stichting
mathematisch
centrum

AFDELING MATHEMATISCHE STATISTIEK SW 82/82 APRIL
(DEPARTMENT OF MATHEMATICAL STATISTICS)

H.C.P. BERBEE

A BOUND ON THE SIZE OF POINT CLUSTERS OF A RANDOM WALK WITH STATIONARY INCREMENTS

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: Primary: 60G10
Secondary: 60C05, 60K05

A bound on the size of point clusters of a random walk with stationary increments

by

H. Berbee

SUMMARY

Consider a random walk on $\mathbb{R}^d$ with stationary, possibly dependent increments. Let $N(V)$ count the number of visits to a bounded set V. We give bounds on the size of $N(t+V)$, uniformly in t, in terms of the behavior of N in a neighborhood of the origin.

KEY WORDS & PHRASES: *Stationary increments, point cluster, point process*

1. INTRODUCTION

Let $(\xi_n)_{n\in\mathbb{Z}}$ be a stationary sequence of random vectors in the d-dimensional Euclidean space $(\mathbb{R}^d, \mathcal{B}^d)$. The process $(S_n)_{n\in\mathbb{Z}}$, determined by

$$S_0 := 0,\ S_n = \xi_n + S_{n-1},\ n \in \mathbb{Z},$$

is called a *random walk with stationary increments*. This definition of S_n for all $n \in \mathbb{Z}$ is uncommon but will be useful in the present context. Define the point process N by

$$N(B) := \sum_{n\in\mathbb{Z}} 1_B(S_n), \qquad B \in \mathcal{B}^k.$$

We assume that the random walk is *transient*, i.e. N is finite on bounded Borel sets B.

For random walks on $\mathbb{R}^1$ with stationary, non-negative increments KAPLAN (1955) proved that $EN(t,t+h) \le EN(-h,h)$ for real t and $h > 0$. In case the increments are independent, this inequality is a simple consequence of the Markov property (see FELLER (1970,VI.10)) and in fact $N(t,t+h)$ is stochastically dominated by $N(-h,h)$. Below we shall see that this domination does not hold without independence.

Let us now consider random walks on $\mathbb{R}^d$. Assume V is a bounded Borel set with translate $V+t := \{s+t : s \in V\}$, and suppose $V_0 := \{s-t : s,t \in V\}$ is also a Borel set. We prove that if $f \ge 0$ is a function, growing not too slowly such that

$$(1) \qquad n(f(n+1)-f(n)) \ge 0 \text{ is non-decreasing}$$

then

$$(2) \qquad Ef(N(V)) \le Ef(N(V_0)).$$

The condition (1) is satisfied for e.g. $f(n) = n^\alpha$, $\alpha > 0$, or $f(n) = \log n$. If (2) were true for any non-decreasing f then N(V) would be stochastically dominated by $N(V_0)$. However we prove

(3) $$P(N(V)\geq p) \leq \gamma P(N(V_0)\geq p) \qquad \text{where } \gamma = 2 - \frac{1}{p}$$

for $p = 1,2,\ldots$. An example will show that γ cannot be smaller without restricting V. The two results above will follow from the more general theorem 1 below. Inequality (3) could also be proved directly using the method of BERBEE (1979), theorem 2.2.3.

Suppose $0 = f(0) \leq f(1) \leq \ldots$ is given. Let $c(n) := \frac{1}{n}\sum_{k=1}^{n} f(k)$ be a Cesaro average and let

$$h(n) := c(n) + \sup_{k\leq n}(f(k)-c(k)).$$

We shall see that (1) implies that f-c is non-decreasing and then $f \equiv h$. In section 2 we show

<u>THEOREM 1</u>. $Ef(N(V)) \leq Eh(N(V_0))$.

This result and also (2), (3) and (5) can be improved slightly if -V is a translate of V. In that case we may replace $N(V_0)$ by

(4) $$\sup_{V'\ni 0} N(V')$$

where V' runs over the translates of V.

In section 3 we pay special attention to random walks on the real line. We prove for an interval $V = (t,t+h)$

(5) $$P(N(V)\geq p) \leq \gamma P(N(V_0)\geq p) \qquad \text{where } \gamma = \frac{3}{2} - \frac{1}{2p}$$

for $p = 1,2,\ldots$. An example shows that γ cannot be smaller.

Replacing V by V+t in the inequalities does not change V_0. As a consequence an important application of our results concerns uniform integrability. Suppose that $EN(U) < \infty$ on a neighborhood U of the origin. Using that the bounded set V is contained in a finite union of translates of U, it is proved easily from our inequalities that N(V+t) is integrable, uniformly in t. This result is used in BERBEE (1979) to obtain Blackwell's theorem for stationary processes. A related integrability problem is solved in DALEY (1971) in connection with the global renewal theorem. A condition for finiteness of EN(U) can be found in LAI (1977) in terms of strong mixing. In the limit theory of semi-Markov chains very complicated integrability conditions are used (see KESTEN (1974)).

2. INEQUALITIES FOR GENERAL V

The proof of theorem 1 is based on a combinatorial lemma. Let $A := (s_0,\ldots,s_n)$ be a finite sequence of points in $\mathbb{R}^k$. Define the *distant cluster* of $s \in A$ as the subsequence $A(s) \equiv A \cap (V+s)$ of points of A in V+s (with the same multiplicities) and the *close cluster* as $A_0(s) \equiv A\cap(V_0+s)$. Let n(s) and $n_0(s)$ denote the number of points in the distant and close cluster of s.

With f and h as in theorem 1 we have the following comparison lemma for the sizes of distant and close clusters.

<u>LEMMA 2</u>. $\sum_s f(n(s)) \le \sum_s h(n_0(s))$.

Here as in the proof below the sums are over the points in A with the right multiplicities.

<u>PROOF</u>. Obviously for $s \in A$

$$f(n(s)) \le c(n_0(s)) + (f(n(s))-c(n_0(s)))^+.$$

Define

$$h_1(s,t) := \frac{1}{n(s)}\, c(n_0(s)), \qquad t \in A(s),$$

$$h_2(s,t) := \frac{1}{n(s)}\, (f(n(s))-c(n_0(s)))^+, \qquad t \in A(s),$$

$$h_1(s,t) = h_2(s,t) := 0, \qquad \text{otherwise.}$$

Because $n(s) = \# A(s)$ we have, rewriting sums,

$$\sum_s f(n(s)) \le \sum_s(\sum_t h_1(s,t) + \sum_r h_2(r,s))$$

and it suffices to prove that the term in brackets is at most $h(n_0(s))$. This term equals

$$(6)\qquad c(n_0(s)) + \sum_{r:s\in A(r)} \frac{1}{n(r)} (f(n(r))-c(n_0(r)))^+.$$

If $s \in A(r)$ then $V + r \subset V_0 + s$ so $n(r) \leq n_0(s)$. Hence (6) is at most

$$c(n_0(s)) + \sum_{r:s\in A(r)} \sup_{n\leq n_0(s)} \frac{1}{n} (f(n)-c(n_0(r)))^+.$$

The sum above is taken over $k := \# A \cap (-V+s)$ terms. If $s \in A(r)$ then $-V + s \subset V_0 + r$, so $k \leq n_0(r)$. Because c is non-decreasing $(f(n)-c(j))^+$ is non-decreasing in j. Hence (6) is at most

$$(7)\qquad c(n_0(s)) + k \sup_{n\leq n_0(s)} \frac{1}{n} (f(n)-c(k))^+.$$

By the definition of c as Cesaro average, the difference

$$\frac{k}{n} (f(n) - c(k)) - \frac{k-1}{n} (f(n)-c(k-1)) = \frac{1}{n} (f(n)-f(k))$$

is non-negative for $k \leq n$ and non-positive for $k \geq n$. So the expression (7) is maximal for $k = n$. Therefore (7) and so also (6) is at most $h(n_0(s))$. □

<u>REMARK 3</u>. If $-V$ is a translate of V we can do better than in lemma 2 by taking

$$(8)\qquad n_0(s) := \sup_{V'\ni s} \# A \cap V'$$

where V' runs over all translates $V + t$ of V. Then lemma 2 holds again (note that in the proof also now $n(r) \leq n_0(s)$ and $k \leq n_0(r)$ if $s \in A(r)$). The assertion concerning (4) is obtained by following the arguments below with the obvious changes.

Theorem 1 follows from lemma 2 using the ergodic theorem as follows.

<u>PROOF of theorem 1</u>. Take $A := (S_0,\ldots,S_n)$ and define

$$\bar{N}(B) := \sum_{k=0}^{n} 1_B(S_k).$$

By lemma 2

$$\text{(9)} \qquad \sum_{k=0}^{n} f(\bar{N}(S_k+V)) \le \sum_{k=0}^{n} h(\bar{N}(S_k+V_0)).$$

Choose some large constant m and define for $-\infty < k < \infty$ a stationary sequence

$$\begin{aligned} N_k^{(m)} &:= N(S_k+V) && \text{if for all } |j| \ge m \text{ holds } S_{j+k} \notin S_k+V, \\ &:= 0 && \text{else.} \end{aligned}$$

With these definitions

$$\begin{aligned} N_k^{(m)} &\le \bar{N}(S_k+V) && \text{for } m \le k \le n-m, \\ &\le 2m-1 && \text{for all } k, \end{aligned}$$

and hence

$$\sum_{k=0}^{n} f(N_k^{(m)}) - 2m\ f(2m-1) \le \sum_{k=0}^{n} f(\bar{N}(S_k+V)).$$

By (9) the right hand side is dominated by

$$\sum_{k=0}^{n} h(\bar{N}(S_k+V_0)) \le \sum_{k=0}^{n} h(N(S_k+V_0)).$$

In the last inequality we used that h is non-decreasing and $\bar{N} \le N$. Hence

$$\sum_{k=0}^{n} f(N_k^{(m)}) - 2m\ f(2m-1) \le \sum_{k=0}^{n} h(N(S_k+V_0)).$$

Divide by n+1, let $n \to \infty$ and apply the ergodic theorem. After taking expectations we obtain

$$Ef(N_0^{(m)}) \le Eh(N(V_0)).$$

Let $m \to \infty$. By the monotone convergence theorem this implies the assertion. □

To get (2) from (1) we apply theorem 1 and the following remark.

REMARK 4. Obviously $h \equiv f$ if and only if $f(n)-c(n)$ is non-decreasing. This property holds under (1). To see this observe that f can be expressed as $f \equiv \sum_1^\infty a_p f_p$ where $a_1 := f(1)$ and

$$(n-1)(f(n)-f(n-1)) = a_2 + \ldots + a_n, \qquad n \geq 2,$$

specifies the other a_p. They are non-negative by (1). Here f_p is defined by

$$\begin{aligned} f_p(n) &:= \sum_p^n \frac{1}{k-1} && n \geq p > 1 \\ &:= 1 && n \geq p = 1 \\ &:= 0 && \text{else.} \end{aligned}$$

That f-c is non-decreasing is checked easily for $f \equiv f_p$. This follows then also for $f \equiv \sum_1^\infty a_p f_p$.

Inequality (3) follows from theorem 1 by using $f \equiv 1_{[p,\infty)}$ and observing that for $n \geq p$

$$(10) \qquad h(n) = 1 - \frac{p-1}{n} + \frac{p-1}{p} \leq \gamma = 2 - \frac{1}{p}.$$

The constant in (3) cannot be smaller because of the following example for $d = 1$.

EXAMPLE 5. Fix some $m \geq 1$. We construct a sequence $\bar{A}$ of reals $x_1 < y_1 < \ldots < x_m < y_m < z$ and a set V such that $y_i \in x_i + V$, $z \in y_i + V$ and $(x_i + V_0) \cap \bar{A} = \{x_i\}$.

Suppose this is done. Let $A = (s_0, \ldots, s_n)$ consist of (p-1)-tuplets at $x_1, \ldots, x_m$ and p-tuplets at $y_1, \ldots, y_m, z$. Then, counting with the right multiplicities

$$\# \{s \in A: n(s) \geq p\} = m(p-1) + mp$$

$$\# \{s \in A: n_0(s) \geq p\} = mp + p.$$

If m is large the ratio γ_m of these numbers is close to $2 - \frac{1}{p}$.

To construct the probabilistic example, let $\omega := (\omega_k)_{k\in\mathbb{Z}}$ have period n+1 such that $\omega_i = s_i - s_{i-1}$, $1 \le i \le n$, and ω_0 is some very large number. Let each element of $\Omega := \{T^i\omega,\ 0 \le i \le n\}$ have equal probability. The identity ξ on Ω is stationary and the ratio of the probabilities in (3) is γ_m as above.

To construct $\bar{A}$ let $2 < p_1 < p_2 < \dots$ be primes. Take $z := 0$ and

$$y_i := -p_1 \cdot \ldots \cdot p_{m+i}$$

$$x_i := y_i - p_1 \cdot \ldots \cdot p_i,\quad 1 \le i \le m,$$

and let $V := \{p_1 \cdot \ldots \cdot p_i : 1 \le i \le 2m\}$. The only property of $\bar{A}$ that is not obvious is $(x_i + V_0) \cap \bar{A} = \{x_i\}$. Let us call products of more than m primes long and the other products short. Each $v \in V_0$ is uniquely represented as difference of two elements in V. Let v_ℓ be obtained by replacing in this difference the short products by 0. Also $(x_i)_\ell := y_i$.

Suppose $x_j \in x_i + V_0$. It is easily proved that for the long products in $x_j - x_i = v \in V_0$ we have $y_j - y_i = v_\ell$ and then we should have $v = v_\ell$. So $y_j - y_i = x_j - x_i$ and $i = j$. Similar considerations disprove y_j or $0 \in x_i + V_0$. Hence $(x_i + V_0) \cap \bar{A} = \{x_i\}$.

3. INEQUALITIES FOR INTERVALS

Let $d = 1$ and assume $V = (t, t+h)$. Let $A := (s_0, \dots, s_n)$ and take $n(s) := \# A \cap (V + s)$ as before but define $n_0(s)$ by (8). Because $-V$ is a translate of V lemma 2 holds. We get (5) from lemma 6 as in the proof of theorem 1. Counting $s \in A$ with its multiplicity, we have

<u>LEMMA 6</u>. $\# \{s \in A : n(s) \ge p\} \le (\frac{3}{2} - \frac{1}{2p})\ \# \{s \in A : n_0(s) \ge p\}$.

<u>PROOF</u>. Let $f \equiv 1_{[p,\infty)}$. Then $h(n) \le \frac{3}{2} - \frac{1}{2p}$ for $n \le 2p$ by (10). Hence if $n_0(s) \le 2p$ for all $s \in A$ then the assertion follows from lemma 2.

Let $\gamma(A) := \#/\#_0$ be the ratio of the numbers at the left and right in the assertion. If $\gamma(A) \le 1$ nothing has to be proved. Otherwise there may exist an interval $I = (x, x+h)$ with more than 2p points of A. We will remove one of these points to get A' and will show $\gamma(A) \le \gamma(A')$. Continuing this procedure we would come in finitely many steps to A" with no such intervals I. For such A" we already obtained the assertion and so $\gamma(A) \le \gamma(A") \le \frac{3}{2} - \frac{1}{2p}$ would complete the proof.

So consider A and I as above and remove $\bar{s} \in A \cap I$ from A such that both in $(x,\bar{s}]$ and $[\bar{s},x+h)$ at least p points of A are left. One checks easily that then $\# A' \cap V' \ge p$ if $\# A \cap V' \ge p$ for any translate V' of V. Hence in $\gamma(A) := \#/\#_0$ the removal of $\bar{s}$ causes the denominator (numerator) to decrease by (at most) 1. Because $\gamma(A) \ge 1$ we may conclude $\gamma(A') \ge \gamma(A)$. □

<u>EXAMPLE 7</u>. The constant γ in (5) cannot be smaller than $\frac{3}{2} - \frac{1}{2p}$. To see this let $0 < \varepsilon_0 < \ldots < \varepsilon_m < 1$. Let A contain p-tuplets at 5k and $5k + \varepsilon_k$ and (p-1)-tuplets at $5k + \varepsilon_k + 1$, $0 \le k \le m$. With V := (5,6) the ratio γ_m of

$$\# \{s \in A: n(s) \ge p\} = (3p-1)m$$

$$\# \{s \in A: n_0(s) \ge p\} = 2p(m+1)$$

is close to $\frac{3}{2} - \frac{1}{2p}$ for large m. Here we may take $n_0(s) := \# A \cap (V_0+s)$. Just as in example 5 we can construct a probability space where the ratio of the probabilities in (5) is γ_m.

<u>ACKNOWLEDGEMENT</u>. I am grateful to a referee for comments that led to much more general results. Also example 5 is due to this referee.

REFERENCES

[1] BERBEE, H.C.P. (1979), *Random walks with stationary increments and renewal theory*, Math. Centre Tracts 112, Math. Centre, Amsterdam.

[2] DALEY, D.J. (1971), *Weakly stationary point processes and random measures*, J. Roy. Statist. Soc. Ser B <u>33</u> 406-428.

[3] FELLER, W. (1970), *An introduction to probability theory and its applications* II. 2nd ed. Wiley, New York.

[4] KAPLAN, E.L. (1955), *Transformations of stationary random sequences*, Math. Scand. <u>3</u> 127-149.

[5] KESTEN, H. (1974), *Renewal theory for functionals of a Markov chain with general state space*, Ann. Probability 2 355-386.

[6] LAI, T.L. (1977), *Convergence rates and r-quick versions of the strong law for stationary mixing sequences*, Ann. Probability 5 693-706.