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J.H. van Zanten

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CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

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Harry van Zanten
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
hvz@cwi.nl

ABSTRACT
We derive limit theorems for diffusion processes that have a finite speed measure. First we prove a number of asymptotic properties of the density $\rho_t = d\mu_t/d\mu$ of the empirical measure μ_t with respect to the normalized speed measure μ. These results are then used to derive finite dimensional and uniform central limit theorems for integrals of the form $\sqrt{t}\int(\rho_t - 1)\,d\nu$, where ν is an arbitrary finite, signed measure on the state space of the diffusion. We also consider a number of interesting special cases, such as uniform central limit theorems for Lebesgue integrals and functional weak convergence of the empirical distribution function.

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1 Introduction

In this paper we present some new contributions to the theory of one-dimensional diffusion processes. For a recent treatment of this subject, see for instance Rogers and Williams (1987), Revuz and Yor (1991) or Kallenberg (1997). The most important definitions and basic results will be recalled in the next section. We consider a regular diffusion process X on an interval $I \subseteq \mathbb{R}$. The state space I may be closed, open or half-open, bounded or unbounded. We denote the speed measure of the diffusion by m. The basic assumption is that m has finite total mass.

We will first be interested in the relation between the probability measure $\mu = m/m(I)$ and the empirical measures μ_t of the process, defined by (2.4). Using the occupation measure formula for diffusions, it is easily seen that $\mu_t \ll \mu$ almost surely. Moreover, the density $\rho_t = d\mu_t/d\mu$ can be expressed in terms of local time processes (see theorem 2.3 below). Using the well-known fact that a diffusion in natural scale is a time-changed Brownian motion and a simple scaling property of Brownian local time (see lemma 3.1), this expression for ρ_t allows us to prove several asymptotic properties.

In section 4 we first prove that the sup-norms $\|\rho_t\|_\infty$ of the ρ_t are asymptotically tight for $t \to \infty$ (see theorem 4.1). We then show that in probability, the random functions ρ_t converge to 1, uniformly on compact intervals (see theorem 4.2). Using these two results we derive that for every finite measure ν on I and $0 < p < \infty$, the $L^p(I, \nu)$-norm of $\rho_t - 1$ converges to 0 in probability (theorem 4.4). A 'weak law of large numbers' follows easily from this result (see corollary 4.5 and the remarks thereafter).

The observation that we have proved this weak law without assuming recurrence leads us to an interesting consequence, formulated below as corollary 4.6. It turns out that a diffusion with finite speed measure is necessarily recurrent. This then also implies ergodicity, since it is well-known that a recurrent diffusion with finite speed measure is ergodic. The measure μ is the unique invariant probability measure of the diffusion.

In section 5 we give an alternative expression for the random densities. We write $\rho_t(x) - 1$ as a stochastic integral plus a remainder term of lower order (see lemma 5.1). If the state space I is an open interval, so that there are no reflecting boundary points (there can be no absorbing boundary points, since the speed measure is finite), the stochastic integrals are local martingales (see the remarks in the beginning of section 6). This fact allows us to prove a number of central limit theorems in sections 6 and 7. The results of section 4 are important ingredients in the proofs of these theorems.

In section 6 we prove a central limit theorem for random vectors of the form

$$\sqrt{t}\left(\int (\rho_t - 1)\, d\nu_1, \ldots, \int (\rho_t - 1)\, d\nu_n\right),$$

where the ν_i are arbitrary finite, signed measures on the state space I (see theorem 6.1). If we take the ν_i for instance equal to Dirac measures, we find that the random vectors $\sqrt{t}(\rho_t(x_1) - 1, \ldots, \rho_t(x_n) - 1)$ are asymptotically normal, provided that the invariant measure μ satisfies a tail condition (see corollary 6.2). The choice $d\nu_i = f_i\, d\mu$ (for functions $f_i \in L^1(\mu)$) yields the well-known central limit theorem for Lebesgue integrals (see corollary 6.3).

The last section of the paper, section 7, is devoted to the proof of uniform central limit theorems. We consider an indexed collection $\{\nu_\theta\}_{\theta \in \Theta}$ of finite, signed measures on I. Theorem 7.1 gives conditions under which for every countable $\Theta^* \subseteq \Theta$, the random maps

$$\theta \mapsto \sqrt{t} \int (\rho_t - 1)\, d\nu_\theta$$

have a weak limit in the space $\ell^\infty(\Theta^*)$ of bounded functions on Θ^*. The theorem says that it suffices to have a metric d on the set Θ such that the entropy condition (7.3) is satisfied and the map

$$(\Theta, d) \ni \theta \mapsto \nu_\theta(l, \cdot] - F(\cdot)\nu_\theta(I) \in L^2(I, s)$$

is Lipschitz, where l is the left endpoint of the interval I, F is the distribution function of the invariant measure μ and s is the scale function of the diffusion. If Θ is finite, the result reduces to the central limit theorem of section 6. We close section 7 with a number of interesting corollaries of theorem 7.1. Corollaries 7.2 and 7.5 state that under the tail condition (6.2), we have functional weak convergence of the random maps $\sqrt{t}(\rho_t - 1)$ and $\sqrt{t}(F_t - F)$, where F_t is the empirical distribution function. Corollary 7.4 gives a uniform central limit theorem for random maps of the form

$$\theta \mapsto \frac{1}{\sqrt{t}} \int_0^t f_\theta(X_u)\, du,$$

under the assumption that the functions $f_\theta(x)$ depend differentiably on a Euclidean parameter θ.

Some of the results that are presented in this paper are already known in special cases, in particular for ergodic solutions of stochastic differential equations. For such diffusions, the empirical measures μ_t and the invariant measure μ have densities f_t and f with respect to the Lebesgue measure on the state space. We then have $\rho_t = f_t/f$ and the results of this paper give a number of asymptotic properties of the ratio f_t/f. Such limit theorems for the so-called empirical densities f_t have proven to be useful tools for the asymptotic analysis of nonparametric estimators (see for instance Kutoyants (1998) and Van Zanten (2000c, 2000d)). Theorem 4.2 and corollary 7.2 extend results of Kutoyants (1998), Van Zanten (2000b) and Van Zanten (2000c). In connection with corollary 7.5 we mention Negri (1998), who studied the functional weak convergence of the empirical distribution function for a certain class of stochastic differential equations (see also remark 7.6). The central limit theorem for Lebesgue integrals (theorem 6.3) was already proved by Mandl (1968), using the strong Markov property and the central limit theorem for i.i.d. random variables. For the special case of ergodic solutions of stochastic differential equations, Florens-Zmirou (1984) proved it by using the central limit theorem for continuous martingales.

2 Regular diffusions with finite speed measure

We consider a diffusion on some interval $I \subseteq \mathbb{R}$. The state space I may be closed, open or half-open, bounded or unbounded. By $\Omega = C(\mathbb{R}^+, I)$ we denote the space of continuous functions on $\mathbb{R}^+$ taking values in I. The canonical process on Ω is denoted by X, so $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$. On Ω we have the canonical filtration $\mathcal{F}_t = \sigma(X_u : u \leq t)$ and the σ-algebra $\mathcal{F} = \sigma(X_u : u \geq 0)$. The family $\{P_z : z \in I\}$ of probability measures on $(\Omega, \mathcal{F})$ is supposed to constitute a (canonical) diffusion on I. This means that the following three properties are supposed to be satisfied: (i) under P_z, the process X starts in z, i.e. $P_z(X_0 = z) = 1$ for every $z \in I$, (ii) for every $B \in \mathcal{B}(I)$, the function $z \mapsto P_z(B)$ is measurable, (iii) the strong Markov property holds, i.e. for every optional time τ, measurable function f on Ω and $z \in I$ it holds that $E_z(f(X_{\tau+\cdot}) \,|\, \mathcal{F}_\tau) = E_{X_\tau} f(X_\cdot)$. As general references for diffusion theory we mention Kallenberg (1997), Revuz and Yor (1991) and Rogers and Williams (1987). In this section we recall some elements of the theory that we need in this paper.

We assume that the diffusion is regular, i.e. for every z in the interior of I and $x \in I$ it holds that $P_z(T_x < \infty) > 0$, where T_x denotes the first hitting time of x. Under this condition, there exists a continuous, strictly increasing function s on I such that for all $z, a, b \in I$ with $a \leq z \leq b$ it holds that

$$P_z(T_b < T_a) = \frac{s(z) - s(a)}{s(b) - s(a)}. \tag{2.1}$$

The function s is called the scale function of the diffusion. It is not unique, but determined up to an affine transformation. If the function $s(x) = x$ is a scale function, the diffusion is said to be in natural scale. Using (2.1) it is easily seen that the diffusion $Y = s(X)$ on $s(I)$ is in natural scale.

With a regular diffusion in natural scale we can associate a unique measure m on I, called the speed measure. It can be introduced via the following theorem, which states that a regular diffusion in natural scale is in fact a time-changed Brownian motion. The speed measure m determines the time change. See for instance Rogers and Williams (1987), theorem V.47.1.

Theorem 2.1. *Suppose that X is in natural scale. There exists a unique measure m on I such that for every $z \in I$, there exists an extension of the probability space $(\Omega, \mathcal{F}, P_z)$, supporting a Brownian motion W that starts in z, such that $X_t = W_{\tau_t}$, where τ is the right-continuous inverse of the process A defined by*

$$A_t = \int_I L_t^W(x)\, m(dx).$$

Here L^W is the local time process of W.

The diffusion X may not be in natural scale, but we just noted that $Y = s(X)$ always is. If m^Y denotes the speed measure of Y on $s(I)$, determined by the preceding theorem, we define the speed measure m of X by

$$m = m^Y \circ s. \tag{2.2}$$

Note that the speed measure of X thus depends on the choice of the scale function. Definition (2.2) allows us to generalize theorem 2.1 as follows.

Corollary 2.2. *For every $z \in I$, there exists an extension of the probability space $(\Omega, \mathcal{F}, P_z)$, supporting a Brownian motion W that starts in z, such that $s(X_t) = W_{\tau_t}$, where τ is the right-continuous inverse of the process A defined by*

$$A_t = \int_I L_t^W(s(x))\, m(dx).$$

Here L^W is the local time process of W.

Proof. Since $Y = s(X)$ is in natural scale, theorem 2.1 implies that $s(X_t) = Y_t = W_{\tau_t}$, where τ is the right-continuous inverse of the process A defined by

$$A_t = \int_{s(I)} L_t^W(x)\, m^Y(dx).$$

Apply (2.2) and a change of variables to see that this process A is equal to the process mentioned in the statement of the theorem. □

We will study the case that the speed measure m of the diffusion has finite total mass. The process A of corollary 2.2 is then continuous in t. Since $A_\infty = \infty$ it therefore holds that

$$A_{\tau_t} = t \tag{2.3}$$

for all $t \geq 0$, since τ is the right-continuous inverse of A. The finiteness of m allows us to define the probability measure $\mu = m/m(I)$. Note that it follows from (2.2) that the speed measure of X is finite if and only if the speed measure of $Y = s(X)$ is finite. If the diffusion is recurrent, then the finiteness of the speed measure implies that it is also μ-ergodic (see Kallenberg (1997), lemma 20.19). This means that for every $z \in I$

$$X_t \overset{P_z}{\Longrightarrow} \mu.$$

as $t \to \infty$ ($\overset{P_z}{\Longrightarrow}$ denotes weak convergence under P_z). The measure μ is then the unique invariant probability measure of the process. (We will see later that finiteness of the speed measure in fact implies recurrence, see corollary 4.6 below).

The empirical measures of the diffusion are denoted by μ_t. So for every $t > 0$ and $B \in \mathcal{B}(I)$ we define

$$\mu_t(B) = \frac{1}{t}\int_0^t 1_B(X_u)\, du. \tag{2.4}$$

The following theorem states that for every $t > 0$, the empirical measure μ_t is absolutely continuous with respect to the invariant probability measure μ. Moreover, the random density $\rho_t = d\mu_t/d\mu$ is expressed in terms of the local time L^Y of $Y = s(X)$ and the local time L^W of the Brownian motion W appearing in corollary 2.2. (Note that since Y is a time-changed Brownian motion, it is a continuous semimartingale, so its local time is well-defined.)

Theorem 2.3. *Let $z \in I$ be fixed and let W and τ_t be as in corollary 2.2. Under P_z, we almost surely have $\mu_t \ll \mu$ for all $t > 0$. Moreover, it holds that $\mu_t(dx) = \rho_t(x)\mu(dx)$, where*

$$\rho_t(x) = m(I)\frac{1}{t}L_t^Y(s(x)) \tag{2.5}$$

$$= m(I)\frac{1}{t}L_{\tau_t}^W(s(x)) \tag{2.6}$$

for every $x \in I$ and $t > 0$.

Proof. In view of (2.3), we can use exactly the same arguments as in the proof of theorem V.49.1 of Rogers and Williams (1987). □

Remark 2.4. Since Brownian local time and the scale function s are continuous, expression (2.6) implies that under every P_z, the random densities ρ_t are almost surely continuous functions on I. From the continuity of the Brownian sample paths it follows that the random functions $x \mapsto L_{\tau_t}^W(x)$ almost surely have compact support. Since s is continuous, the functions ρ_t have the same property. So under each P_z, the functions ρ_t are continuous, compactly supported functions on I. In particular, they are uniformly bounded.

3 A scaling property of Brownian local time

The following lemma gives a simple, but very useful property of Brownian motion. We will use it in several proofs.

Lemma 3.1. *Let W be a Brownian motion starting in z. For every $c > 0$ we have*

$$\left\{\tfrac{1}{c}L_{c^2t}^W(x) : x \in \mathbb{R}, t \geq 0\right\} \overset{\mathrm{d}}{=} \left\{L_t^B\left(\frac{x-z}{c}\right) : x \in \mathbb{R}, t \geq 0\right\},$$

where B is a standard Brownian motion, i.e. $B_0 = 0$.

Proof. Put $B = W - z$ and define the process B' by $B'_t = B_{c^2t}/c$. Since W starts in z, the process B is a standard Brownian motion. By the scaling property of Brownian motion, the same holds for B'. Now fix $t \geq 0$ and take an arbitrary measurable function f on $\mathbb{R}$ that is bounded and nonnegative. Using two times a basic property of local time and a change of variables we see that

$$\begin{aligned}\int_{\mathbb{R}} f(x)\tfrac{1}{c}L^B_{c^2t}(x)\,dx &= \tfrac{1}{c}\int_0^{c^2t} f(B_u)\,du = c\int_0^t f(B_{c^2v})\,dv \\ &= c\int_0^t f(cB'_v)\,dv = c\int_{\mathbb{R}} f(cx)L^{B'}_t(x)\,dx = \int_{\mathbb{R}} f(y)L^{B'}_t\left(\frac{y}{c}\right)\,dy.\end{aligned}$$

Since f is arbitrary and Brownian local time is continuous, it follows that

$$\tfrac{1}{c}L^B_{c^2t}(x) = L^{B'}_t\left(\frac{x}{c}\right)$$

for all $t \geq 0$ and $x \in \mathbb{R}$. From the definition of B we have that $L^W_t(x) = L^B_t(x - z)$ for all $t \geq 0$ and $x \in \mathbb{R}$. Consequently, we have

$$\tfrac{1}{c}L^W_{c^2t}(x) = L^{B'}_t\left(\frac{x-z}{c}\right)$$

for all $t \geq 0$ and $x \in \mathbb{R}$. This completes the proof of the lemma, since B' has the same distribution as B. □

The first application of lemma 3.1 is a weak convergence theorem for the time-change τ_t of corollary 2.2.

Theorem 3.2. *For every $z \in I$ we have*

$$\frac{\tau_t}{t^2} \xrightarrow{P_z} \frac{1}{m^2(I)\chi^2},$$

where, as usual, χ^2 denotes the distribution of the square of a standard normal random variable.

Proof. Consider the Brownian motion W and the process A of corollary 2.2. Using lemma 3.1 we see that under P_z

$$\tfrac{1}{\sqrt{t}}A_t = \int_I \tfrac{1}{\sqrt{t}}L^W_t(s(x))\,m(dx) \overset{\mathrm{d}}{=} \int_I L^B_1\left(\frac{s(x)-z}{\sqrt{t}}\right)\,m(dx),$$

where B is a standard Brownian motion. By the finiteness of m, the integral on the right hand side converges almost surely to $m(I)L^B_1(0)$. It follows that

$$\tfrac{1}{\sqrt{t}}A_t \xrightarrow{P_z} m(I)L^B_1(0). \tag{3.1}$$

The process τ is the right-continuous inverse of A, so for every $t, T \geq 0$ it holds that $\tau_t < T$ if and only if $A_T > t$. Using (3.1) we see that for every $x \in \mathbb{R}$

$$\begin{aligned}P_z(\tau_t/t^2 < x) &= P_z(A_{t^2x} > t) = P_z\left(\tfrac{1}{t\sqrt{x}}A_{t^2x} > \tfrac{1}{\sqrt{x}}\right) \\ &\to P_z\left(m(I)L^B_1(0) > \tfrac{1}{\sqrt{x}})\right) = P_z\left(\frac{1}{m^2(I)(L^B_1(0))^2} < x\right).\end{aligned}$$

Use the well-known fact that $(L^B_1(0))^2$ has a χ^2-distribution to complete the proof. □

4 Asymptotic properties of the random densities

In this section we derive a number of asymptotic properties of the densities ρ_t. The random densities are uniformly bounded on I (see remark 2.4). We will prove that the sup-norms

$$\|\rho_t\|_\infty = \sup_{x\in I} |\rho_t(x)|$$

are asymptotically tight under each P_z, i.e. for every $\varepsilon > 0$ there exists an $a > 0$ such that

$$\limsup_{t\to\infty} P_z(\|\rho_t\|_\infty > a) < \varepsilon.$$

As usual, we abbreviate this by writing $\|\rho_t\|_\infty = O_{P_z}(1)$.

Theorem 4.1. *For every $z \in I$ we have $\|\rho_t\|_\infty = O_{P_z}(1)$.*

Proof. Let W and τ be as in corollary 2.2. By relation (2.6) of theorem 2.3 we have for all $a, b > 0$

$$\begin{aligned} P_z(\|\rho_t\|_\infty > a) &\leq P_z\left(\sup_{x\in\mathbb{R}} \tfrac{1}{t} L^W_{t^2 \frac{\tau_t}{t^2}}(x) > a/m(I)\right) \\ &\leq P_z\left(\sup_{x\in\mathbb{R}\,,\, u\leq b} \tfrac{1}{t} L^W_{t^2 u}(x) > a/m(I)\right) + P_z\left(\frac{\tau_t}{t^2} > b\right). \end{aligned}$$

By lemma 3.1 we have a standard Brownian motion B such that under P_z

$$\sup_{x\in\mathbb{R}\,,\, u\leq b} \tfrac{1}{t} L^W_{t^2 u}(x) \overset{\mathrm{d}}{=} \sup_{x\in\mathbb{R}\,,\, u\leq b} L^B_u\left(\frac{x-z}{t}\right) = \sup_{x\in\mathbb{R}} L^B_b(x).$$

We thus find that

$$P_z(\|\rho_t\|_\infty > a) \leq P_z\left(\sup_{x\in\mathbb{R}} L^B_b(x) > a/m(I)\right) + P_z\left(\frac{\tau_t}{t^2} > b\right) \tag{4.1}$$

Now fix $\varepsilon > 0$. By theorem 3.2 the ratio τ_t/t^2 has a weak limit. In particular, it is asymptotically tight. As a consequence, we can choose b so large that

$$\limsup_{t\to\infty} P_z\left(\frac{\tau_t}{t^2} > b\right) < \tfrac{1}{2}\varepsilon. \tag{4.2}$$

By continuity of the Brownian sample paths the random function $x \mapsto L^B_b(x)$ has compact support. Since Brownian local time is continuous, it follows that $x \mapsto L^B_b(x)$ is uniformly bounded, almost surely. Therefore, we can choose a so large that

$$P_z\left(\sup_{x\in\mathbb{R}} L^B_b(x) > a/m(I)\right) < \tfrac{1}{2}\varepsilon. \tag{4.3}$$

Combine relations (4.1), (4.2) and (4.3) to find that

$$\limsup_{t\to\infty} P_z(\|\rho_t\|_\infty > a) < \varepsilon.$$

This concludes the proof. □

We have the following result regarding the uniform convergence of the densities.

Theorem 4.2. *For every $z \in I$ and compact $J \subseteq I$ we have*

$$\sup_{x\in J} |\rho_t(x) - 1| \xrightarrow{P_z} 0.$$

Proof. Let W, A and τ be as in corollary 2.2 and recall that we have (2.3). So by relation (2.6) of theorem 2.3 we have for every $x \in J$ and $t > 0$

$$\rho_t(x) = m(I)\frac{L^W_{\tau_t}(s(x))}{A_{\tau_t}} = Z_t(\tau_t/t^2, x),$$

where the random map $Z_t : \mathbb{R}^+ \times J \to \mathbb{R}$ is defined by

$$Z_t(u,x) = m(I)\frac{L^W_{t^2u}(s(x))}{A_{t^2u}}.$$

Lemma 3.1 implies that under P_z it holds that $Z_t \overset{\mathrm{d}}{=} Z'_t$, where Z'_t is defined by

$$Z'_t(u,x) = \frac{L^B_u\left(\frac{s(x)-z}{t}\right)}{\int_I L^B_u\left(\frac{s(y)-z}{t}\right)\,\mu(dy)}$$

and B is a standard Brownian motion. For every $a \geq 0$ we have

$$\sup_{u\leq a, x\in J}\left|L^B_u\left(\frac{s(x)-z}{t}\right) - L^B_u(0)\right| = \sup_{u\leq a, x\in\frac{1}{t}(s(J)-z)}\left|L^B_u(x) - L^B_u(0)\right| \overset{\text{as}}{\to} 0,$$

by the joint continuity of Brownian local time. Using the dominated convergence theorem we see in particular that

$$\sup_{u\leq a}\left|\int_I L^B_u\left(\frac{s(y)-z}{t}\right)\,\mu(dy) - L^B_u(0)\right| \overset{\text{as}}{\to} 0.$$

Combine the last three displays to conclude that for every $a \geq 0$ we have

$$\sup_{u\leq a, x\in J}|Z'_t(u,x) - 1| \overset{\text{as}}{\to} 0. \tag{4.4}$$

Now fix $\varepsilon, \eta > 0$. Note that for every $a \geq 0$ we have

$$\begin{aligned}
&P_z\left(\sup_{x\in J}|\rho_t(x) - 1| > \varepsilon\right)\\
&= P_z\left(\sup_{x\in J}|Z_t(\tau_t/t^2, x) - 1| > \varepsilon\right)\\
&\leq P_z\left(\sup_{x\in J, u\leq a}|Z_t(u,x) - 1| > \varepsilon\right) + P_z\left(\tau_t/t^2 > a\right)\\
&= P_z\left(\sup_{x\in J, u\leq a}|Z'_t(u,x) - 1| > \varepsilon\right) + P_z\left(\tau_t/t^2 > a\right).
\end{aligned}$$

By theorem 3.2 the ratio τ_t/t^2 has a weak limit, in particular it is asymptotically tight. We can thus choose a so large that

$$\limsup_{t\to\infty} P_z\left(\frac{\tau_t}{t^2} > a\right) < \tfrac{1}{2}\eta.$$

By (4.4) it holds that

$$P_z\big(\sup_{x\in J, u\le a} |Z_t'(u,x) - 1| > \varepsilon\big) < \tfrac{1}{2}\eta$$

for all t large enough. Consequently, we have

$$\limsup_{t\to\infty} P_z(|\rho_t(x) - 1| > \varepsilon) < \eta.$$

To complete the proof, let $\eta \downarrow 0$. □

Remark 4.3. If I is not compact, the support of the function ρ_t is a true subset of I (see remark 2.4). In that case, we almost surely have

$$\sup_{x\in I} |\rho_t(x) - 1| \ge 1.$$

So the convergence of ρ_t to 1 can only be uniform on the entire state space I if I is compact.

By theorem 4.2 and remark 4.3 we have $\|\rho_t - 1\|_\infty \xrightarrow{P} 0$ if and only if the state space I is compact. The following theorem shows that for convergence of L^p-norms with $p < \infty$, compactness of I is not necessary.

Theorem 4.4. *Let ν be a finite measure on I. Then for all $z \in I$ and $0 < p < \infty$*

$$\|\rho_t - 1\|_{L^p(I,\nu)} \xrightarrow{P_z} 0.$$

Proof. For every interval $J \subseteq I$ we have

$$\begin{aligned}\int_I |\rho_t(x) - 1|^p\,\nu(dx) &= \int_J |\rho_t(x) - 1|^p\,\nu(dx) + \int_{J^c} |\rho_t(x) - 1|^p\,\nu(dx)\\ &\le \nu(I)\left(\sup_{x\in J}|\rho_t(x) - 1|\right)^p + \nu(J^c)\,\|\rho_t - 1\|_\infty^p.\end{aligned}$$

Hence, for every $\varepsilon > 0$,

$$\begin{aligned}P_z\left(\|\rho_t - 1\|^p_{L^p(I,\nu)} > \varepsilon\right) &\le P_z\left(\sup_{x\in J}|\rho_t(x) - 1| > \left(\frac{\varepsilon}{2\nu(I)}\right)^{1/p}\right)\\ &\quad + P_z\left(\|\rho_t - 1\|_\infty > \left(\frac{\varepsilon}{2\nu(J^c)}\right)^{1/p}\right).\end{aligned}$$

Now let $\varepsilon, \eta > 0$ be arbitrary. Since ν has finite total mass, the number $(\varepsilon/2\nu(J^c))^{1/p}$ can be made arbitrarily large by choosing a sufficiently large compact interval $J \subseteq I$. By theorem 4.1, we thus choose a compact interval $J \subseteq I$ such that

$$\limsup_{t\to\infty} P_z\left(\|\rho_t - 1\|_\infty > \left(\frac{\varepsilon}{2\nu(J^c)}\right)^{1/p}\right) \le \eta.$$

For this particular compact interval J, theorem 4.2 implies that

$$\lim_{t\to\infty} P_z\left(\sup_{x\in J}|\rho_t(x)-1| > \left(\frac{\varepsilon}{2\nu(I)}\right)^{1/p}\right) = 0.$$

By combining the last three displays we find that

$$\limsup_{t\to\infty} P_z\left(\|\rho_t - 1\|^p_{L^p(I,\nu)} > \varepsilon\right) \le \eta.$$

To complete the proof, let $\eta \downarrow 0$. □

The following result is a simple consequence of theorem 4.4.

Corollary 4.5. *Let ν be a measure on I and suppose that $f \in L^1(I,\nu)$. Then for all $z \in I$*

$$\int_I f\rho_t\,d\nu \xrightarrow{P_z} \int_I f\,d\nu.$$

Proof. Since f is ν-integrable, the measure ν' given by $d\nu' = |f|\,d\nu$ is finite. Obviously,

$$\left|\int_I f\rho_t\,d\nu - \int_I f\,d\nu\right| \le \int_I |\rho_t - 1||f|\,d\nu = \|\rho_t - 1\|_{L^1(I,\nu')}.$$

Now apply theorem 4.4. □

If we take $\nu = \mu$ in corollary 4.5, we get the weak law of numbers for Lebegue integrals: for all $z \in I$ and $f \in L^1(\mu)$ we have

$$\frac{1}{t}\int_0^t f(X_u)\,du \xrightarrow{P_z} \int_I f\,d\mu.$$

The fact that we did not have to assume recurrence to get this result allows us to prove the following interesting corollary.

Corollary 4.6. *A regular diffusion with finite speed measure m is recurrent and therefore ergodic. The measure $\mu = m/m(I)$ is the unique invariant probability measure.*

Proof. Since X is recurrent on I if and only if $Y = s(X)$ is recurrent on $s(I)$, we can restrict ourselves to diffusions that are in natural scale. By theorem 20.15 of Kallenberg (1997), two things can happen if the speed measure is finite. Either the diffusion is recurrent and ergodic, or it converges almost surely to a boundary point of the state space I. We prove that the second situation can not occur. Suppose on the contrary that it does and take a compact subset J in the interior of I. Then almost surely, the diffusion X is outside J from a certain finite time on, so

$$\frac{1}{t}\int_0^t 1_J(X_u)\,du \xrightarrow{\text{as}} 0.$$

But on the other hand, the weak law of large numbers for Lebesgue integrals implies that

$$\frac{1}{t}\int_0^t 1_J(X_u)\,du \xrightarrow{P} \mu(J).$$

Since the speed measure gives positive mass to compact intervals in the interior of I (see for instance Kallenberg (1997), theorem 20.9), this leads to a contradiction. □

5 A useful expression for the densities

The following lemma gives a useful expression for the random functions $\rho_t - 1$. It will be used in the next two sections. We denote the distribution function of the invariant measure μ by F.

Lemma 5.1. *For every $x \in I$, define the function π_x on $\mathbb{R}$ by*

$$\pi_x(y) = 2m(I)(1_{\{y>s(x)\}} - F(s^{-1}(y))1_{s(I)}(y)$$

and let Π_x be an arbitrary primitive function of π_x. Then for all $z \in I$ we have

$$\rho_t(x) - 1 = \frac{1}{t}(\Pi_x(Y_t) - \Pi_x(s(z))) - \frac{1}{t}\int_0^t \pi_x(Y_u)\, dY_u \tag{5.1}$$

under P_z, where $Y = s(X)$.

Proof. The function π_x is the difference of to nonnegative functions, so Π_x is the difference of two convex functions. Consider the signed measure $\nu = 2m(I)(\delta_{s(x)} - \mu \circ s^{-1})1_{s(I)}$ on $\mathbb{R}$, where $\delta_{s(x)}$ denotes the Dirac measure at $s(x)$. Clearly, we have the relation $\pi_x(b) - \pi_x(a) = \nu(a, b]$ for all $a \leq b$. So by the generalized Itô formula

$$\Pi_x(Y_t) - \Pi_x(Y_0) = \int_0^t \pi_x(Y_s)\, dY_s + \tfrac{1}{2}\int_{\mathbb{R}} L_t^Y(y)\, \nu(dy).$$

By relation (2.5) of theorem 2.3 we have

$$\begin{aligned}\tfrac{1}{2}\int_{\mathbb{R}} L_t^Y(y)\, \nu(dy) &= m(I)L_t^Y(s(x)) - m(I)\int_{s(I)} L_t^Y(y)\, dF(s^{-1}(y)) \\ &= m(I)L_t^Y(s(x)) - m(I)\int_I L_t^Y(s(y))\, \mu(dy) \\ &= t(\rho_t(x) - 1).\end{aligned}$$

This completes the proof of the lemma. □

In the proofs of the central limit theorems that we derive in the next two sections, we will use the following lemma to deal with the first term on the right hand side of (5.1).

Lemma 5.2. *The primitives Π_x of the functions π_x defined in the statement of lemma 5.1 can be chosen in such a manner that for all $z \in I$*

$$\sup_{x\in I} \tfrac{1}{\sqrt{t}} |\Pi_x(Y_t) - \Pi_x(s(z))| \xrightarrow{P_z} 0.$$

Proof. Since the functions π_x are bounded in absolute value by $2m(I)$, we can choose the primitives Π_x in such a manner that for every $x \in I$ we have $|\Pi_x| \leq |\Pi|$, where Π is some function that does not depend on x. We then get

$$\sup_{x\in I} \tfrac{1}{\sqrt{t}} |\Pi_x(Y_t) - \Pi_x(s(z))| \leq \tfrac{1}{\sqrt{t}}(\Pi(Y_t) + \Pi(s(z)) \xrightarrow{P_z} 0.$$

since Y is ergodic. □

6 Finite-dimensional central limit theorems

If the state space I is an open interval (bounded or unbounded), then $Y = s(X)$ is an ergodic diffusion in natural scale on the open interval $s(I)$. Therefore, by theorem 20.15 of Kallenberg (1997), we must have $s(I) = \mathbb{R}$. The process Y is then a local martingale (see for instance Kallenberg (1997), theorem 20.7), which allows us to prove the results below. We use the notation ds to denote Stieltjes-integration with respect to the scale function s. As usual, $N_n(0, \Sigma)$ denotes the n-dimensional normal distribution with mean vector 0 and covariance matrix Σ. If ν is a signed measure on I, we can write $\nu = \nu' - \nu''$, where ν' and ν'' are true measures on I. We say that ν is finite if both ν' and ν'' are finite.

Theorem 6.1. *Suppose that $I = (l, r)$ is an open interval. let $\nu_1, \ldots, \nu_n$ be finite signed measures on I such that*

$$\int_I (\nu_i(l, x] - F(x)\nu_i(I))^2 \, ds(x) < \infty \tag{6.1}$$

for every i. Then for every $z \in I$ we have

$$\sqrt{t}\left(\int (\rho_t - 1)\, d\nu_1, \ldots, \int (\rho_t - 1)\, d\nu_n\right) \xrightarrow{P_z} N_n(0, \Sigma),$$

where

$$\Sigma_{i,j} = 4\, m(I) \int_I (\nu_i(l, x] - F(x)\nu_i(I))\, (\nu_j(l, x] - F(x)\nu_j(I))\, ds(x).$$

Proof. By lemma 5.1 we have

$$\begin{aligned}\sqrt{t}\int (\rho_t - 1)\, d\nu_i = {} & \frac{1}{\sqrt{t}}\int_I (\Pi_x(Y_t) - \Pi_x(s(z)))\, \nu_i(dx) \\ & - \frac{1}{\sqrt{t}}\int_I \left(\int_0^t \pi_x(Y_u)\, dY_u\right)\, \nu_i(dx).\end{aligned}$$

Since ν_i is finite, lemma 5.2 implies that the first term on the right hand side converges to 0 in probability. By the stochastic Fubini theorem (see Protter (1990)), the second term is equal to $-M_t^i/\sqrt{t}$, where M^i is defined by

$$M_t^i = \int_0^t \left(\int_I \pi_x(Y_u)\, \nu_i(dx)\right)\, dY_u.$$

It thus remains to prove that

$$\tfrac{1}{\sqrt{t}}\, (M_t^1, \ldots, M_t^n) \xrightarrow{P_z} N_n(0, \Sigma).$$

In view of the remarks preceding the theorem, every M^i is a continuous local martingale. Using relation (2.5) of theorem 2.3 we find that

$$
\begin{aligned}
&\tfrac{1}{t}\left\langle M^i, M^j\right\rangle_t = \\
&= \tfrac{1}{t}\int_0^t \Big(\int_I \pi_x(Y_u)\,\nu_i(dx)\Big)\Big(\int_I \pi_x(Y_u)\,\nu_j(dx)\Big)\,d\langle Y\rangle_u \\
&= \int_{\mathbb{R}} \Big(\int_I \pi_x(y)\,\nu_i(dx)\Big)\Big(\int_I \pi_x(y)\,\nu_j(dx)\Big)\,\tfrac{1}{t}L_t^Y(y)\,dy \\
&= \frac{1}{m(I)}\int_{\mathbb{R}} \Big(\int_I \pi_x(y)\,\nu_i(dx)\Big)\Big(\int_I \pi_x(y)\,\nu_j(dx)\Big)\,\rho_t(s^{-1}(y))\,dy \\
&= 4\,m(I)\int_I (\nu_i(l,y] - F(y)\nu_i(I))\,(\nu_j(l,y] - F(y)\nu_j(I))\,\rho_t(y)\,ds(y).
\end{aligned}
$$

Corollary 4.5 thus implies that

$$\tfrac{1}{t}\left\langle M^i, M^j\right\rangle_t \xrightarrow{P_z} \Sigma_{i,j}.$$

The statement of the theorem now follows from the central limit theorem for continuous local martingales (see for instance Van Zanten (2000a), where the CLT is proved by means of a time-change argument). □

Interesting corollaries of theorem 6.1 can be obtained by specifying the measures ν_i. If we take $\nu_i = \delta_{x_i}$ (the Dirac measure concentrated at x_i) for $i = 1, \ldots, n$, we get a multivariate central limit theorem for the densities ρ_t themselves. Condition (6.1) then reads as

$$\int_I (1_{(x_i,r)} - F)^2\,ds < \infty$$

for $i = 1, \ldots, n$. This is clearly a condition on the tails of the invariant measure μ. It is easily seen to be equivalent to the single condition

$$\int F^2(1-F)^2\,ds < \infty. \tag{6.2}$$

Hence, we arrive at the following result.

Corollary 6.2. *Suppose that $I = (l, r)$ is an open interval and that (6.2) holds. Then for all $z \in I$ and $x_1, \ldots, x_d \in I$ we have*

$$\sqrt{t}(\rho_t(x_1) - 1, \ldots, \rho_t(x_d) - 1) \overset{P_z}{\Longrightarrow} N_d(0, \Sigma),$$

where

$$\Sigma_{i,j} = 4\,m(I)\int_I (1_{(x_1,r)} - F)(1_{(x_d,r)} - F)ds.$$

If we take a function $f_i \in L^1(\mu)$, the signed measure ν_i defined by $d\nu_i = f_i\,d\mu$ is finite and it holds that

$$\int_I (\rho_t - 1)\,d\nu_i = \frac{1}{t}\int_0^t f_i(X_u)\,du - \int_I f_i\,d\mu.$$

Theorem 6.1 thus yields the following central limit theorem for Lebesgue integrals.

Corollary 6.3. *Suppose that $I = (l, r)$ is an open interval. Let $f_1, \ldots, f_n \in L^1(\mu)$ be such that $\int f_i \, d\mu = 0$ and*

$$\int_I \left(\int_l^x f_i(y) \, \mu(dy) \right)^2 ds(x) < \infty$$

for every i. Then for every $z \in I$ we have

$$\frac{1}{\sqrt{t}} \left(\int_0^t f_1(X_u) \, du, \ldots, \int_0^t f_n(X_u) \, du \right) \xrightarrow{P_z} N_d(0, \Sigma),$$

where

$$\Sigma_{i,j} = 4 \, m(I) \int_I \left(\int_l^x f_i(y) \, \mu(dy) \right) \left(\int_l^x f_j(y) \, \mu(dy) \right) ds(x).$$

7 Uniform central limit theorems

Suppose that we have a collection $\{\nu_\theta\}_{\theta \in \Theta}$ of finite signed measures on I, indexed by a set Θ. Consider the random maps Z_t on Θ given by

$$Z_t(\theta) = \sqrt{t} \int (\rho_t - 1) \, d\nu_\theta. \tag{7.1}$$

If ν is a signed measure on I, say $\nu = \nu' - \nu''$, where ν' and ν'' are true measures on I, then we write $\|\nu\| = \nu'(I) + \nu''(I)$. Under the assumption that

$$\sup_{\theta \in \Theta} \|\nu_\theta\| < \infty, \tag{7.2}$$

each Z_t is a random element of the space $\ell^\infty(\Theta)$ of bounded functions on Θ. In this section we prove a result regarding the weak convergence of the random maps Z_t in this space (see Van der Vaart and Wellner (1996) for the basic theory of weak convergence in ℓ^∞-spaces). For uncountable index sets Θ we will prove weak convergence in $\ell^\infty(\Theta^*)$ for every countable subset $\Theta^* \subseteq \Theta$, instead of weak convergence in $\ell^\infty(\Theta)$ itself. That way, we do not have to impose awkward continuity conditions on the Z_t. If Θ is finite, the result reduces to that of theorem 6.1. Given a metric d on Θ, we denote by $N(\varepsilon, \Theta, d)$ the minimal number of balls of d-radius ε that are needed to cover Θ.

Theorem 7.1. *Suppose that $I = (l, r)$ is an open interval. Let $\{\nu_\theta\}_{\theta \in \Theta}$ be a collection of finite signed measures on I, indexed by a set Θ. Suppose that (7.2) holds and that*

$$\int_I \left(\nu_\theta(l, x] - F(x) \nu_\theta(I) \right)^2 ds(x) < \infty$$

for every $\theta \in \Theta$. Moreover, assume that there exists a metric d on Θ such that (Θ, d) is bounded,

$$\int_0^1 \sqrt{\log N(\varepsilon, \Theta, d)} \, d\varepsilon < \infty \tag{7.3}$$

and

$$\int_I ((\nu_\theta - \nu_\psi)(l,x] - F(x)(\nu_\theta - \nu_\psi)(I))^2 \, ds(x) \le d^2(\theta,\psi) \tag{7.4}$$

for all $\theta, \psi \in \Theta$. Then for all $z \in I$ and countable $\Theta^* \subseteq \Theta$ we have $Z_t \overset{P_z}{\Longrightarrow} G$ in $\ell^\infty(\Theta^*)$, where G is a zero-mean Gaussian random map with covariance function

$$EG(\theta)G(\psi) = 4m(I)\int_I (\nu_\theta(l,x] - F(x)\nu_\theta(I))\,(\nu_\psi(l,x] - F(x)\nu_\psi(I))\; ds(x).$$

Proof. Using lemmas 5.1 and 5.2 and the stochastic Fubini theorem as in the proof of theorem 6.1, we see that it suffices to prove the weak convergence in $\ell^\infty(\Theta^*)$ of the random maps

$$\theta \mapsto \tfrac{1}{\sqrt{t}}\, M_t^\theta, \tag{7.5}$$

where the local martingales M^θ are defined by

$$M_t^\theta = \int_0^t \left(\int_I \pi_x(Y_u)\, \nu_\theta(dx)\right) dY_u.$$

The finite dimensional convergence follows from theorem 6.1, so we only have to show asymptotic equicontinuity. For that purpose we use a result of Nishiyama (2000), who gives sufficient conditions for equicontinuity of random maps of the form (7.5) in terms of the brackets of the local martingales M^θ. Under the conditions of the present theorem, theorem 3.4.2 of Nishiyama (2000) implies that it suffices to show that

$$\sup_{\theta\neq\psi\in\Theta^*} \frac{\frac{1}{t}\left\langle M^\theta - M^\psi\right\rangle_t}{d^2(\theta,\psi)} = O_{P_z}(1). \tag{7.6}$$

Using (2.5) and (7.4) we see that for $\theta, \psi \in \Theta$

$$\begin{aligned}
\frac{1}{t}\left\langle M^\theta - M^\psi\right\rangle_t &= \frac{1}{t}\int_0^t \left(\int_I \pi_x(Y_u)\,(\nu_\theta - \nu_\psi)(dx)\right)^2 d\left\langle Y\right\rangle_u \\
&= \int_{\mathbb{R}} \left(\int_I \pi_x(y)\,(\nu_\theta - \nu_\psi)(dx)\right)^2 \tfrac{1}{t}L_t^Y(y)\,dy \\
&= \frac{1}{m(I)}\int_I \left(\int_I \pi_x(s(y))\,(\nu_\theta - \nu_\psi)(dx)\right)^2 \rho_t(y) ds(y) \\
&= 4m(I)\int_I ((\nu_\theta - \nu_\psi)(l,y] - F(y)(\nu_\theta - \nu_\psi)(I))^2\, \rho_t(y)\, ds(y) \\
&\le 4m(I)\|\rho_t\|_\infty\, d^2(\theta,\psi).
\end{aligned}$$

We thus find that

$$\sup_{\theta\neq\psi\in\Theta^*} \frac{\frac{1}{t}\left\langle M^\theta - M^\psi\right\rangle_t}{d^2(\theta,\psi)} \le 4m(I)\|\rho_t\|_\infty.$$

By theorem 4.1 this is $O_{P_z}(1)$, so we have (7.6). □

The first corollary of theorem 7.1 extends the finite dimensional result of corollary 6.2. If B is a compact set in Euclidean space, we denote by $C(B)$ the space of continuous functions on B, endowed with the supremum norm. By remark 2.4 the (restrictions of the) densities ρ_t are random elements of the space $C(J)$ for every compact interval $J \subseteq I$. We will prove that the random functions $\sqrt{t}(\rho_t - 1)$ converge weakly in this space. The functions $\sqrt{t}(\rho_t - 1)$ can also be viewed as random elements of the space $C_b(I)$ of bounded, continuous functions on the whole interval I. But by remark 4.3 they can not have a weak limit in this space, since that would imply that $\|\rho_t - 1\|_\infty \to 0$ in probability.

Corollary 7.2. *Suppose that $I = (l, r)$ is an open interval and that (6.2) holds. Then for every $z \in I$ and compact $J \subseteq I$ we have the weak convergence*

$$\sqrt{t}(\rho_t - 1) \overset{P_z}{\Longrightarrow} G$$

in $C(J)$, where G is a zero-mean Gaussian random map with covariance function

$$EG(x)G(y) = 4\,m(I) \int_I (1_{(x,r)} - F)(1_{(y,r)} - F)ds.$$

Proof. Since the random functions $\sqrt{t}(\rho_t - 1)$ are continuous, it suffices to show that for some countable, dense $J^* \subseteq J$, they converge weakly in $\ell^\infty(J^*)$. To prove that this is the case we apply theorem 7.1 with $\Theta = J$ and $\nu_\theta = \delta_\theta$, the Dirac measure at θ. We define the metric d on Θ by putting $d(\theta, \psi) = (|s(\theta) - s(\psi)|)^{1/2}$. Since $s : I \to s(I)$ is a homeomorphism and Θ is compact, the set $s(\Theta)$ is a compact interval again. This implies first of all that (Θ, d) is a bounded metric space. Note also that

$$N(\varepsilon, d, \Theta) = N(\varepsilon^2, |\cdot|, s(\Theta)) \le C/\varepsilon^2$$

for some constant $C > 0$, so that the entropy condition (7.3) is satisfied. To prove that we have (7.4), note that for $\theta \le \psi$ the left hand side of (7.4) is in this case equal to

$$\int_I 1_{[\theta,\psi)}(x)\, ds(x) = s(\psi) - s(\theta).$$

This completes the proof of the corollary. □

The following simple corollary gives a uniform version of the central limit theorem for Lebesgue integrals of corollary 6.3.

Corollary 7.3. *Suppose that $I = (l, r)$ is an open interval. Let $\{f_\theta : \theta \in \Theta\}$ be a collection of functions that is bounded in $L^1(\mu)$, such that $\int f_\theta\, d\mu = 0$ and*

$$\int_I \left(\int_l^x f_\theta(y)\, \mu(dy) \right)^2 ds(x) < \infty$$

for every $\theta \in \Theta$. Moreover, assume that there exists a metric d on Θ such that (Θ, d) is bounded, condition (7.3) holds, and

$$\int_I \left(\int_l^x (f_\psi(w) - f_\theta(w))\, \mu(dw) \right)^2 ds(x) \le d^2(\theta, \psi) \tag{7.7}$$

for all $\theta, \psi \in \Theta$. Then for all $z \in I$ and countable $\Theta^* \subseteq \Theta$ the random maps

$$\theta \mapsto \frac{1}{\sqrt{t}} \int_0^t f_\theta(X_u)\, du$$

converge weakly in $\ell^\infty(\Theta^*)$ to G under P_z, where G is a zero-mean Gaussian random map with covariance function

$$EG(\theta)G(\psi) = 4\, m(I) \int_I \left(\int_l^x f_\theta(y)\, \mu(dy) \right) \left(\int_l^x f_\psi(y)\, \mu(dy) \right) ds(x).$$

Proof. The result follows immediately from theorem 7.1 by taking $d\nu_\theta = f_\theta\, d\mu$. □

The result of corollary 7.3 is still fairly general. In concrete cases, an additional analysis will be needed to verify the conditions of the corollary. We close this section with two examples. For functions that depend in a differentiable manner on a Euclidean parameter, we have the following uniform central limit theorem.

Corollary 7.4. *Suppose that $I = (l, r)$ is an open interval. Let $\Theta \subseteq \mathbb{R}^n$ be compact, convex set and let $\{f_\theta : \theta \in \Theta\}$ be a collection of functions that is bounded in $L^1(\mu)$, such that $\int f_\theta\, d\mu = 0$ and*

$$\int_I \left(\int_l^x f_\theta(y)\, \mu(dy) \right)^2 ds(x) < \infty$$

for every $\theta \in \Theta$. Moreover, assume that $f_\theta(x)$ is continuously differentiable in θ for every $x \in I$, the partial derivatives $\frac{\partial}{\partial \theta_i} f_\theta(x)$ are jointly measurable and μ-integrable in x for every $\theta \in \Theta$, and

$$\sup_{\theta \in \Theta} \int_I \left(\int_l^x \tfrac{\partial}{\partial \theta_i} f_\theta(w)\, \mu(dw) \right)^2 ds(x) < \infty$$

for every i. Then for all $z \in I$, the random maps

$$\theta \mapsto \frac{1}{\sqrt{t}} \int_0^t f_\theta(X_u)\, du$$

converge weakly in $C(\Theta)$ to G under P_z, where G is a zero-mean Gaussian random map with covariance function

$$EG(\theta)G(\psi) = 4\, m(I) \int_I \left(\int_l^x f_\theta(y)\, \mu(dy) \right) \left(\int_l^x f_\psi(y)\, \mu(dy) \right) ds(x).$$

Proof. The random maps are clearly continuous in θ, so it suffices to prove that for some countable, dense $\Theta^* \subseteq \Theta$, the weak convergence takes place in $\ell^\infty(\Theta^*)$. The convexity of Θ and the fact that $f_\theta(w)$ is continuously differentiable in θ imply that

$$f_\psi(w) - f_\theta(w) = \sum_{i=1}^n (\psi_i - \theta_i) \int_0^1 \tfrac{\partial}{\partial \theta_i} f_{(1-t)\theta + t\psi}(w)\, dt.$$

Using the regularity of the derivatives, Fubini's theorem, the Cauchy-Schwarz inequality and Jensen's inequality, it follows that

$$\left(\int_l^x (f_\psi(w) - f_\theta(w))\,\mu(dw)\right)^2$$
$$\leq \|\psi - \theta\|^2 \sum_{i=1}^n \int_0^1 \left(\int_l^x \tfrac{\partial}{\partial\theta_i} f_{(1-t)\theta+t\psi}(w)\,\mu(dw)\right)^2 dt.$$

Another application of Fubini's theorem yields

$$\begin{aligned}
&\int_I \left(\int_l^x (f_\psi(w) - f_\theta(w))\,\mu(dw)\right)^2 ds(x) \\
&\leq \|\psi - \theta\|^2 \sum_{i=1}^n \int_0^1 \left(\int_I \left(\int_l^x \tfrac{\partial}{\partial\theta_i} f_{(1-t)\theta+t\psi}(w)\,\mu(dw)\right)^2 ds(x)\right) dt \\
&\leq \|\psi - \theta\|^2 \sum_{i=1}^n \sup_{\theta\in\Theta} \int_I \left(\int_l^x \tfrac{\partial}{\partial\theta_i} f_\theta(w)\,\mu(dw)\right)^2 ds(x) \\
&= C^2 \|\psi - \theta\|^2,
\end{aligned}$$

with

$$C = \left(\sum_{i=1}^n \sup_{\theta\in\Theta} \int_I \left(\int_l^x \tfrac{\partial}{\partial\theta_i} f_\theta(w)\,\mu(dw)\right)^2 ds(x)\right)^{1/2}.$$

So if we define the metric d on Θ by $d(\theta, \psi) = C\|\theta - \psi\|$, all conditions of corollary 7.3 are satisfied. □

The usefulness of corollary 7.3 is not restricted to classes of functions that depend smoothly on a parameter. To illustrate this we prove a functional central limit theorem for the empirical distribution functions F_t defined by $F_t(x) = \mu_t(l, x]$. The result requires the tail condition (6.2) again. Note that (6.2) holds if and only if

$$\int_I (F(u \wedge x) - F(u)F(x))^2\, ds(u) < \infty$$

for all $x \in I$.

Corollary 7.5. *Suppose that $I = (l, r)$ is an open interval and that (6.2) holds. Then for all $z \in I$ and compact $J \subseteq I$ we have the weak convergence*

$$\sqrt{t}(F_t - F) \overset{P_z}{\Longrightarrow} G$$

in $\ell^\infty(J)$, where G is a zero-mean Gaussian random map with covariance function

$$EG(x)G(y) = 4\,m(I) \int_I (F(u \wedge x) - F(u)F(x))(F(u \wedge y) - F(u)F(y))ds(u).$$

Proof. Since the random functions $\sqrt{t}(F_t - F)$ are continuous from the right, it suffices to show that for some countable, dense subset $J^* \subseteq J$, they converge weakly in $\ell^\infty(J^*)$. We apply corollary 7.3 with $\Theta = J$ and $f_\theta = 1_{(l,\theta]} - F(\theta)$. Some elementary calculations show that in this case

$$\begin{aligned}\int_I \left(\int_l^x (f_\psi(w) - f_\theta(w))\, \mu(dw) \right)^2 ds(x) \\ = (F(\theta) - F(\psi))^2 \int_I (T_{\theta,\psi}(x) - F(x))^2 \, ds(x),\end{aligned} \tag{7.8}$$

where for $\theta \leq \psi$, the function $T_{\theta,\psi}$ is defined by

$$T_{\theta,\psi}(x) = \begin{cases} 0 & , x \leq \theta, \\ \frac{F(x)-F(\theta)}{F(\psi)-F(\theta)} & , \theta < x \leq \psi, \\ 1 & x > \psi. \end{cases}$$

Since J is compact and therefore bounded, the assumption on the tails of the invariant measure implies that

$$\sup_{\theta,\psi \in J} \int_I (T_{\theta,\psi}(x) - F(x))^2 \, ds(x) < \infty. \tag{7.9}$$

So we have a finite constant $C > 0$ such that

$$\int_I \left(\int_l^x (f_\psi(w) - f_\theta(w))\, \mu(dw) \right)^2 ds(x) \leq C^2 \left(F(\theta) - F(\psi)\right)^2.$$

Since m gives positive mass to non-empty intervals (see for instance Kallenberg (1997), theorem 20.9), F is strictly increasing and therefore injective. It follows that $d(x,y) = C|F(x) - F(y)|$ defines a metric on J. Obviously, (J,d) is bounded. Since F maps I into $[0,1]$ we have

$$N(\varepsilon, J, d) \leq N(\varepsilon/C, [0,1], |\cdot|) \leq C/\varepsilon,$$

so that the entropy condition (7.3) is satisfied. □

Remark 7.6. Note that in the proof of the last corollary, we only used the compactness of J to show (7.9). There are examples of diffusions for which

$$\sup_{\theta,\psi \in I} \int_I (T_{\theta,\psi}(x) - F(x))^2 \, ds(x) < \infty$$

(see Negri (1998)). In that case, the weak convergence of $\sqrt{t}(F_t - F)$ takes place in the space $\ell^\infty(I)$.

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