times\gamma, \beta G\times\beta G)$, where $(\gamma, \beta G)$ is the Stone-Čech compactification of G (c.f. 2.2). Let $p : G \times G \to G$ denote the multiplication mapping, that is, $p(x,y) = xy$ for $x, y \in G$. Because p is continuous, p has a continuous extension $\bar{p} : \beta(G\times G) \to \beta G$. By the identification, mentioned above, $\bar{p}$ may be regarded as a continuous mapping from $\beta G \times \beta G$ into βG. Since the restriction of $\bar{p}$ to G × G is p, and G × G × G is dense in $\beta G \times \beta G \times \beta G$, the fact that p is associative implies that $\bar{p}$ is associative. In the same way it can be seen, that $\bar{p}(e,x) = \bar{p}(x,e) = x$ for all $x \in \beta G$. Hence βG is a topological semigroup with identity e, containing G as a dense subgroup.
By a similar procedure, the continuous mapping $x \mapsto x^{-1}$ from G onto G has a continuous extension to βG, which we also denote by $x \mapsto x^{-1} : \beta G \to \beta G$. Because the continuous mappings $x \mapsto \bar{p}(x,x^{-1})$, $x \mapsto \bar{p}(x^{-1},x)$ and $x \mapsto e$ from βG into βG coincide on the dense subset G of βG, they coincide on all of βG. Hence

$$\bar{p}(x,x^{-1}) = \bar{p}(x^{-1},x) = e$$

for all $x \in \beta G$. Thus βG is a topological group in which G is a dense subgroup.

3.2 Let $\{K_\alpha | \alpha\in A\}$ be any set of topological groups, and let $\sum_{\alpha\in A} K_\alpha$ denote the subspace of the cartesian productspace $\prod_{\alpha\in A} K_\alpha$, consisting of all points $x = (x_\alpha)_{\alpha\in A}$ such that, for at most countably many $\alpha \in A$, $x_\alpha \neq e_\alpha$, the identity of K_α. It is easy to see that $\sum_{\alpha\in A} K_\alpha$

is a dense subgroup of $\prod_{\alpha\in A} K_\alpha$. Moreover, if, for all $\alpha \in A$, K_α is compact, then $\sum_{\alpha\in A}$ is countably compact, hence pseudocompact (any sequence in $\sum_{\alpha\in A} K_\alpha$ has a cluster point $y \in \prod_{\alpha\in A} K_\alpha$, and there cannot be more than countably many $\alpha\in A$ such that $y_\alpha \neq e_\alpha$, so that $y \in \sum_{\alpha\in A} K_\alpha$).

As far as we know the first proof of our next proposition is given by Glicksberg in [1]. Glicksberg makes use of a technique which enables him to state that any $f \in C(\sum_{\alpha\in A} K_\alpha)$ does depend on not too many coördinates. A similar technique is used by Kister in [5] to prove the same proposition. We shall give another proof which avoids these techniques (we use only 3.1, that is, we use only Theorem 1 of [1] for products of finitely many factors).

3.3 PROPOSITION. If $\{K_\alpha \mid \alpha \in A\}$ is any uncountable set of compact topological groups, then the proper subgroup $\sum_{\alpha\in A} K_\alpha$ of $\prod_{\alpha\in A} K_\alpha$ is an S-group.

PROOF. Since A is uncountable, $\sum_{\alpha\in A} K_\alpha$ is a proper subgroup of $\prod_{\alpha\in A} K_\alpha$. Because $\sum_{\alpha\in A} K_\alpha$ is dense in $\prod_{\alpha\in A} K_\alpha$, this implies that $\sum_{\alpha\in A} K_\alpha$ is not compact. By 3.1 we need only to prove, that $(\sum_{\alpha\in A} K_\alpha) \times (\sum_{\alpha\in A} K_\alpha)$ is pseudocompact. This follows immediately from the pseudocompactness of $\sum_{\alpha\in A} (K_\alpha \times K_\alpha)$ (c.f. the remark, preceding our proposition) and the fact that the mapping

$$((x_\alpha)_{\alpha\in A}, (y_\alpha)_{\alpha\in A}) \longmapsto ((x_\alpha, y_\alpha))_{\alpha\in A}$$

defines a homeomorphism from $(\prod_{\alpha\in A} K_\alpha) \times (\prod_{\alpha\in A} K_\alpha)$ onto $\prod_{\alpha\in A} (K_\alpha \times K_\alpha)$, which sends $(\sum_{\alpha\in A} K_\alpha) \times (\sum_{\alpha\in A} K_\alpha)$ onto $\sum_{\alpha\in A} (K_\alpha \times K_\alpha)$.

3.4 COROLLARY. If $\{K_\alpha \mid \alpha\in A\}$ is any uncountable set of compact topological groups, then $\prod_{\alpha\in A} K_\alpha$ is the Stone-Čech compactification of $\sum_{\alpha\in A} K_\alpha$.

PROOF. Since $\sum_{\alpha\in A} K_\alpha$ is an S-group, its Stone-Čech compactification and its Bohr compactification coïncide (c.f. (i) $\Longrightarrow$ (iii) in 2.4). Since $\sum_{\alpha\in A} K_\alpha$ is a dense subgroup of the compact group $\prod_{\alpha\in A} K_\alpha$, the desired result follows from 2.3.

3.5 REMARK. We provided a class of S-groups G such that G and $G \times G$ are pseudocompact, even countably compact. There are, however, S-groups which are not countably compact.

To get an example, we first observe that if G_0 is any S-group, and if H is a subgroup of βG_0 such that $G_0 \subseteq H \subsetneq \beta G_0$, then $\beta H = \beta G_0$, hence H is an S-group. Now let A be an uncountable set, $K_\alpha = \mathbb{T}$ [)1] for all $\alpha \in A$ and $G_0 = \sum_{\alpha\in A} K_\alpha$. Let H be the subset of $\prod_{\alpha\in A} K_\alpha$, consisting of all $x = (x_\alpha)_{\alpha\in A}$ such that $x_\alpha = \exp(2\pi i t_\alpha)$ with $t_\alpha \in \mathbb{Q}$, except for at most countably many $\alpha \in A$. By 3.4, $\prod_{\alpha\in A} K_\alpha = \beta G_0$, hence $G_0 \subseteq H \subsetneq \beta G_0$. Thus H is an S-group. But it is easy to see that H is not countably compact (this example is due to J.M. Kister [5]).

4. A characterization of S-groups.

4.1 In this section we show that the class of S-groups is exactly the class of non-compact, pseudocompact topological groups. As a motivation for the method of proof that a non-compact, pseudocompact group is an S-group we wish to make the following remarks.

The result follows easily from 3.1 as soon as we have established that an arbitrary product of pseudocompact topological groups is pseudocompact. One of the characterizations of pseudocompactness for a completely regular space X is 2.1 (iv) : every non-void closed G_δ-set in βX meets X. If one tries to prove that a product $\prod_{\alpha\in A} X_\alpha$ has this pro-

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)1 $\mathbb{T}$ always denotes the circle group $\{\lambda \mid \lambda \in \mathbb{C} \;\&\; |\lambda| = 1\}$.

perty if each X_α has it, the proof breaks down on the fact that $\beta(\prod_{\alpha\in A} X_\alpha)$ is not equal to $\prod_{\alpha\in A} \beta X_\alpha$ (unless we know already that $\prod_{\alpha\in A} X_\alpha$ has the desired property).

Now it is natural, indeed, to consider Stone-Čech compactifications in connection with pseudocompactness. However, a pseudocompact topological group G turns out to be totally bounded (c.f. the proof of 4.2 below), and in view of 2.3 it may be usefull to characterize pseudocompactness of G in terms of the Bohr compactification G_c of G. This has the advantage that the Bohr compactification of a product is the product of the Bohr compactifications (for totally bounded groups this is trivial in view of 2.3, but it is true for arbitrary topological groups; c.f. [4]). Such a characterization can be achieved:

4.2 PROPOSITION. Let G be a topological group. The following are equivalent:

(i) G is pseudocompact.

(ii) G is a dense subgroup of a compact group $\hat{G}$ and every non-void G_δ-set in $\hat{G}$ meets G.

PROOF. (i) $\Longrightarrow$ (ii). If G is pseudocompact, then G is totally bounded. Suppose not. Then there is a neighbourhood U of e and a sequence $x_1, x_2, \ldots$ in G such that the sets x_nU are mutually disjoint. Let V be a symmetrical neighbourhood of e such that $V^2 \subseteq U$. Then the sequence $\{x_nV\}_{n\in\mathbb{N}}$ is locally finite, contradicting the pseudocompactness of G. By 2.3, G is a dense subgroup of a compact topological group $\hat{G}$. Suppose a non-void G_δ-subset of $\hat{G}$ is contained in $\hat{G}\setminus G$. Then it is easy to construct a continuous function $f : \hat{G} \to [0,1]$ such that

$$\{ x \mid x \in \hat{G} \;\&\; f(x) = 0 \} \subseteq F \subseteq G_c \setminus G.$$

The function $x \mapsto 1/f(x)$ is continuous and not bounded on G, contradicting the pseudocompactness of G.

(ii) $\Rightarrow$ (i). Let $\{W_n\}_{n\in\mathbb{N}}$ be a descending sequence of open sets in G. We must prove that $\bigcap_{n=1}^{\infty} \tilde{W}_n \neq \emptyset$, where $\tilde{W}_n$ is the closure of W_n in G (cf (i) $\Longleftrightarrow$ (ii) of 2.1).

Now G is a dense subgroup of a compact group $\hat{G}$, and for all $n \in \mathbb{N}$

there is an open set U_n in $\hat{G}$ such that $W_n = G \cap U_n$. We may assume that $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. If we denote the closure of a set A in $\hat{G}$ with $\overline{A}$, then

$$\tilde{W}_n = G \cap \overline{W}_n = G \cap \overline{U}_n$$

for all $n \in \mathbb{N}$. Since $\bigcap_{n=1}^{\infty} \tilde{W}_n = G \cap \bigcap_{n=1}^{\infty} \overline{U}_n$, it will clearly be sufficient to prove that $\bigcap_{n=1}^{\infty} \overline{U}_n$ contains a non-void G_δ-set.

Take, for all $n \in \mathbb{N}$, $x_n \in U_n$, and let V_n be an open neighbourhood of the identy in $\hat{G}$ such that $x_n V_n \subseteq U_n$. Then $N := \bigcap_{n=1}^{\infty} V_n$ is a non-void G_δ in $\hat{G}$ (it is even a normal subgroup of $\hat{G}$, and if one takes $\overline{V}_{n+1} \subseteq V_n$, N is closed in $\hat{G}$). Since $\hat{G}$ is compact, the sequence $\{x_n\}_{n=1}^{\infty}$ has a clusterpoint x in $\hat{G}$.

Let $y \in N$. Then xy is clusterpoint of the sequence $\{x_n y\}_{n=1}^{\infty}$ in $\hat{G}$. For all $n \in \mathbb{N}$ we have $y \in V_n$, hence $x_n y \in x_n V_n \subseteq U_n$. Thus any neighbourhood O of xy in $\hat{G}$ meets U_n for infinitely many $n \in \mathbb{N}$. Because $U_{k+1} \subseteq U_k$ for all k, this implies that $O \cap U_n \neq \emptyset$ for all $n \in \mathbb{N}$, so that $xy \in \overline{U}_n$ for all $n \in \mathbb{N}$. We have proved that the non-void G_δ-set xN is contained in $\bigcap_{n=1}^{\infty} \overline{U}_n$.

4.3 THEOREM. Let A be a non-void set and, for all $\alpha \in A$, let G_α be a topological group. Then $\prod_{\alpha \in A} G_\alpha$ is pseudocompact if and only if, for all $\alpha \in A$, G_α is pseudocompact.

PROOF. Since every G_{α_0} is a continuous image of $\prod_{\alpha \in A} G_\alpha$, pseudocompactness of the product implies pseudocompactness of the factors.
Conversely, suppose each G_α is pseudocompact. Then for all $\alpha \in A$, G_α is a dense subgroup of a compact group $\hat{G}_\alpha$, and every non-void G_δ-set in $\hat{G}_\alpha$ meets G_α. Hence $\prod_{\alpha \in A} G_\alpha$ is a dense subgroup of the compact group $\prod_{\alpha \in A} \hat{G}_\alpha$, and the proof is finished if we show, that given any non-void G_δ-set F in $\prod_{\alpha \in A} \hat{G}_\alpha$, $F \cap \prod_{\alpha \in A} G_\alpha \neq \emptyset$: then we know by 4.2 that $\prod_{\alpha \in A} G_\alpha$ is pseudocompact.

Fix any $x \in F$. It is easy to see that there are open sets U_n in $\prod_{\alpha\in A} \hat{G}_\alpha$ such that

$$x \in U_n = \prod_{\alpha\in A} U_{n,\alpha}$$

for all $n \in \mathbb{N}$, with $U_{n,\alpha}$ open in $\hat{G}_\alpha$ for all $\alpha \in A$ and, in addition $U: = \bigcap_{n\in\mathbb{N}} U_n \subseteq F$ (we do not need the fact that the $U_{n,\alpha}$ can be chosen such that $U_{n,\alpha} = \hat{G}_\alpha$ for all $\alpha \in A \setminus A_n$, where A_n is some finite subset of A).

For each $\alpha \in A$, $F_\alpha: = \bigcap_{n\in\mathbb{N}} U_{n,\alpha}$ is a G_δ-set in $\hat{G}_\alpha$, and $x_\alpha \in F_\alpha$, so that $F_\alpha \neq \emptyset$. Then we know that $F_\alpha \cap G_\alpha \neq \emptyset$, say $p_\alpha \in F_\alpha \cap G_\alpha$. Let $p: = (p_\alpha)_{\alpha\in A}$; then

$$p \in U \cap \prod_{\alpha\in A} G_\alpha \subseteq F \cap \prod_{\alpha\in A} G_\alpha .$$

Thus $F \cap \prod_{\alpha\in A} G_\alpha \neq \emptyset$, and $\prod_{\alpha\in A} G_\alpha$ is pseudocompact.

4.4 THEOREM. A topological group G is pseudocompact if and only if either G is an S-group or G is compact.

PROOF. Let G be pseudocompact and non-compact. By 4.3, $G \times G$ is pseudocompact, hence G is an S-group by 3.1.

Conversely, suppose G is an S-group. From (i) $\Longrightarrow$ (iv) of 2.4 we know that G is totally bounded and that every $f \in C(G)$ is uniformly continuous. Suppose G is not pseudocompact. Then there is a sequence of pairwise disjoint open sets in G which is locally finite. Hence there is a sequence $\{x_n\}_{n=1}^\infty$ in G and a sequence $\{W_n\}_{n=1}^\infty$ of neighbourhoods of the identity e of G such that the sequence $\{x_n W_n\}_{n=1}^\infty$ is locally finite, and the sets $x_n W_n$ are pairwise disjoint. In addition, we may suppose that $W_{n+1} \subseteq W_n$ for all $n \in \mathbb{N}$.

Because G is completely regular, for every $n \in \mathbb{N}$ there is a continuous function $f_n : G \to [0,1]$ such that

$$f_n(x) = \begin{cases} 1 & \text{for } x = x_n \\ 0 & \text{for } x \notin x_n W_{n+1} \end{cases}$$

Note that $x_n W_{n+1} \subseteq x_n W_n$ ($n \in \mathbb{N}$). Since the sequence $\{x_n W_n\}_{n=1}^{\infty}$ is locally finite, the well defined, bounded function $f := \sum_{n=1}^{\infty} f_n$ is continuous, hence uniformly continuous. This implies that there is a neighbourhood U of e in G such that

$$|f(x) - f(y)| < \tfrac{1}{2}$$

for all $(x,y) \in G \times G$, $x^{-1}y \in U$. In particular, for all $n \in \mathbb{N}$, we have

$$\forall y \in G : y \in x_n U \implies f(y) > \tfrac{1}{2}.$$

However, for all $y \in x_n W_n$, we have $f(y) = f_n(y)$, thus

$$x_n U \cap x_n W_n \subseteq x_n W_{n+1} \qquad (n \in \mathbb{N}).$$

Hence, for all $n \in \mathbb{N}$, $U \cap W_n \subseteq U \cap W_{n+1}$, and this implies

$$U \cap W_1 \subseteq U \cap \bigcap_{n=1}^{\infty} W_n \subseteq \bigcap_{n=1}^{\infty} W_n.$$

In particular, for every $n \in \mathbb{N}$, $x_n(U \cap W_1) \subseteq x_n W_n$. Consequently, the non-void open sets $x_n(U \cap W_1)$ are mutually disjoint, contradicting the total boundedness of G. Thus, if G is an S-group, G is pseudocompact. Finally, it is trivial that compactness of G implies pseudocompactness.

4.5 REMARK. The behaviour of the class of S-groups under the forming of products, subgroups and quotients is the same as the behaviour of pseudocompact spaces under these operations. But in some cases we can say more. We state explicitly:

4.6 PROPOSITION. Let G be an S-group. A non-compact subgroup H of G is an S-group if and only if H is C^*-embedded in G.

PROOF. Note first, that H is a dense subgroup of the compact topological group $\overline{H}$, where $\overline{H}$ denotes the closure of H in the compact topological group βG. If H is an S-group, it follows from 2.3 and 2.4 that any compact topological group in which H is a dense subgroup, may be identified with βH. In particular, $\beta H = \overline{H}$. Hence, by [2], 6.9, H is C^*-embedded in G.

The converse statement may be derived also from [2], 6.9: if H is C^*-embedded in G, $\beta H = \overline{H}$, hence βH is a topological group.

4.7 REMARKS. 1^o. In general, a pseudocompact subspace of a pseudocompact space need not be C^*-embedded, and a C^*-embedded subspace of a pseudocompact space need not be pseudocompact.

2^o. In general, a non-compact subgroup of an S-group is not an S-group (hence not C^*-embedded). Let A be an uncountable set, $K_\alpha = \mathbb{T}$ for all $\alpha \in A$, H_0 a dense proper subgroup of $\mathbb{T}$ and $G = \sum_{\alpha\in A} K_\alpha$. Then G is an S-group, and G contains a copy H of H_0 as a subgroup. Since H_0 is not an S-group (c.f.[6], 3.10), H is not an S-group.

3^o. An open subgroup of a topological group G is C^*-embedded in G. This follows immediately from the fact that G is the disjoint union of the distinct left cosets of H in G, each of which is open in G and homeomorphic with H.

4.8 PROPOSITION. Let $\{G_\alpha \mid \alpha \in A\}$ be a set of non-compact topological groups. Then $\prod_{\alpha\in A} G_\alpha$ is an S-group if and only if, for all $\alpha \in A$, G_α is an S-group.

PROOF. Follows immediately from 4.3 and 4.4

4.9 REMARK. If in 4.8 the non-compactness of the G_α's is not given, then the fact that $\prod_{\alpha\in A} G_\alpha$ is an S-group implies that every G_α is pseudocompact, hence for all $\alpha \in A$, G_α is either compact or G_α is an S-group. Since $\prod_{\alpha\in A} G_\alpha$ is not compact, there is at least one α for which G_α is not compact. In any case, the following is true: a non-compact topological group G is an S-group if and only if $G \times K_0$ is an S-group for some compact topological group K. In that case $G \times K$ is an S-group for every compact topological group K.

4.10. PROPOSITION. Let G be an S-group, H a non-compact topological group and ϕ a continuous homomorphism of G onto H. Then H is an S-group.

PROOF. Immediate from 4.4 and the fact that continuous images of pseudocompact spaces are pseudocompact. Nevertheless, we will give an amusing alternative proof, that does not make use of the pseudocompactness criterion for S-groups, but of the equivalence of (ii) and (i) of 2.4.
If K is any topological group, let A(K) denote the space of almost periodic functions on K. In addition, if $a \in K$ and $g \in C(K)$, let g_a denote the function $x \mapsto g(xa) : K \to \mathbb{C}$. Let $\tilde{\phi} : C(H) \to C(G)$ denote the mapping induced by ϕ :

$$\tilde{\phi}(f) := f \circ \phi \qquad (f \in C(H)).$$

Then, for all $a \in G$ and $f \in C(H)$, we have $[\tilde{\phi}(f)]_a = \tilde{\phi}(f_{\phi(a)})$. Since ϕ is a surjection, this implies that for every $f \in C(H)$ the equality

$$\{f_b \mid b \in H\} = \tilde{\phi}^{-1}\{[\tilde{\phi}(f)]_a \mid a \in G\}$$

holds. Now $\tilde{\phi}$ is an isometrical isomorphism from C(H) onto a closed subalgebra of C(G). Hence $\{f_b \mid b \in H\}$ is totally bounded in C(H) if and only if $\{\tilde{\phi}(f)_a \mid a \in G\}$ is totally bounded in C(G). Thus, if $f \in C(H)$, then $f \in A(H)$ if and only if $\tilde{\phi}(f) \in A(G)$.
In general only the "only if" part of this statement is usefull, but in this case A(G) = C(G), that is, for all $f \in C(H)$ we have $\tilde{\phi}(f) \in A(G)$, hence $f \in A(H)$. This means, that A(H) = C(H).

In [3], 4.21(d), (e) a class of 0-dimensional groups is described which are all totally bounded. We shall show now, that all 0-dimensional S-groups belong to this class.

4.11 PROPOSITION. If G is a 0-dimensional S-group, then there is an open basis at the identity e consisting of open normal subgroups of G with finite index.

PROOF. If G is 0-dimensional, then βG is 0-dimensional (cf [2], 6L). Now let U be an arbitrary neighbourhood of e in G, say $U = U_0 \cap G$, with U_0 a neighbourhood of e in βG. Then, by [3], 7.7 , U_0 contains an open normal subgroup N_0 of βG, and U contains the open normal subgroup $N_0 \cap G$ of G. Since G is totally bounded by 2.3, this open subgroup of G must have a finite index in G.

4.12 EXAMPLE. Let A be an uncountable set and, for all $\alpha \in A$, let K_α be the multiplicative discrete group $\{-1,1\}$. By 3.3, the group $G: = \sum_{\alpha\in A} K_\alpha$ is an abelian S-group. An open basis at the identity is formed by the collection of all sets

$$H_B : = \{x \mid x \in G \,\&\, x_\alpha = 1 \text{ for all } \alpha \in B\}$$

with B a finite subset of A. Then it is easy to see that for every finite subset B of A, H_B is an open normal subgroup of G with finite index (the index of H_B in G equals the number of elements in B).

4.13 PROPOSITION. If G is a 0-dimensional S-group and H a closed normal subgroup of G, then G/H is a 0-dimensional topological group.

PROOF. (Compare [3], 7.11). Let B denote an open basis at the identity e of G, and let $q : G \to G/H$ denote the quotient mapping of G onto G/H. Then $\{q(U) \mid U \in B\}$ is an open basis at q(e) in G/H. By 4.11 we are allowed to suppose that every $U \in B$ is an open normal subgroup of G. Hence, for each $U \in B$, q(U) is an open normal subgroup of G/H. Since open subgroups of a topological group (i.c. G/H) are closed, we have proved that there is an open basis at q(e) consisting of open-and-closed subsets.

4.14 PROPOSITION. Let G be an S-group. The following are equivalent:

(i) G is connected.

(ii) G has no proper open subgroups.

(iii) For every neighbourhood U of the identity e in G we have $\bigcup_{n\in\mathbb{N}} U^n = G$.

PROOF (Compare [3], 7.9). The implications (i) $\Rightarrow$ (ii) $\Longleftrightarrow$ (iii) are generally valid. We prove (ii) $\Rightarrow$ (i).

Suppose G is not connected. Then βG is not connected ([2], 6L 1). Since βG is a compact topological group, the connected component of the identity in βG is the intersection of all open subgroups of βG (c.f [3], 7.8). Hence there is an open subgroup H in βG such that $H \neq \beta G$. Then $H \cap G$ is an open subgroup of G, and $H \cap G \neq G$: if $H \cap G = G$, then $G \subseteq H$, hence $H = \beta G$ because H is closed in βG and G is dense in βG, contradicting the fact that $H \neq \beta G$. Thus (ii) $\Rightarrow$ (i).

4.15 EXAMPLE. There do exist connected S-groups. Let A be an uncountable set, and let, for all $\alpha \in A$, K_α be a connected compact topological group.

Then $G := \sum_{\alpha\in A} K_\alpha$ is connected (the proof that a cartesian product of connected spaces is connected works also for $\sum_{\alpha\in A} K_\alpha$). Hence G is a connected S-group. If, in addition, every K_α is locally connected, then G is connected and locally connected. Finally, if K is a compact, locally connected, but non-connected group, $G \times K$ is a locally connected, non-connected S-group (c.f. 4.9).

4.16 REMARK. Let G be an S-group and let C be the connected component of the identity in G. We prove: if G/C is 0-dimensional, then

$$(*) \qquad C = \bigcap\{H \mid H \text{ is an open subgroup of } G\}.$$

Since trivially C is contained in every open subgroup H of G, we prove that $C \supseteq \bigcap\{H \mid H \text{ is an open subgroup of } G\}$. Take $x \in G\backslash C$. Then the canonical image $\tilde{x}$ of x in G/C is different from $\tilde{e}$, hence by 4.11 (if G/C is an S-group) or [3], 7.7 (if G/C is compact) there is an open subgroup $\tilde{H}$ in G/C such that $\tilde{x} \notin \tilde{H}$. Consequently, there is an open subgroup H in G with $C \subseteq H$ and $x \notin H$.

Notice, that there exist S-groups G such that G/C is non-trivial and 0-dimensional and G is not: if G is a locally connected, non-connected S-group, then C is an open subgroup in G and G/C is discrete and non-trivial (but finite, since C has finite index in G). However, in this example, (*) is trivially valid.

<u>Problem</u>: If G is an S-group and C is the connected component of the identity of G, has G/C to be 0-dimensional? Equivalently: is every totally disconnected S-group 0-dimensional? If not, is (*) equivalent with the fact that G/C is 0-dimensional?

REFERENCES.

[1]. Glicksberg, I., Stone-Čech compactifications of products, *Trans. Amer. Math. Soc.* *90* (1959), 369-382.

[2]. Gillman, L., and Jerison, M., *Rings of Continuous Functions*, Princeton, 1960.

[3]. Hewitt, E. and Ross, K.A., *Abstract Harmonic Analysis*, Vol.I, II, Berlin, Heidelberg, New York, 1963.

[4]. Holm, P., On the Bohr Compactification, *Math. Annalen* *156* (1964), 34-46.

[5]. Kister, J.M., Uniform continuity and compactness in topological groups, *Proc. Amer. Math. Soc.* *13* (1962), 37-40.

[6]. de Vries, J., Topological groups which do not have the PETER - WEYL property. To appear.