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TOPOLOGICAL SPACES WITH COMPLETE UNIFORMITIES

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Topological spaces with complete uniformities

Miroslav Hušek

It was proved by Shirota in [23] that a uniformizable Hausdorff space having no closed discrete subspace of measurable cardinality admits a complete uniformity if and only if it can be embedded as a closed subspace in a product of real lines (i.e. if it is realcompact - see [4], [11], [26]). The aim of this paper is to prove analogous assertion also for spaces having closed discrete subspaces of measurable cardinalities (if they exist) and to state new modifications of the Shirota's theorem. Several comments concerning Herrlich's α-compact spaces and van der Slot's α-ultracompact spaces with complete uniformities are stated at the end of this paper. The main results are Theorems 2, 3 and 4.

§1. Introduction.

All the topological spaces under consideration are supposed to be uniformizable Hausdorff. Mainly the terminology of [1] and of [4] is used throughout this paper. The symbol (CH) will mean that we assume the continuum hypothesis.

Cardinals will be denoted by letters α,β and identified with initial ordinal numbers; arbitrary are denoted by ξ,η.
If α is a cardinal, then α^+ is the immediate cardinal successor of α. Relatively measurable cardinals are infinite cardinals α admitting nontrivial two-valued measures which are β-additive for all $\beta < \alpha$ (see [3], [18]). If we arrange the class M' of all relatively measurable cardinals into a transfinite sequence $\{\alpha_\xi\}$, then $\alpha_0 = \omega_0$ and α_ξ, $\xi > 0$, is the first cardinal admitting a nontrivial two-valued measure which is α_η-additive for all $\eta < \xi$. It should be noticed that an existence of a nontrivial two-valued β-additive measure on a set D is equivalent to an existence of a free ultrafilter on D with β^+-intersection property (i.e., any subcollection of cardinality less than β^+ has a non-empty

intersection). It is not known whether M' can contain more than one element in a model of set-theory. In the sequel, the transfinite sequence $\{\alpha_\xi\}$ will be used only as a substitute of M' or of M, where $M = M'$ if M' is a proper class and $M = M' \cup (\alpha_\eta)$ if M' is a set $\{\alpha_\xi | \xi<\eta\}$, where α_η is a symbol greater than any ordinal number. The reason for this convention is to ensure that any cardinal is followed by a member of M. By a nonmeasurable cardinal we shall mean any cardinal smaller than α_1.

Another important concept used frequently throughout this paper is that of E-compact space introduced in [2]. The E-compact spaces are homeomorphs of closed subspaces of powers E^α; thus I-compact spaces or R-compact spaces are just compact or realcompact spaces, respectively (here $I = [0,1]$ and R is the space of real numbers).
The class of all E-compact spaces will be denoted by $K(E)$. Every space P has an E-compactification $\beta_E P$, i.e., a reflection in $K(E)$ (see [2]); van der Slot and Herrlich in [10], [24] have proved that if any compact set is E-compact, then the reflection $\beta_E P$ can be naturally embedded into the Čech-Stone compactification βP of P - all these results were generalized by Herrlich in [7] on the case when E is a class of spaces. A class of spaces is equal to a $K(\mathcal{E})$ for a class $\mathcal{E}$ if and only if the class is productive and closed-hereditary; if a class of spaces is equal to a $K(E)$ for a single space E, then the class is called simple (see [7]). Further results of this sort and their generalization can be found in [8], [9] and [25].

Many generalizations of compact spaces can be obtained by generalization of definitions of compactness. If we want for the obtained spaces to form a class closed under products and closed subspaces, then mainly two such generalizations were studied. The first one is the class of all α-compact spaces in the sense of Herrlich [6], [7] and the second one the class of all α-ultracompact spaces of van der Slot [25]: a space P is said to be α-compact or α-ultracompact, where α is an infinite cardinal, if each ultrafilter in P converges whenever any of its subcollections of cardinalities less than α and composed of zero-sets or closed sets, respectively, has a nonvoid intersection. We can obtain another generalization of compactness by requiring that maximal filters of zero-sets are fixed provided any of its monotone subcollections of ordinality α has a nonvoid intersection (i.e., R_α-spaces, where R_α is the long line till α - see [4]). This last generalization has the disadvantage that the union of all R_α-compact spaces is not the class of all spaces as is the case for α-compact and α-ultracompact spaces. We denote by K_α or U_α the class of all α-compact spaces or α-ultracompact spaces, respectively. It is almost obvious that $K_\alpha \subset U_\alpha$, that $K_{\omega_0} = U_{\omega_0}$ are all compact spaces and K_{ω_1} are all realcompact spaces (K_{ω_1} is a proper subclass of U_{ω_1} - see below). All the classes K_α are simple (see [12]) and can be characterized by algebraic properties of sets of continuous mappings into generating spaces (see [13]). But for $\alpha \notin M$, K_α are not stable under perfect images (see [20], [21] for $\alpha = \omega_1$, [15] for $\alpha \notin M$). On the other hand each class U_α is stable under perfect images (see [25]) but need not be simple (see [15]).

Thus many properties of compact spaces are inherited in K_α or U_α. But there is an important characterization of compact and realcompact spaces by means of completeness which cannot carried over to K_α or U_α for $\alpha > \omega_1$: A space is compact or realcompact if and only if it has a complete uniformity and each of its closed discrete subspaces is finite or of nonmeasurable cardinality, respectively (see [4], [23]). Both classes K_α, U_α, $\alpha > \omega$, contain a space without complete uniformity (e.g. the space T_{ω_1} of all countable ordinals endowed with the order-topology) - the same is true for $K(R_\alpha)$. It follows that if we put S_ξ to be the class of all spaces with complete uniformities and containing no closed discrete subspace of cardinality α_ξ, then we get new classes of spaces. We shall see that classes S_ξ are productive and closed-hereditary. We will study relations between S_ξ, K_α and U_α and prove that any S_ξ is simple. We then deduce the class of all spaces with complete uniformities is simple if and only if the class M is a set.

The further concept used in the sequel is that of a pseudo-α-compact space introduced by Frolik in [5] and Isbell in [16] (we use the term from [16]: a space P is said to be pseudo-α-compact if every uniformizable covering of P has a uniformizable refinement of cardinality less than α (i.e., the covering character of the fine uniformity of P is at most α). Evidently, pseudo-ω_0-compact spaces are just pseudocompact spaces. It is easy to characterize pseudo-α-compact spaces as those having no discrete family of open sets which has cardinality α (see [5], [22]). Many other characterizations of pseudo-α-compact spaces are contained in [5] and, for α to be the first uncountable measurable cardinal, in [14].

The symbol $S(\alpha)$ will denote the metrizable hedgehog with α prickles, i.e., α copies of the closed unit interval $I = [0,1]$ sewed together in the point 0. The points of the β-th copy of I are then $\langle\beta,x\rangle$, $x \in I$, or 0 if $x = 0$. We shall use the following complete metric on $S(\alpha)$:

$$d\langle\langle\beta,x\rangle, \langle\beta',x'\rangle\rangle = \begin{cases} |x-x'| & \text{if } \beta = \beta' \\ x+x' & \text{if } \beta \neq \beta' . \end{cases}$$

Finally we denote $S_0 = I = [0,1]$ and for $\alpha_\xi \in M$, $\xi > 0$, $S_\xi = \Pi\{S(\alpha_\eta) | \eta<\xi\}$.

§2. S_ξ-compact spaces.

In this section we shall investigate a generalization of the Shirota's theorem mentioned at the beginning of this paper; S_ξ-compact spaces (i.e., homeomorphs of closed subsets of powers S_ξ^α) will be convenient objects for this investigation. We shall show that S_ξ-compact spaces coincide with members of the class $\mathcal{S}_\xi$ introduced in §1.
As in the realcompact case (i.e. when $\xi=1$) we must investigate discrete spaces at first. Recall that a discrete space D is of cardinality less than an $\alpha_\xi \in M$ if and only if any ultrafilter in P with α_η^+-intersection property for any $\eta < \xi$ is fixed, i.e., if and only if D is α_η^+-compact (or, which is the same, α_η^+-ultracompact for any $\eta < \xi$.

Theorem 1. A discrete space is S_ξ-compact if and only if it is of cardinality less than α_ξ.

Proof. We will suppose that $\xi > 0$ because for $\xi = 0$ the result is well-known (S_0-compact spaces are just compact spaces). Assume first that D is a discrete space with card $D < \alpha_\xi$. We wish to prove that $\beta_{S_\xi} D = D$, i.e., that for any $X \in \beta D-D$ there is a mapping $f:D \to S_\xi$ which cannot be continuously extended to X. Since X is a free ultrafilter on D, there is a monotone subsystem $\{X_\xi \mid \xi<\alpha_\eta\}$ of X for a $\eta < \xi$ such that $X_0 = D$, $\bigcap\{X_{\xi'} \mid \xi'<\xi\} = X_\xi$ for any limit ordinal $\xi < \alpha_\eta$ and $\bigcap\{X_\xi \mid \xi<\alpha_\eta\} = \emptyset$. But $fx = <\xi,1\}$ whenever $x \in X_\xi - X_{\xi+1}$. Certainly, f is a mapping on D onto a closed discrete subspace D' of $S(\alpha_\eta)$ with card $D' \leqq \alpha_\eta$. If f had a continuous extension to X (into D' then) with a value $<\xi,1>$, then $f^{-1}[<\xi,1>] \in X$, which is impossible.

We shall now assume conversely, that card $D \geqq \alpha_\xi$. There is a free ultrafilter X on D with α_η^+-intersection property for each $\eta < \xi$. We shall prove that any mapping $f: D \to S_\xi$ can be continuously extended to X (as a point of βD); hence $X \in \beta_{S_\xi} D-D$ and, consequently, D is not S_ξ-compact. Since $S_\xi = \Pi\{S(\alpha_\eta) \mid \eta<\xi\}$ we may and shall suppose that $f:D \to S(\alpha_\eta)$ for a $\eta < \xi$. There is a ξ-th prickle I_ξ such that $f^{-1}[I_\xi] \in X$ (X has α_η^+-intersection property); consequently, $f[[X]]$ converges to a point of I_ξ and, thus, f has a continuous extension to X. The proof is complete.

In fact we have shown more in the proof just accomplished: a discrete space is of cardinality smaller than α_ξ if and only if it can be embedded as a closed subspace into a product of discrete spaces with cardinalities at most α_η, $\eta < \xi$.

Now the main theorem of this section:

Theorem 2. Let P be a space and $\alpha_\xi \in M$. Then the following conditions are equivalent:

(1) P is S_ξ-compact;

(2) P has a complete uniformity and no closed discrete subspace of cardinality α_ξ;

(3) P has a complete uniformity and no closed discrete C^*-embedded subspace of cardinality α_ξ;

(4) P has a complete uniformity and no closed discrete C-embedded subspace of cardinality α_ξ;

(5) P has a complete uniformity and no uniformly discrete subspace of cardinality α_ξ (i.e., P has a complete uniformity and is pseudo-α_ξ-compact);

(6) P has a complete uniformity with covering character at most α_ξ;

(7) P has a complete uniformity with covering character at most $\sup\{\alpha_\eta^+ | \eta<\xi\}$.

Proof. Evidently, (2)$\Longrightarrow$(3)$\Longrightarrow$(4)$\Longrightarrow$(5), (7)$\Longrightarrow$(6). If P is S_ξ-compact, then any of its closed discrete subspaces is S_ξ-compact and, thus by Theorem 1, is of cardinality less than α_ξ; it follows that (1)$\Longrightarrow$(2). If P fulfils (5), then the fine uniformity of P satisfies the condition (6). Since the metrizable uniformity of $S(\alpha)$ has the covering character equal to α^+, every S_ξ-compact space has a complete uniformity with the covering character at most the least cardinal greater than any α_η, $\eta < \xi$ - i.e., (1)$\Longrightarrow$(7). Therefore it remains to prove the implication (6)$\Longrightarrow$(1).

Suppose that P has a complete uniformity $\mathcal{U}$ with covering character smaller or equal to α_ξ. We wish to prove that $\beta_{S_\xi} P = P$. Pick out an $\mathcal{X} \in \beta_{S_\xi} P \subset \beta P$ ($\mathcal{X}$ is to be understood as a maximal filter of zero-sets in P) and an arbitrary uniform cover C of $\langle P,\mathcal{U}\rangle$. We shall show that there is an element of C containing a member of $\mathcal{X}$; hence $\mathcal{X}$ is a Cauchy filter in $\langle P,\mathcal{U}\rangle$ and thus it is convergent (i.e., $\mathcal{X} \in P$). By our assumption, C has a refinement C' in $\mathcal{U}$ of cardinality less than α_ξ. The covering C' has a δ-uniformly (in the fine uniformity of P) discrete refinement C'', i.e., $C'' = \bigcup\{C_n | n \in N\}$, where each C_n is a uniformly discrete collection of zero-sets in P - a Stone's theorem (see [16], VII 4). Since card $C' < \alpha_\xi$, we may and shall suppose that card $C'' < \alpha_\xi$.

The filter X has countable intersection property ($\alpha > 0$ and hence $\beta_{S_\xi} P \subset \beta_{S_1} P = \nu P$) and, consequently, there is an $n \in N$ such that $\bigcup C_n \in X$ (union of a uniformly discrete collection of zero-sets is a zero-set). Let $C_n = \{V_a | a \in A\}$, card $A < \alpha_\xi$, and let W be a uniformizable neighborhood of the diagonal in $P \times P$ such that $\{W[V_a] | a \in A\}$ is a uniformly discrete family. Now we can construct (see [14], proposition 1) a continuous mapping $f\colon P \to S(\text{card } A)$ with $f[V_a] = \langle a, 1\rangle$, $f[P - W[\bigcup \{V_a | a \in A\}]] = 0$. Denote by $\tilde{f}$ the continuous extension of f on $\beta_{S_\xi} P$ into $\beta_{S_\xi} S(\text{card } A)$. If we denote by D the closed discrete subspace $\{\langle a, 1\rangle | a \in A\}$ in $S(\text{card } A)$, then $\tilde{f}X$ must belong to the closure $\overline{D}$ of D in $\beta_{S_\xi} S(\text{card } A)$. But we have $\overline{D} = \beta_{S_\xi} D = D$; the second equality follows by Theorem 1 because card $D < \alpha_\xi$ and the first one by the fact that any mapping on D into S_ξ can be continuously extended on S(card A) - this extension property suffices to check only for mappings into $S(\alpha_\eta)$ for arbitrary $\eta < \xi$ and this is Proposition 1 in [14] (stated for other cardinalities but the proof is the same also in our case - see also similar assertion in [4], 3L.1). Therefore there is an $a \in A$ such that $\tilde{f}X = \langle a, 1\rangle$ and hence, $V_a \in X$. This concludes the proof since $V_a \subset U$ for a $U \in C$.

The second part of the foregoing proof is similar to that of Shirota's theorem given in [4]. The main difference is in using of the hedgehog instead of real line.

Since we use the spaces S_ξ only for the purpose to investigate S_ξ-compact spaces, we could define $S_\xi = \Pi\{S(\alpha_\eta) | \eta \in C_\xi\}$, where C_ξ is a cofinal set in the set of all ordinals smaller than ξ-especially we could define $S_{\xi+1} = S(\alpha_\xi)$. This is a consequence of the fact that, for $\eta < \xi$, $S(\alpha_\eta)$ can be embedded as a closed subspace into $S(\alpha_\xi)$. All the assertions of this paper remain true after this change.

The next result is an easy consequence of Theorem 2 but because of its importance we will state it as Theorem.

Theorem 3. The class of spaces with complete uniformities is simple if and only if the class of all relatively measurable cardinals is a set.

Proof. If M is a set with the last member α_ξ, then the class of all spaces with complete uniformities coincides with S_ξ, i.e., with the class of all S_ξ-compact spaces, by Theorem 2. Let M be a proper class and take an arbitrary space P with a complete uniformity. There is an ordinal ξ such that P is S_ξ-compact; thus $S_{\xi+1}$ is not P-compact because it is not S_ξ-compact by Theorem 2,(2).

There is a problem to characterize simple categories (see [7], [8], [9], [15]); Theorem 3 shows that a solution of this problem is closely connected with the problem of measurability of cardinals.

Now we will consider conditions of Theorem 2 but stated for cardinals not belonging to M. It is clear from the conditions (6) and (7) that if α_ξ is the first member of M greater or equal to a cardinal α, then P has a complete uniformity with covering character at most α if and only if P is S_ξ-compact. The following two Propositions show that a similar case occur for the condition (1) but not for those remaining.

Proposition 1. If α_ξ is the first member of M greater than a cardinal α, then $S(\alpha)$-compact spaces coincide with S_ξ-compact spaces.

Proof. Of course it suffices to show that the space $S(\alpha)$ is S_ξ-compact and that the space S_ξ is $S(\alpha)$-compact. The latter condition is trivial since α is greater or equal to any α_η, $\eta < \xi$. Conversely, since $\alpha < \alpha_\xi$, the space $S(\alpha)$ is S_ξ-compact by Theorem 2,(2).

Denote by $E_i(\alpha)$ the class of all spaces satisfying the condition (i), i = 2,3,4,5, of Theorem 2 for α instead of α_ξ.

Proposition 2. *Let* $i = 2,3,4,5$. *The class* $\mathcal{E}_i(\alpha)$ *is productive and closed-hereditary if and only if* $\alpha \in M$. *If* α_ξ *is the first element of* M *greater or equal to* α, *then* $\mathcal{E}_i(\alpha)$*-compact spaces coincide with* S_ξ*-compact spaces*.

Proof. Evidently $\mathcal{E}_2(\alpha) \subset \mathcal{E}_3(\alpha) \subset \mathcal{E}_4(\alpha) \subset \mathcal{E}_5(\alpha) \subset \mathcal{K}(S_\xi)$ (the last inclusion follows by the implication (5) $\Longrightarrow$(1) Theorem 2). Since $\alpha_\eta < \alpha$ for any $\eta < \xi$, the spaces $S(\alpha_\eta)$ belong to $\mathcal{E}_2(\alpha)$ and, hence, the space S_ξ is $\mathcal{E}_2(\alpha)$-compact. Consequently, $\mathcal{E}_2(\alpha)$-compact spaces (and therefore $\mathcal{E}_i(\alpha)$-compact spaces for other i) coincide with $\mathcal{K}(S_\xi)$. If $\alpha < \alpha_\xi$, then $S(\alpha) \in \mathcal{K}(S_\xi) - \mathcal{E}_5(\alpha)$ and thus $\mathcal{E}_i(\alpha) = \mathcal{K}(\mathcal{E}_i(\alpha))$ for no i. It follows that in this case neither of $\mathcal{E}_i(\alpha)$ can be both productive and closed-hereditary.

Obviously, the class $\mathcal{E}_2(\alpha)$ is closed-hereditary. It need not be finitely productive: if we take the space $\tilde{R}$ of real numbers with the topology generated by the base $[a,b)$, then $\tilde{R} \in \mathcal{E}_2(\omega_1)$ because any Lindelöf space belongs to $\mathcal{E}_2(\omega_1)$, and $\tilde{R}\times\tilde{R} \notin \mathcal{E}_2(\omega_1)$ (the diagonal $\{<x,-x> | x \in R\}$ is closed discrete of cardinality 2^{ω_0}.

(CH): The product $\tilde{R} \times \tilde{R}$ belongs to $\mathcal{E}_3(\omega_1)$ ($\tilde{R}\times\tilde{R}$ is separable and hence card $C^*(\tilde{R}\times\tilde{R}) \leq 2^{\omega_0}$; by (CH), $2^{\omega_0} < 2^{\omega_1}$; if $\tilde{R}\times\tilde{R}$ contained an uncountable discrete C^*-embedded subspace D, then card $C^*(\tilde{R}\times\tilde{R}) \geq$ card $C^*(D) \geq 2^{\omega_1}$). From this result we can deduce that $\mathcal{E}_2(\omega_1)$ is a proper subclass of $\mathcal{E}_3(\omega_1)$ and that neither of $\mathcal{E}_3(\omega_1)$, $\mathcal{E}_4(\omega_1)$, $\mathcal{E}_5(\omega_1)$ need be closed-hereditary ($\tilde{R}\times\tilde{R}$ has a closed subspace $\{<x,-x> | x \in R\}$ not belonging to $\mathcal{E}_5(\omega_1)$).

Now we shall show that the classes $\mathcal{E}_3(\omega_1)$, $\mathcal{E}_4(\omega_1)$ and $\mathcal{E}_5(\omega_1)$ need not even be finitely productive. In [19], Michael has constructed under (CH) a Lindelöf space Y the square Y×Y of which is paracompact and not Lindelöf. As a Lindelöf space, $Y \in \mathcal{E}_2(\omega_1)$. It is easy to show that $Y\times Y \notin \mathcal{E}_5(\omega_1)$; indeed since Y×Y is paracompact, uniformizable covers coincide with interior covers and because Y×Y is not Lindelöf, there is an open cover with no countable open refinement - i.e., Y×Y is not pseudo-ω_1-compact.

As for the relations between $E_3(\omega_1)$, $E_4(\omega_1)$ and $E_5(\omega_1)$ the author conjectures that $E_3(\omega_1)$ is a proper subclass of $E_4(\omega_1)$ and that $E_4(\omega_1)$ is a proper subclass of $E_5(\omega_1)$.

§3. Relations between classes S_ξ, K_α and U_α.

First we recal that the class K_α of all α-compact spaces is a subclass of the class U_α of all α-ultracompact spaces and that $K_\alpha \subset K_\beta$, $U_\alpha \subset U_\beta$ provided $\alpha \leqq \beta$. Evidently, α-ultracompact space P is α-compact if each closed subset of P is a zero-set, i.e., if P is perfectly normal (especially if P is metrizable). We shall prove now that α-ultracompactness and α-compactness coincide also for spaces with complete uniformities. First we will investigate discrete spaces.

Proposition 3. <u>A discrete space D is</u> α-<u>compact if and only if</u> $\alpha > \alpha_\eta$ <u>for all</u> $\eta < \xi$, <u>where</u> α_ξ <u>is the first member of M greater than card D</u>.

Proof. Assume that D is α-compact. Since D is not α_η-compact for any $\eta < \xi$ (D contains a free ultrafilter with α_η-intersection property, because card $D \geqq \alpha_\eta$), the cardinal α must be greater than any such α_η. Suppose now conversely that D is not α-compact. Then there is a free ultrafilter on D with α-intersection property and therefore $\alpha \leqq \alpha_\eta$ for a $\eta < \xi$ because otherwise card $D \geqq \alpha_\xi$.

Corollary 1. <u>If P is α-ultracompact and</u> $\alpha_\xi \in M$ <u>is greater or equal to</u> <u>α, then P is pseudo-</u>α_ξ<u>-compact (moreover any closed discrete subspace of P is of cardinality less than</u> α_ξ).

Proof. If D is a closed subspace of P, then D is α-ultracompact and, hence by Proposition 3, if D is discrete, then α must be greater than any α_η, $\eta < \xi_0$, where α_{ξ_0} is the first member of M greater than card D. It follows that $\alpha_\xi \geqq \alpha_{\xi_0}$ because $\alpha_\xi \geqq \alpha_\eta$, $\eta < \xi_0$. Consequently, card $D < \alpha_{\xi_0} \leqq \alpha_\xi$.

Corollary 2. *If α_ξ is the first member of M greater or equal to an infinite cardinal α, then for any $\beta < \alpha_\xi$ there is an α-compact space which is not pseudo-β-compact.*

Proof. By Proposition 3, a discrete space of cardinality β is α-compact and certainly is not pseudo-β-compact.

Several times there appeared a condition "α is greater than any α_η, $\eta < \xi$" in the preceding assertions; it is easy to see that it is equivalent to the condition "$\alpha \geqq \sup\{\alpha_\eta^+ | \eta<\xi\}$", which is the same as $\alpha \geqq \sup\{\alpha_\eta | \eta<\xi\}$ if ξ is limit and $\alpha > \alpha_{\xi-1}$ if ξ is isolated.

Now we turn our attention to the hedgehogs.

Proposition 4. *The hedgehog $S(\beta)$ is α-compact if and only if $\alpha > \alpha_\eta$ for all $\eta < \xi$, where α_ξ is the first member of M greater than β.*

Proof. Since $S(\beta)$ contains a closed discrete subspace of cardinality β, the necessity follows by Proposition 3. For the other implication assume that α is greater than any α_η, $\eta < \xi$. By Proposition 1, the hedgehog $S(\beta)$ is S_ξ-compact and, thus, it suffices to prove that S_ξ is α-compact, which is the same as that $S(\alpha_\eta)$ is α-compact for any $\eta < \xi$. Let $\eta < \xi$ and take an arbitrary maximal filter of zero-sets in $S(\alpha_\eta)$ with α-intersection property. Since $\alpha_\eta < \alpha$, there must be a prickle in $S(\alpha_\eta)$ belonging to the ultrafilter; it follows that the ultrafilter contains a compact set and thus converges.

Corollary 1. *The space S_ξ is α-compact or α-ultracompact if and only if $\alpha > \alpha_\eta$ for all $\eta < \xi$.*

Proof follows at once from Proposition 4 because S_ξ is α-compact if and only if all the $S(\alpha_\eta)$, $\eta < \xi$ are α-compact and the same for ultracompactness.

Corollary 2. *Let $\alpha > \alpha_\eta$ for all $\eta < \xi$. Then any pseudo-α_ξ-compact space with a complete uniformity is α-compact.*

Proof. If P is pseudo-α_ξ-compact and has a complete uniformity, then it is S_ξ-compact by Theorem 2. By the foregoing Corollary, P is α-compact.

The next theorem states other generalized modifications of the Shirota's theorem:

Theorem 4. Let $\alpha_\xi \in M$ and $\alpha_\xi \geq \alpha > \alpha_\eta$ for all $\eta < \xi$. Then the following conditions on a space P are equivalent:

(a) *P is S_ξ-compact;*

(b) *P has a complete uniformity and is α-compact;*

(c) *P has a complete uniformity and is α-ultracompact.*

Proof. Obviously (b)$\Longrightarrow$(c); by Corollary 2 of Proposition 4, (a)$\Longrightarrow$(b). The last implication (c)$\Longrightarrow$(a) follows by Corollary 1 of Proposition 3 (P is α-ultracompact and hence pseudo-α_ξ-compact; since in addition it has a complete uniformity, it is S_ξ-compact by Theorem 2).

Corollary 1. *If a space has a complete uniformity, then it is α-compact if and only if it is α-ultracompact.*

Evidently, the converse implication of Corollary 1 does not hold (the space T_{ω_1} of all countable ordinals with order-topology is both ω_2-compact and ω_2-ultracompact (see [6], [25]) and has no complete uniformity).

Corollary 2. *If P has a complete uniformity and $\alpha_\xi \geq \alpha > \alpha_\eta$ for all $\eta < \xi$, then the following conditions are equivalent:*

(a) *P is α-compact;*

(b) *P is α_ξ-compact;*

(c) *P is $\sup\{\alpha_\eta^+ | \eta<\xi\}$-compact;*

(d) *P is pseudo-α_ξ-compact.*

Proof. The implication (a)$\Longrightarrow$(b) is obvious, (b)$\Longrightarrow$(c) follows by Theorem 4, (c)$\Longrightarrow$(d) is Corollary 1 of Proposition 3 and (d)$\Longrightarrow$(a) follows at once from Theorem 2, implication (5)$\Longrightarrow$(1), and from Theorem 4, implication (a)$\Longrightarrow$(b).

The same assertion can be stated for α-ultracompact spaces.

As an application of Theorem 4 we shall give another proof of a van der Slot's result proved in [25]:

Corollary 3. A coutably paracompact normal ω_1-ultracompact space is realcompact.

Proof. By Theorem 4 or by its Corollary 1 it suffices to prove that an arbitrary countably paracompact normal ω_1-ultracompact space P has a complete uniformity. A space P is countably paracompact normal if and only if all its countable open coverings form a base for a uniformity on P. Denote this uniformity by $\mathcal{U}$. We shall prove that $\mathcal{U}$ is complete. Let $\mathcal{X}$ be an ultrafilter on P which is $\mathcal{U}$-Cauchy and assume that $\mathcal{X}$ does not converge. Since P is ω_1-ultracompact, there are countably many closed sets $A_i \in \mathcal{X}$ with an empty intersection. But this is a contradiction because the open cover $\{P-A_i\}$ is a member of $\mathcal{U}$ and no its element belongs to the $\mathcal{U}$-Cauchy filter $\mathcal{X}$.

Now we shall summarize our results for the case $\xi = 1$. All the conditions in the next theorem which contain a direct assumption on completeness can be regarded as modifications of the Shirota's theorem mentioned at the beginning of this paper. Of course, some of them are well-known - see [4], [16], [17], [23].

Theorem 5. The following conditions are equivalent for a space P:

(a) P is realcompact;

(b) P can be embedded as a closed subspace into a power $S^A(\omega_0)$ of the metrizable hedgehog $S(\omega_0)$;

(c) P has a complete uniformity and each of its closed discrete subspaces is of nonmeasurable cardinality;

(d) P has a complete uniformity and each of its closed discrete C^*-embedded subspaces is of nonmeasurable cardinality;

(e) P has a complete uniformity and each of its closed discrete C-embedded subspaces is of nonmeasurable cardinality;

(f) P has a complete uniformity and each of its uniformly discrete subspaces is of nonmeasurable cardinality (i.e., P has a complete uniformity and is pseudo-α_1-compact;

(g) P has a complete uniformity with covering character at most α_1:

(h) P has a complete uniformity with covering character at most ω_1;

(i) P has a complete uniformity and each maximal filter of zero-sets in P with nonmeasurable intersection property is fixed (i.e., P has a complete uniformity and is α-compact);

(j) P has a complete uniformity and each ultrafilter with countable intersection property for closed sets converges (i.e., P has a complete uniformity and is ω_1-ultracompact);

(k) P has a complete uniformity and each ultrafilter with nonmeasurable intersection property for closed sets converges (i.e., P has a complete uniformity and is α_1-ultracompact).

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