DOI: 10.1002/jgt.22484

ARTICLE

WILEY

On packing spanning arborescences with matroid constraint

Quentin Fortier[1] | **Csaba Király**[2] | **Zoltán Szigeti**[1] | **Shin-ichi Tanigawa**[3,4]

[1]G-SCOP, Univ. Grenoble Alpes, Grenoble, France

[2]MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Department of Operations Research, ELTE Eötvös Loránd University, Budapest, Hungary

[3]Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan

[4]Centrum Wiskunde & Informatica (CWI), Amsterdam, The Netherlands

Correspondence
Csaba Király, MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Department of Operations Research, ELTE Eötvös Loránd University, Pázmány Péter sétány 1/C, Budapest 1117, Hungary.
Email: cskiraly@cs.elte.hu

Funding information
New National Excellence Program of the Ministry of Human Capacities of Hungary, Grant/Award Number: ÚNKP-17-4; JSPS Postdoctoral Fellowships for Research Abroad JSPS Grant-in-Aid for Scientific Research (B), Grant/Award Number: 5280004; Országos Tudományos Kutatási Alapprogramok, Grant/Award Numbers: FK_18 NKFI-128673, K109240; Laboratory G-SCOP, Grant/Award Number: RIME; Hausdorff Research Institute, University of Bonn: Hausdorff Trimester Pogram on Combinatorial Optimization

Abstract

Let $D = (V + s, A)$ be a digraph with a designated root vertex s. Edmonds' seminal result (see J. Edmonds [4]) implies that D has a packing of k spanning s-arborescences if and only if D has a packing of $k\,(s, t)$-paths for all $t \in V$, where a packing means arc-disjoint subgraphs. Let $\mathcal{M}$ be a matroid on the set of arcs leaving s. A packing of (s, t)-paths is called $\mathcal{M}$-based if their arcs leaving s form a base of $\mathcal{M}$ while a packing of s-arborescences is called $\mathcal{M}$-based if, for all $t \in V$, the packing of (s, t)-paths provided by the arborescences is $\mathcal{M}$-based. Durand de Gevigney, Nguyen, and Szigeti proved in [3] that D has an $\mathcal{M}$-based packing of s-arborescences if and only if D has an $\mathcal{M}$-based packing of (s, t)-paths for all $t \in V$. Bérczi and Frank conjectured that this statement can be strengthened in the sense of Edmonds' theorem such that each s-arborescence is required to be spanning. Specifically, they conjectured that D has an $\mathcal{M}$-based packing of spanning s-arborescences if and only if D has an $\mathcal{M}$-based packing of (s, t)-paths for all $t \in V$. In this paper we disprove this conjecture in its general form and we prove that the corresponding decision problem is NP-complete. We also prove that the conjecture holds for several fundamental classes of matroids, such as graphic matroids and transversal matroids. For all the results presented in this paper, the undirected counterpart also holds.

KEYWORDS
connectivity, Edmonds' branching theorem, matroid, packing arborescences

1 | INTRODUCTION

The packing problem in digraphs is one of the fundamental topics in graph theory and combinatorial optimization, where the goal is to find the largest family of disjoint subgraphs satisfying a specified property in a given digraph. In this paper, by packing subgraphs, we always mean a set of arc-disjoint subgraphs.

Suppose that we are given a *rooted digraph*, that is, a digraph $D = (V + s, A)$ with a designated root vertex s. An *s-arborescence* is a directed tree $\overrightarrow{T}$ rooted at s, that is, the underlying undirected graph T is a tree and every vertex except s has in-degree one in $\overrightarrow{T}$. An s-arborescence $\overrightarrow{T}$ is said to be *spanning* if it contains all the vertices of D. If D has a packing of k spanning s-arborescences, then D has a packing of k (s, t)-paths for every $t \in V$, since each of the arborescences contains an (s, t)-path. The celebrated Edmonds theorem gives the exact relation between spanning arborescence packings and path packings as follows.

Theorem 1.1 (Edmonds [4]). *There exists a packing of k spanning s-arborescences in a rooted digraph $D = (V + s, A)$ if and only if there exists a packing of k (s, t)-paths in D for every $t \in V$.*

The problem of packing k (s, t)-paths is equivalent to asking whether one can send k distinct commodities from s to t by assuming that each arc can transmit at most one commodity. Then what happens if commodities have an involved independence structure? Here we are interested in a situation that each commodity c_i is assigned to some vertex s_i at the beginning, and we would like to know whether every vertex can receive a sufficient amount of independent commodities to understand the whole structure. By adding an auxiliary root vertex s and arcs from s to s_i for each i, we may convert the situation such that all commodities are assigned to the root s and each arc from the root can be used to transmit only a particular commodity.

More formally, suppose that we are given a *matroid-rooted digraph* $(D = (V + s, A), \mathcal{M})$, that is, a matroid $\mathcal{M}$ is given on the set of arcs leaving the root s that we call *root arcs*. We are interested in a packing of (s, t)-paths whose root arcs form a base of $\mathcal{M}$. Such a packing is said to be an *$\mathcal{M}$-based packing* of (s, t)-paths. A packing of s-arborescences is called *$\mathcal{M}$-based* if, for all $t \in V$, the packing of (s, t)-paths provided by the arborescences that contain t is $\mathcal{M}$-based. Figure 1 illustrates an example.

A natural question is whether Edmonds' theorem can be extended for $\mathcal{M}$-based packings. The result of Durand de Gevigney, Nguyen, and Szigeti [3] gives a partial answer to this question.

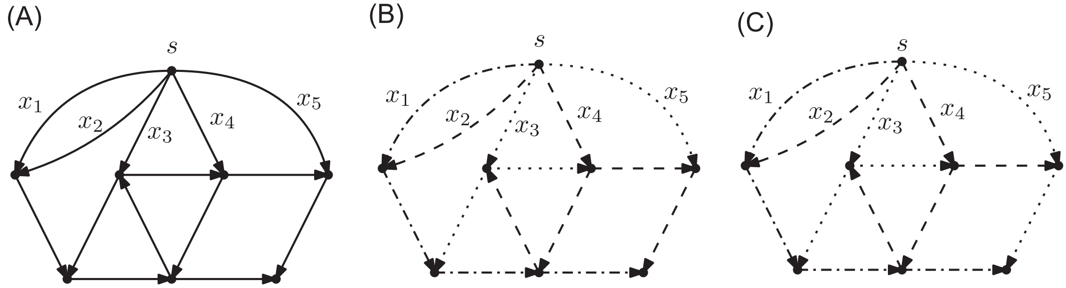

FIGURE 1 An example of $\mathcal{M}$-based packings of arborescences. Suppose that a given digraph is as in (A) and $\mathcal{M}$ is a matroid on the set $\{x_1,\ldots,x_5\}$ of the root arcs. Suppose also that $\mathcal{M}$ has rank two, has no loop, and has two circuits $\{x_1, x_5\}$ and $\{x_2, x_3\}$ of rank one. (B) and (C) illustrate the two distinct packings of three arc-disjoint s-arborescences. Observe that (C) is not $\mathcal{M}$-basic as the corresponding two paths from s to the right bottom vertex is not $\mathcal{M}$-basic (because $\{x_1, x_5\}$ is dependent). On the other hand, (B) is $\mathcal{M}$-basic

Theorem 1.2 (Durand de Gevigney et al [3]). *Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. Then there exists an $\mathcal{M}$-based packing of s-arborescences in D if and only if there exists an $\mathcal{M}$-based packing of (s, t)-paths in D for every $t \in V$.*

Notice that at the quantitative level, Theorem 1.1 always guarantees the existence of k spanning s-arborescences while the number of s-arborescences in Theorem 1.2 may be more than the rank of $\mathcal{M}$ since these arborescences are not necessarily spanning.

Bérczi and Frank [8] conjectured that Theorem 1.2 can be strengthened in the sense of Edmonds' theorem. This conjecture appeared also in a paper of Bérczi, Király, and Kobayashi [2]. More formally, the conjecture is the following.

Conjecture 1.3 (Bérczi et al [2]). *Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D if and only if there exists an $\mathcal{M}$-based packing of (s, t)-paths in D for every $t \in V$.*

The main result of this paper is that Conjecture 1.3 is false in its general form. We will even prove that the following decision problem is NP-complete, which was conjectured by Bérczi-Kovács [8].

Problem 1.4. Given a matroid-rooted digraph $(D = (V + s, A), \mathcal{M})$, decide whether there exists an $\mathcal{M}$-based packing of spanning s-arborescences in D.

As positive results, we will prove that Conjecture 1.3 is true for several fundamental classes of matroids such as graphic and transversal matroids.

1.1 | Related works

Connectivity is one of the most well-studied properties of graphs. The earliest results related to our main interest on packing problems concerning connectivity are the papers of Nash-Williams [17] and Tutte [20] on packing trees in undirected graphs from 1961. The topic of packing arborescences has been extensively studied in the seventies by Edmonds [4] and Frank [6]. The connection between these problems was pointed out in a work of Frank [7] on orientations of graphs.

The hypergraphic counterparts of the above packing results were discovered by Frank et al [9,10]. A surprising extension of Edmonds' result was given by Katoh, Kamiyama, and Takizawa [13] and Fujishige [11] for the case when no spanning arborescences exist. Szegő [19] gave an abstract version of Edmonds' result that was extended to an abstract version of the result of [13] in a paper of Bérczi and Frank [1].

Investigations in rigidity theory inspired an extensive research on possible extensions of Nash-Williams' and Tutte's result. Katoh and Tanigawa [14] introduced the concept of matroid-based packing of rooted trees and presented several applications of this result in rigidity theory. Durand de Gevigney, Nguyen, and Szigeti [3] used the techniques of Frank to show that, by an extension of Edmonds' result, an alternative proof of the packing result of [14] can be obtained. These breakthrough results inspired an intensive research in the last few years on this topic to extend the above mentioned results, see [2,5,15,16].

2 | DEFINITIONS

We will use some basic terms from matroid theory listed below. For details, we refer to [18]. Recall that, for a set function $r: 2^{\mathsf{S}} \to \mathbb{Z}_+$, $\mathcal{M} = (\mathsf{S}, r)$ is called a *matroid* if $r(\emptyset) = 0$, monotone nondecreasing, subcardinal ($r(\mathsf{Q}) \le |\mathsf{Q}|$) and submodular ($r(\mathsf{P}) + r(\mathsf{Q}) \ge r(\mathsf{P} \cap \mathsf{Q}) + r(\mathsf{P} \cup \mathsf{Q})$). The members of $\mathcal{I} = \{\mathsf{Q} \subseteq \mathsf{S}: r(\mathsf{Q}) = |\mathsf{Q}|\}$ are called *independent* sets of the matroid and r is called the *rank function* of the matroid. It is well known that a matroid can also be defined by its independent sets. Let $\mathsf{Q} \subseteq \mathsf{S}$. The maximal independent sets in Q are called *bases* of Q. Note that all bases are of the same size. The bases of S are called the bases of $\mathcal{M}$. The *rank* of $\mathcal{M}$, denoted by $\boldsymbol{r}(\mathcal{M})$, is the size of a base of $\mathcal{M}$. We define $\mathbf{Span}(\mathsf{Q}) := \{\mathsf{s} \in \mathsf{S}: r(\mathsf{Q} \cup \{\mathsf{s}\}) = r(\mathsf{Q})\}$. Note that Span is monotone. Two elements $\mathsf{a}, \mathsf{a}' \in \mathsf{S}$ are said to be *parallel* in $\mathcal{M} = (\mathsf{S}, r)$ (in notation, $\mathsf{a} \| \mathsf{a}'$) if $r(\{\mathsf{a}\}) = r(\{\mathsf{a}'\}) = r(\{\mathsf{a}, \mathsf{a}'\}) = 1$.

The following classes of matroids will be discussed in this paper:

1. *graphic matroid*: given a graph $G = (V, E)$ with a bijection $\pi: E \to \mathsf{S}$, $\mathcal{I} := \{\pi(F): F$ is the edge set of a forest of $G\}$;
2. *Fano matroid*: a rank-three matroid derived from the Fano plane (the smallest projective plane with seven points) on a seven element ground set (the points of the Fano plane) where every set of cardinality three is a base except the lines of the Fano plane;
3. *transversal matroid*: given a bipartite graph $G = (S, T; E)$ with a bijection $\pi: S \to \mathsf{S}$, $\mathcal{I} := \{\pi(X): X \subseteq S$ that can be covered by a matching in $G\}$;
4. *linear matroid*: given a finite set of vectors $A \subseteq \mathbb{F}^d$ for a field $\mathbb{F}$ and a positive integer d, $\mathcal{I} := \{X \subseteq A$: the vectors in X are independent$\}$.

A special class of the transversal matroids where G is the complete bipartite graph $K_{n,k}$ is called the *uniform matroid* $\boldsymbol{U_{k,n}}$. It is well known that a graphic matroid $\mathcal{M}$ is always representable by a connected graph on $r(\mathcal{M}) + 1$ vertices and a transversal matroid $\mathcal{M}$ is always representable by a bipartite graph $G = (S, T; E)$ where $|S| = |\mathsf{S}|$ and $|T| = r(\mathcal{M})$. It is also well known that a matroid of rank at most three is not graphic if and only if it has a minor isomorphic to the Fano matroid or $U_{2,4}$ (see, eg, [18]).

Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph, that is, a pair of a digraph D with a designated vertex s and a matroid $\mathcal{M}$ on the set of arcs leaving s. For an s-arborescence T in D and a vertex $v \neq s$ of T, we denote the unique path from s to v by $\boldsymbol{T}[\boldsymbol{s}, \boldsymbol{v}]$, and its first arc by $\boldsymbol{e}_{\boldsymbol{T}[\boldsymbol{s},\boldsymbol{v}]}$. With this definition, a packing $\{T_1,...,T_k\}$ of s-arborescences in a digraph D is $\mathcal{M}$-based if and only if $\{e_{T_i[s,v]} : i \in \{1,...,k\}, v \in V(T_i)\}$ is a base of $\mathcal{M}$ for every $v \in V$.

For disjoint sets $X, Y \subseteq V + s$, we denote by $\boldsymbol{\partial^D_X(Y)}$ the subset of arcs in D with tail in X and head in Y. The superscript D will be omitted, when it is clear from the context. The in-degree of a set $X \subseteq V + s$ is denoted by $\boldsymbol{\varrho_D(X)} := |\partial^D_{V+s-X}(X)|$.

We say that a matroid-rooted digraph $(D, \mathcal{M})$ is *rooted $\mathcal{M}$-arc-connected* if there exists an $\mathcal{M}$-based packing of (s, t)-paths for all vertices t in V. One can easily prove a Menger-type theorem saying that D is rooted $\mathcal{M}$-arc-connected if and only if

$$r(\partial_s(X)) + \varrho_{D-s}(X) \geq r(\mathcal{M}) \quad \text{for all} \quad X \subseteq V. \tag{1}$$

We say that a packing of arborescences *covers* $\partial_s(V)$ if every root arc is contained in some arborescence in the packing. For simplicity, we will call an $\mathcal{M}$-based packing of spanning s-arborescences in D that covers $\partial_s(V)$ a *feasible packing*.

3 | POSITIVE RESULTS

In this section, we prove Conjecture 1.3 for several special cases. The necessity of Conjecture 1.3 is always true by Theorem 1.2 (and is easy to prove anyway), so we will only prove the sufficiency in each case.

3.1 | Overview of the proof of Theorem 1.2

Some of our positive results are obtained by extending the proof of Theorem 1.2 given by [3], and hence we shall first review it by introducing several key ingredients used later. In [3], Theorem 1.2 was proved in a slightly stronger form by imposing an extra technical condition as follows. Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. D is called *$\mathcal{M}$-independent* if $\partial_s(v)$ is independent in $\mathcal{M}$ for every $v \in V$. This condition ensures that each root arc can be used in an $\mathcal{M}$-based packing of s-arborescences in D, as follows.

Theorem 3.1 (Durand de Gevigney et al [3]). *Let $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ be a matroid-rooted digraph. There exists an $\mathcal{M}$-based packing of s-arborescences in D that covers $\partial_s(V)$ if and only if D is rooted $\mathcal{M}$-arc-connected and $\mathcal{M}$-independent.*

Observe that Theorem 3.1 implies Theorem 1.2. Indeed, if $(D, \mathcal{M})$ is rooted $\mathcal{M}$-arc-connected, then by omitting some root arcs of $(D, \mathcal{M})$, one can get a rooted $\mathcal{M}'$-arc-connected and $\mathcal{M}'$-independent digraph, where $\mathcal{M}'$ is a submatroid of $\mathcal{M}$ with the same rank. Applying Theorem 3.1 to the resulting instance, we get an $\mathcal{M}'$-based packing, which is also an $\mathcal{M}$-based packing in the original instance.

Let $(D, \mathcal{M})$ be as in Theorem 3.1. The following graphical operation is frequently used in the proof of Theorem 3.1 in [3]: for a nonroot arc uv and $x \in \partial_s(u)$, *shifting* of $(D, \mathcal{M})$ along (uv, x)

is a new instance $(D', \mathcal{M}')$ obtained from $(D, \mathcal{M})$ by removing uv and inserting a new root arc $x' = sv$ such that x' is a parallel element to x in the underlying matroid (Figure 2).

We want to construct the new instance $(D', \mathcal{M}')$ such that the conditions of Theorem 3.1 are satisfied. We say that a nonroot arc uv is *good* if $\partial_s(u) \not\subseteq \mathrm{Span}_{\mathcal{M}}(\partial_s(v))$. A pair (uv, x) is called *good* if $x \in \partial_s(u) - \mathrm{Span}_{\mathcal{M}}(\partial_s(v))$. We call $X \subseteq V$ *tight* if (1) holds with equality. A pair (uv, x) is said to be *admissible* if there is no tight set X with $v \in X$ and $u \notin X$ such that x is in the span of $\partial_s(X)$. The key observations proved in [3] are the following (see case 2 in the proof of Theorem 1.6 in [3])

Lemma 3.2 (Durand de Gevigney et al [3]). *For a matroid-rooted digraph $(D, \mathcal{M})$, the following hold:*

- **(a)** *The shifting $(D', \mathcal{M}')$ along (uv, x) is $\mathcal{M}'$-independent (resp. rooted $\mathcal{M}'$-arc-connected) if and only if (uv, x) is good (resp. admissible).*
- **(b)** *If no good arc exists, then the set of root arcs form a feasible packing.*
- **(c)** *If D has a good arc, then D has a good admissible pair (uv, x).*

The proof of the sufficiency of Theorem 3.1 is done by induction on the number of nonroot arcs. By Lemma 3.2(b), if no good arc exists, then the set of root arcs forms a feasible packing. Otherwise, by Lemma 3.2(c), there exists a good admissible pair (e, x), and hence, by Lemma 3.2(a), the shifting $(D', \mathcal{M}')$ along (e, x) is $\mathcal{M}'$-independent and rooted $\mathcal{M}'$-arc-connected. By induction, there exists an $\mathcal{M}'$-based packing $\mathcal{T}$ of s-arborescences in D' such that it covers $\partial_s^{D'}(V)$. We can suppose that each s-arborescence in $\mathcal{T}$ has exactly one root arc since otherwise we can split it into several s-arborescences to satisfy this condition. Let $T \in \mathcal{T}$ be the arborescence covering x and $T' \in \mathcal{T}$ be the arborescence covering the new root arc x' in D'. Then $(\mathcal{T} - \{T, T'\}) \cup \{T \cup (T' - x') + e\}$ is a desired $\mathcal{M}$-based packing of s-arborescences in D that covers $\partial_s(V)$, and this completes the proof of Theorem 3.1.

Now consider applying the proof to Conjecture 1.3. In the same manner, by induction, one gets an $\mathcal{M}'$-based packing $\mathcal{T}$ of *spanning* s-arborescences in D' that covers $\partial_s^{D'}(V)$. Our goal is to construct a feasible packing in D based on $\mathcal{T}$. Let $T \in \mathcal{T}$ be the arborescence that covers the new root arc x' of D'. If T also contains x, then $(\mathcal{T} - \{T\}) \cup \{T - x' + e\}$ is an $\mathcal{M}$-based packing of spanning s-arborescences in D that covers $\partial_s(V)$, and we are done. The difficult case is when T does not contain x. We will show how to overcome this difficulty by new ideas if $\mathcal{M}$ has rank at most two or is graphic.

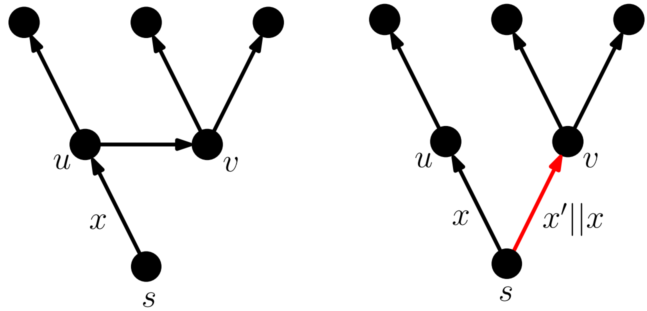

FIGURE 2 The operation shifting [Color figure can be viewed at wileyonlinelibrary.com]

3.2 | Matroids of rank at most 2

In this section we prove that Conjecture 1.3 is true when $r(\mathcal{M}) \leq 2$. We first prove the following technical lemma on changing spanning arborescences. We do not need its most general version but it may be of some interest for later applications.

Lemma 3.3. *Let T_1 and T_2 be two spanning s-arborescences on $V + s$. For $i = 1, 2$, let $F_i \subseteq \partial_s^{T_i}(V)$, and let $V_i := \{v \in V: e_{T_i[s,v]} \in F_i\}$. Let T_1^* and T_2^* be obtained from T_1 and T_2 by exchanging the arcs entering v for every $v \in V_1 \cap V_2$. Then T_1^* and T_2^* are spanning s-arborescences on $V + s$.*

Proof. It suffices to prove the statement for T_1^*. Suppose that T_1^* is not an s-arborescence. Since $\varrho_{T_1^*}(v) = \varrho_{T_1}(v) = 1$ for every $v \in V$, there exists a directed circuit C in T_1^*. Since neither T_1 nor T_2 contains a directed circuit, C contains at least one arc from each arborescence T_1 and T_2. It follows that there exist not necessarily distinct arcs uv and wz of C with $u \neq z$ such that uv and wz belong to T_2 and the path of C from z to u belongs to T_1. Note then that the arc a_1 entering u in T_1 is the same as that in T_1^* as T_1^* contains C and $a_1 \in C$ by the choice of uv. This implies $u \notin V_1 \cap V_2$.

Since uv belongs to T_2 and to T_1^*, v is in $V_1 \cap V_2$ and hence in V_2, and thus u is also in V_2. Since wz belongs to T_2 and to T_1^*, z is in $V_1 \cap V_2$ and hence in V_1, and thus, since the path of C from z to u belongs to T_1, u is also in V_1. It follows that u is in $V_1 \cap V_2$, and we have a contradiction to $u \notin V_1 \cap V_2$. □

Theorem 3.4. *Let $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ be a matroid-rooted digraph with $r(\mathcal{M}) \leq 2$. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D that covers $\partial_s(V)$ if and only if D is $\mathcal{M}$-independent and rooted $\mathcal{M}$-arc-connected.*

Proof. The proof is done by induction on the number of nonroot arcs.

If no good arc exists, then, by Lemma 3.2(b), $\partial_s(v)$ is a base of $\mathcal{M}$ for all $v \in V$. Then we can define $r(\mathcal{M})(=1$ or $2)$ arc-disjoint spanning s-arborescences by distributing the $r(\mathcal{M})$ arcs of $\partial_s(v)$ arbitrarily between them for all $v \in V$. This way we obtain a feasible packing of D.

Hence we assume that D has a good arc. Then, by Lemma 3.2(c), there exists a good admissible pair (u_0v_0, x), and by Lemma 3.2(a), for the shifting $(D', \mathcal{M}')$ along (u_0v_0, x), D' is $\mathcal{M}'$-independent and rooted $\mathcal{M}'$-arc-connected. Now, by induction, there exists a feasible packing in D'. Let x' be the new root arc in D' from s to v_0. We have the following two cases:

Case 1. If x and x' are contained in the same arborescence T of the packing, then substituting T with $T - x' + u_0v_0$ in the packing one gets a feasible packing in D.

Case 2. Otherwise, the packing consists of two arborescences T_1 and T_2 (thus the rank of $\mathcal{M}'$ is two), and we can assume that x is in T_1 and x' is in T_2. Let F_1 be the set of the root arcs used in T_1, and F_2 be the set of the root arcs used in T_2 and parallel to x (including x'). As in Lemma 3.3, we take $V_i = \{v \in V: e_{T_i[s,v]} \in F_i\}$ and consider T_1^* and T_2^* that arise from T_1 and T_2 by exchanging the arcs entering v for every $v \in V_1 \cap V_2$. (Note that $V_1 = V$ as T_1 is spanning, and $u_0 \notin V_2$ as, otherwise, $e_{T_1[s,u_0]} = x \| e_{T_2[s,u_0]}$ which contradicts our assumption that T_1 and T_2 form a feasible packing.) We claim the following. □

Claim 3.5. $\{T_1^*, T_2^*\}$ is an $\mathcal{M}'$-based packing of spanning s-arborescences covering the root arcs in D'.

Proof. By Lemma 3.3, T_1^* and T_2^* are spanning s-arborescences. We show that the packing is indeed $\mathcal{M}'$-based by showing that $\{e_{T_1^*[s,v]}, e_{T_2^*[s,v]}\}$ is a base for every $v \in V$. The proof is split into two cases for each $v \in V$.

Suppose that $V(T_1^*[s, v]) \cap V_2 = \emptyset$. Then $v \notin V_2$. Since V_2 is the set of all vertices that are reachable from s in T_2 through the root arcs in F_2, no arc leaves V_2 neither in T_2 nor in T_2^*, and hence we have $V(T_2^*[s, v]) \cap V_2 = \emptyset$ by $v \notin V_2$. Therefore we have $T_i^*[s, v] = T_i[s, v]$ for each $i = 1, 2$, and $\{e_{T_1^*[s,v]}, e_{T_2^*[s,v]}\}$ is a base as $\{e_{T_1[s,v]}, e_{T_2[s,v]}\}$ is a base.

Suppose that $V(T_1^*[s, v]) \cap V_2 \neq \emptyset$. Let u be the first vertex in V_2 when tracing back from v in $T_1^*[s, v]$. Then, as all the vertices in $T_2[s, u]$ (except s) are in V_2, $T_1^*[s, v]$ includes $T_2[s, u]$, and we have $e_{T_1^*[s,v]} = e_{T_2[s,u]}$, which is parallel to x by $u \in V_2$. On the other hand, T_2^* includes no root arc parallel to x by definition, which in particular implies that $e_{T_2^*[s,v]}$ is not parallel to x. Therefore, $e_{T_1^*[s,v]}$ and $e_{T_2^*[s,v]}$ are not parallel to each other, and they form a base as $\mathcal{M}'$ has rank two.

By Claim 3.5, $\{T_1^*, T_2^*\}$ is also a feasible packing in D'. Moreover since $u_0 \in V_1 - V_2$ and $v_0 \in V_1 \cap V_2$, x and x' are contained in T_1^*. Thus we are in case 1. This completes the proof of Theorem 3.4. □

3.3 | Graphic matroids

We prove that Conjecture 1.3 is true for graphic matroids.

Theorem 3.6. *Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph where $\mathcal{M} = (\partial_s(V), r)$ is a graphic matroid of rank k. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D covering $\partial_s(V)$ if and only if D is $\mathcal{M}$-independent and rooted $\mathcal{M}$-arc-connected.*

Proof. Let $\boldsymbol{G} = (\{\mathbf{0}, \mathbf{1},\ldots,\boldsymbol{k}\}, \boldsymbol{E})$ be a connected undirected graph with a bijection $\boldsymbol{\pi}: E \to \partial_s(V)$ representing $\mathcal{M}$. From now on, we will refer to the matroid-rooted digraph $(D, \mathcal{M})$ as (D, G, π). For an edge $e \in E$, let $\boldsymbol{x_e} = \pi(e)$. For $X \subseteq V$, let $\boldsymbol{E_X} = \pi^{-1}(\partial_s(X))$. Please note that the vertex set $V(G)$ of G is $\{0, 1,\ldots,k\}$ while the vertex set $V(D)$ of D is $V + s$. For each $v \in V$, let $\boldsymbol{C_v}$ be the vertex set of the connected component $\boldsymbol{Q_v}$ of $(V(G), E_v)$ that contains the vertex 0. Note that, since D is $\mathcal{M}$-independent, $k - |E_v| \geq 0$ and Q_v is a tree. For $v \in V$, let us orient each edge e of Q_v to $\overrightarrow{\boldsymbol{e}}$ so that Q_v becomes an arborescence $\overrightarrow{\boldsymbol{Q_v}}$ rooted at 0 (see Figure 3).

We prove the theorem by imposing the following extra property for the packing $\{T_1,\ldots,T_k\}$:

$$\text{for every } \ v \in V \ \text{ and every } \ \overrightarrow{e} = ij \ \text{ in } \ \overrightarrow{Q_v}, \quad x_e \ \text{ belongs to } \ T_j. \tag{2}$$

Let (D, G, π) be a counterexample for the theorem minimizing $k|V| - \sum_{v\in V} |E_v| \geq 0$. We take $\boldsymbol{v}^*$ such that $|C_{v^*}|$ is as small as possible.

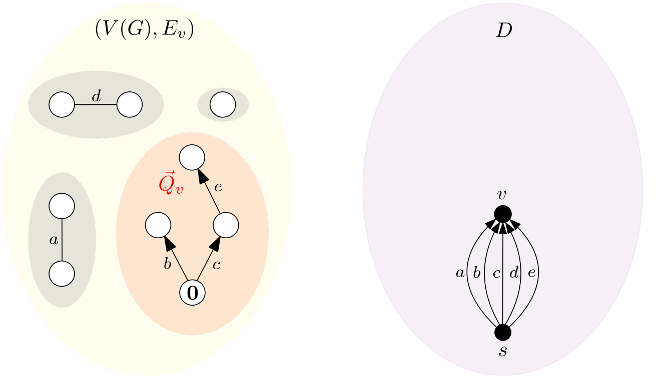

FIGURE 3 The definition of $\overrightarrow{Q_v}$ [Color figure can be viewed at wileyonlinelibrary.com]

If $C_{v^*} = V(G)$, then for every $v \in V$, Q_v is a spanning tree of G, that is, $\partial_s(v)$ is a base of the graphic matroid (G, π). The property (2) uniquely determines the set T_i to which each root arc x_e belongs, so there is a unique partition $\{A_1,\ldots,A_k\}$ of the set of root arcs of D that satisfies (2). Since in $\overrightarrow{Q_v}$ each vertex of $\{1,\ldots,k\}$ has in-degree 1, by (2), the k arcs of $\partial_s(v)$ are associated to different members of $\{A_1,\ldots,A_k\}$. So each A_i contains an arc sv for all $v \in V$, and hence $\{T_1 := (V, A_1),\ldots,T_k := (V, A_k)\}$ is a packing of k spanning s-arborescences. Moreover, the set of root arcs of the paths provided by $\{T_1,\ldots,T_k\}$ for all $v \in V$ consists of $\partial_s(v)$, that is $\{T_1,\ldots,T_k\}$ is a matroid-based packing. Finally, each root arc belongs to some T_i, thus we have a feasible packing of D satisfying (2). Figure 4 illustrates the base case.

From now on, we suppose that C_{v^*} is a proper subset of $V(G)$. Let $\boldsymbol{W} = \{v \in V: C_v = C_{v^*}\}$. Then the vertex set C_W of the connected component that contains 0 in $(V(G), E_W)$ is equal to C_{v^*}. For $p \in V - W$, an element $e \in E_p$ is called *critical* if $\vec{e}$ belongs to $\overrightarrow{Q_p}$ and $\vec{e}$ leaves C_W. By the minimality of $|C_{v^*}|$ and $p \in V - W$, we have $C_p - C_W \neq \emptyset$. Hence the following claim follows from the fact that $\overrightarrow{Q_p}$ is a spanning 0-arborescence on C_p. □

Claim 3.7. For $p \in V - W$, E_p contains a critical element.

Claim 3.8. Let pq be an arc in D with $p \in V - W$ and $q \in W$ and e a critical element in E_p. Then (pq, x_e) is not admissible.

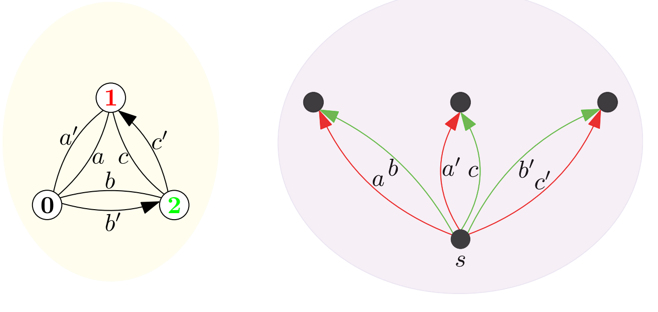

FIGURE 4 The base case [Color figure can be viewed at wileyonlinelibrary.com]

Proof. Suppose that (pq, x_e) is admissible. Since e is critical, $\vec{e}$ and hence e leaves $C_W = C_q$. Hence, x_e is not spanned by $\pi(E_q)$, that is, $x_e \in \partial_s(p) - \mathrm{Span}_{(G,\pi)}(\partial_s(q))$. Thus the pair (pq, x_e) is good. By Lemma 3.2(a), the shifting (D', G', π') of (D, G, π) along (pq, x_e) is $\mathcal{M}'$-independent and rooted $\mathcal{M}'$-arc-connected. Since (D, G, π) is a minimum counterexample, we have a feasible packing $T_1', ..., T_k'$ for (D', G', π') satisfying (2). Let e' be the new edge parallel to e assigned to the new arc $x_{e'}$ from s to q in the shifting. As e is critical, e and hence e' leaves C_W, so (2) implies that $\vec{e}$ and $\vec{e'}$ are parallel arcs and thus x_e and $x_{e'}$ belong to the same spanning s-arborescences T_j' of D. Therefore, by setting T_ℓ $(1 \leq \ell \leq k)$ with $T_\ell = T_\ell'$ for $\ell \neq j$ and $T_j = T_j' - x_{e'} + pq$, we obtain a feasible packing $T_1, ..., T_k$ for (D, G, π) satisfying (2). This contradicts that (D, G, π) is a counterexample.

Since C_W is a proper subset of $V(G)$, $r(\pi(E_W)) < k$. Therefore, by the rooted $\mathcal{M}$-arc-connectivity of D, (1) implies that D has an arc pq with $p \in V - W$ and $q \in W$. By Claim 3.7, E_p contains a critical element e, and then Claim 3.8 says that (pq, x_e) is not admissible. In other words, there exists a tight set $X \subseteq V$ with $q \in X$ and $p \notin X$ such that x_e is contained in the span of $\pi(E_X)$.

We shall take such a pair (pq, x_e) such that X is minimal. Since $\pi(E_X)$ spans x_e while, as e is critical, $\pi(E_W)$ does not span x_e, we have $r(\pi(E_{X\cap W})) < r(\pi(E_X))$. Hence, by the rooted $\mathcal{M}$-arc-connectivity of D and the tightness of X, $\varrho_{D-s}(X \cap W) \geq k - r(\pi(E_{X\cap W})) > k - r(\pi(E_X)) = \varrho_{D-s}(X)$. Hence $D - s$ has an arc $p'q'$ with $p' \in X - W$ and $q' \in X \cap W$. Since $E_{p'}$ contains a critical element e' by Claim 3.7, $(p'q', x_{e'})$ is not admissible by Claim 3.8, that is, there exists a tight set $X' \subseteq V$ with $q' \in X'$ and $p' \notin X'$ such that $x_{e'} \in \mathrm{Span}(\pi(E_{X'}))$. Since $p' \in X - W$, $E_{p'} \subseteq E_X$ and hence $e' \in E_X$. Durand de Gevigney et al [3] says that $X \cap X'$ is tight and $x_e' \in \mathrm{Span}\left(\pi\left(E_{X\cap X'}\right)\right)$. Furthermore, $q' \in X \cap X'$, $p' \notin X \cap X'$, and $e' \in E_{p'}$ is critical, contradicting the minimal choice of X, since $p' \in X - X'$. □

3.4 | Transversal matroids

The case when $\mathcal{M}$ is transversal can be solved by a completely different idea, by reducing the problem to a packing problem of reachability branchings. Let $D^* = (V^*, A)$ be a digraph. For a nonempty set $R \subseteq V^*$, an *R-branching* is a subgraph of D^* that consists of $|R|$ vertex-disjoint arborescences in D^* whose roots are in R. An R-branching B is a *reachability R-branching* if $V(B)$ is exactly the set of vertices that are reachable from some vertex in R in D^*. The following surprising generalization of Edmonds' theorem was discovered by Kamiyama, Katoh, and Takizawa [13].

Theorem 3.9 (Kamiyama, Katoh, and Takizawa [13]). *Let $D^* = (V^*, A^*)$ be a digraph and $\mathcal{R} = \{R_1, ..., R_k\}$ a family of nonempty subsets of V^*. There exists a packing of reachability $\mathcal{R}$-branchings in D^* if and only if*

$$\varrho_{D^*}(X) \geq p_{\mathcal{R}}(X) \quad \text{for every} \quad \emptyset \neq X \subseteq V^*, \tag{3}$$

where $p_{\mathcal{R}}(X)$ denotes the number of R_i's for which $R_i \cap X = \emptyset$ and there exists a path from a vertex in R_i to a vertex in X.

We prove now that Conjecture 1.3 is true for transversal matroids.

Theorem 3.10. *Let* $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ *be a matroid-rooted digraph, where* $\mathcal{M}$ *is a transversal matroid. There exists an* $\mathcal{M}$*-based packing of spanning* s*-arborescences in* D *if and only if* D *is rooted* $\mathcal{M}$*-arc-connected.*

Proof. We only prove the sufficiency. Let k be the rank of $\mathcal{M}$, $G = (S, T; E)$ a bipartite graph representing $\mathcal{M}$ such that $T = \{1,...,k\}$, and $\pi: S \to \partial_s(V)$ a bijection. We subdivide each root arc e in D by inserting a new vertex r_e, and then remove s. Let $D^* = (V^*, A^*)$ be the resulting digraph.

We use R^* to denote the set of new vertices r_e, and let $R_i = \{r_e \in R^*: \pi^{-1}(e)$ is adjacent to i in $G\}$ for $i \in T$ and $\mathcal{R} = \{R_1,...,R_k\}$. With this setting of D^* and $\mathcal{R}$, we apply Theorem 3.9. To do this we have to check if (3) holds. □

Claim 3.11. Condition (3) of Theorem 3.9 holds.

Proof. Let X be a set of vertices in D^*. If X is a subset of R^* then the claim is obvious. Otherwise, let v be a vertex of $X - R^*$. By rooted $\mathcal{M}$-arc-connectivity, there exist an $\mathcal{M}$-based packing of (s, v)-paths $\{P_1,...,P_k\}$ in D. Hence, for every i with $R_i \cap X = \emptyset$, there exists an arc of P_i that enters X in D^*. Hence, by the arc-disjointness of the paths, (3) is satisfied.

We also claim the following to guarantee that the resulting branchings are spanning.

Claim 3.12. For each i and each $v \in V^* - R^*(=V)$, D^* has a path from some vertex in R_i to v.

Proof. By rooted $\mathcal{M}$-arc-connectivity, there exist k arc-disjoint paths in D from s to any other vertex v such that the set of their first arcs $\{e_1,...,e_k\}$ is a base of $\mathcal{M}$. As G has a matching covering $\{\pi^{-1}(e_1),...,\pi^{-1}(e_k)\}$ and T, the set $\{r_{e_1},...,r_{e_k}\}$ intersects R_i for $i = 1, ...,k$.

By Claim 3.11 and Theorem 3.9, there exists a packing of reachability $\{R_1,...,R_k\}$-branchings $B_1, ...,B_k$ in D^*. By Claim 3.12, each reachability R_i-branching B_i covers V^*. By contracting R^* into s, we obtain k pairwise arc-disjoint spanning s-arborescences $T_i = B_i/R^*$ in D. The construction implies that, for each root arc e in T_i, G has an edge between $\pi^{-1}(e) \in S$ and $i \in T$. Therefore, for each $v \in V$ and each $i \in \{1,...,k\}$, $\pi^{-1}\left(e_{T_i[s,v]}\right) \in S$ is connected to $i \in T$ in G, implying that these root arcs over all i form a base of $\mathcal{M}$. Hence $T_1, ...,T_k$ indeed form an $\mathcal{M}$-based packing of spanning s-arborescences. □

3.5 | Fano matroid—when D is acyclic

If D is acyclic, the condition (1) for rooted $\mathcal{M}$-arc-connectivity can be significantly simplified as follows.

Lemma 3.13. *Let* $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ *be a matroid-rooted digraph, where* D *is acyclic. Then* D *is rooted* $\mathcal{M}$*-arc-connected if and only if*

$$\varrho_{D-s}(v) + r(\partial_s(v)) \geq r(\mathcal{M}) \quad \text{for all} \quad v \in V. \tag{4}$$

Proof. Condition (4) is strictly weaker than condition (1). Hence we just need to prove the sufficiency. Let $X \subseteq V$. As D is acyclic, there exists a vertex v_0 of $D[X]$ with $\varrho_{D[X]}(v_0) = 0$. For such v_0, we have $\varrho_{D-s}(X) \geq \varrho_{D-s}(v_0)$. Hence, by the monotonicity of the rank function r and (4) we get

$$\varrho_{D-s}(X) + r(\partial_s(X)) \geq \varrho_{D-s}(v_0) + r(\partial_s(v_0)) \geq r(\mathcal{M}).$$

Thus (1) follows. □

In view of Lemma 3.13 one can consider the following strategy to prove Conjecture 1.3 for acyclic digraphs. Consider proving Conjecture 1.3 by induction on $|V|$. Without loss of generality we may assume that D is $\mathcal{M}$-independent. Note that in this case (4) is equivalent to saying that each vertex v is of in-degree at least $r(\mathcal{M})$. Since the claim is obvious when $|V| = 0$, we also assume $|V| \geq 1$. As D is acyclic, it has a vertex $v \in V$ with outdegree 0. Let $k = r(\mathcal{M})$. By Lemma 3.13, $D - v$ is rooted $\mathcal{M}|_{\partial_s(V-v)}$-arc-connected and there exist ℓ arcs entering v in $D - s$ for some $0 \leq \ell \leq k$ along with $k - \ell$ root arcs entering v which are independent in $\mathcal{M}$. By induction, there exists an $\mathcal{M}|_{\partial_s(V-v)}$-based packing of spanning s-arborescences $\{T_1,\ldots,T_k\}$ in $D - v$. Consider extending this packing in $D - v$ to a packing of D. For each nonroot arc $e = uv$ entering v, let $B_e = \{e_{T_i[s,u]} | u \in V(T_i),\ 1 \leq i \leq k\}$. To extend the packing of $D - v$ to an $\mathcal{M}$-based packing of D, we need to choose one element from B_e for each nonroot arc e entering v such that the chosen elements form a base of $\mathcal{M}$ with the $k - \ell$ root arcs entering v. The following lemma claims that this is always possible in the Fano matroid.

Lemma 3.14. *Let ℓ be an integer with $\ell \leq 3$, $B_1, \ldots, B_\ell$ be bases of the Fano matroid with a three-coloring of $\bigcup_{i=1}^{\ell} B_i$ such that each base B_i is colorful for $i = 1, \ldots, \ell$, and $a_{\ell+1}, \ldots, a_3$ be $3 - \ell$ independent elements of the Fano matroid that are not in $\bigcup_{i=1}^{\ell} B_i$. Then there is a doubly colorful base of the Fano matroid, that is, one can choose elements $a_i \in B_i$ for $i \in \{1,\ldots,\ell\}$ of different colors such that $\{a_1, a_2, a_3\}$ is a base of the Fano matroid.*

Proof. The statement is obvious when $\ell = 0$ and also when $\ell = 1$ as in the latter case there exists an element of B_1 which is not on the a_2a_3-line of the Fano plane (see Figure 5 for a figure of the Fano plane). Similarly, when $\ell = 2$ then we can take any element $a_2 \in B_2$ and an element a_1 of B_1 which is not on the a_2a_3-line. If we have at least two such choice for a_1, then we can chose it to have different color than a_2. Otherwise, the other two elements of B_1 are the elements on the a_2a_3-line different from a_3. Hence a_2 is an element of B_1 and a_1 has a different color than a_2 by the colorfulness of B_1.

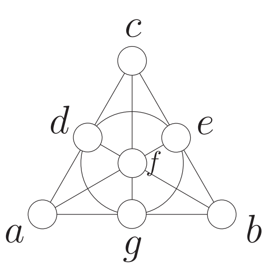

FIGURE 5 The elements of the Fano matroid

Let now $\ell = 3$. The three bases cannot be disjoint, otherwise the Fano matroid should contain nine distinct elements and it has just has seven elements. By relabeling the bases, we can assume that $B_1 \cap B_2 \neq \emptyset$. Assume that $B_1 = B_2$. Since B_3 is a base, $B_3 \neq \bigcup\{\text{Span}_{\mathcal{M}}(B_1 - b) - B_1: b \in B_1\} =: L$ as L is a line. Take $a_3 \in B_3 - L$ and $a_1, a_2 \in B_1 = B_2$ with three different colors. Then, as $a_3 \notin L$ and $a_3 \neq a_1$ nor a_2, $\{a_1, a_2, a_3\}$ is a colorful base. Therefore, we can assume $B_1 \neq B_2$. Let $y_1 \in B_1 - B_2$ and $y_2 \in B_2 - B_1$ with the same color and let $x \in B_1 \cap B_2$. Take $a_3 \in B_3$ with the third color. As the line xa_3 may contain only one of y_1 and y_2, we can assume that, say, y_2 is not on this line. Therefore, we can take $a_1 := x$ and $a_2 := y_2$ such that $\{a_1, a_2, a_3\}$ is a colorful base. □

Thus we have the following for the Fano matroid.

Theorem 3.15. *Let $(D, \mathcal{M})$ be a matroid-rooted digraph where $D = (V + s, A)$ is acyclic and $\mathcal{M} = (\partial_s(V), r)$ is a submatroid of the Fano matroid. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D if and only if D is rooted $\mathcal{M}$-arc-connected.*

4 | NEGATIVE RESULTS

In this section, we will give a counterexample to Conjecture 1.3 and prove that Problem 1.4 is NP-complete for acyclic digraphs and a certain class of matroids. The precise statements are given as follows.

Theorem 4.1. *There exist an acyclic digraph $D = (V + s, A)$ and a matroid $\mathcal{M}$ of rank three such that $(D, \mathcal{M})$ is a counterexample to Conjecture 1.3.*

Theorem 4.2. *Problem 1.4 is NP-complete even if $D = (V + s, A)$ is acyclic and $\mathcal{M}$ is a linear matroid of rank three with a given linear representation.*

As we noted before, the matroid $\mathcal{M}$ used in the construction, which we call a *parallel extension of the Fano matroid*, will arise from the Fano matroid by adding some parallel copies of its elements.

The proof is done by defining several gadget constructions, each of which restricts possible packings. Each construction step is referred to as an *operation* below. In each construction, we insert new vertices one by one together with three new arcs entering it and no arc leaving it. A new root arc will always be added keeping the $\mathcal{M}$-independence as well as the fact that $\mathcal{M}$ is a parallel extension of the Fano matroid (or its submatroid). Thus, an instance $(D = (V + s, A), \mathcal{M})$ constructed by a sequence of operations always satisfies the following properties:

(i) D is acyclic and, by Lemma 3.13, $(D, \mathcal{M})$ is rooted $\mathcal{M}$-arc-connected.
(ii) If $(D, \mathcal{M})$ is constructed from $(D', \mathcal{M}')$ by an operation, then every feasible packing for $(D, \mathcal{M})$ is an extension of a feasible packing for $(D', \mathcal{M}')$.

By the property (i), in the subsequent discussion we omit to mention that $(D, \mathcal{M})$ is $\mathcal{M}$-independent and rooted $\mathcal{M}$-arc-connected. By using the property (ii), we shall be able to control possible extensions of feasible packings.

We say that a vertex $v \in V$ *gets* a base B in a feasible packing $\{T_1, T_2, T_3\}$ if $B = \{e_{T_1[s,v]}, e_{T_2[s,v]}, e_{T_3[s,v]}\}$. We also say that v *gets* $e_{T_i[s,v]}$ *from* u if u is on the path $T_i[s, v]$ ($i = 1, 2, 3$). T_1, T_2, and T_3 will be called the red, blue, and black arborescences, respectively. We say that an element of $\mathcal{M}$ is *colored* by λ if it is in the arborescence of color λ. We will use the notation $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ to denote an ordered (multi)set and use this notation to say $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ *is colored by* $(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3})$ if a is colored by λ_1, b is colored by λ_2 and c is colored by λ_3.

In the following, the elements of $\mathcal{M}$ will be denoted by the first seven letters of the alphabet (see Figure 5) and primes, superscripts (when we would need too many primes) or subscripts will be used when we consider a parallel element of a previously used one (that may be also an identical element to this previous one). It is well known that the Fano plane have automorphisms moving arbitrary three points not lying on a line to any three points in general position.

Each operation is best described with figures, which are illustrated by the following rule (see, eg, Figure 6A). The root vertex s is not shown in the figures. A vertex will be represented as a big circle in which Fano plane is illustrated with three particular elements (empty circles) which represent the base that the vertex will get in every feasible packing. Existing vertices in the original digraph will be denoted by thicker circles, in which the elements of the bases that they get in every feasible packing will be assigned by their letters. For a vertex w which is added in an operation, a letter x may be assigned to a point in the Fano plane, which means that a new root arc sw is added with a new element x in the underlying matroid. Sometimes a new vertex will be represented by just a back point for simplicity (see Figure 6B for example).

Operation 4.3. *Given $(D, \mathcal{M})$, suppose that a vertex $v \in V$ gets the base $\{a, b, c\}$ in every feasible packing. Force-color $\boldsymbol{FC_{(a,b,c)}(v)}$ extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w to D along with two incoming root arcs a' and d and one nonroot arc vw, where $a' \| a$ and $\{a, c, d\}$ is a line of the Fano plane (see Figure 6A).*

Note that, by the automorphisms of the Fano plane, $FC_{(x,y,z)}(v)$ is also defined for any base $\{x, y, z\}$ (and the same remark is applied for other operations given below).

Lemma 4.4. *With the notation as in Operation 4.3, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$. Moreover, in every feasible packing of $(D', \mathcal{M}')$, w gets the base $\{a', b, d\}$, that is, the arc vw will be in the same arborescence as the root arc b.*

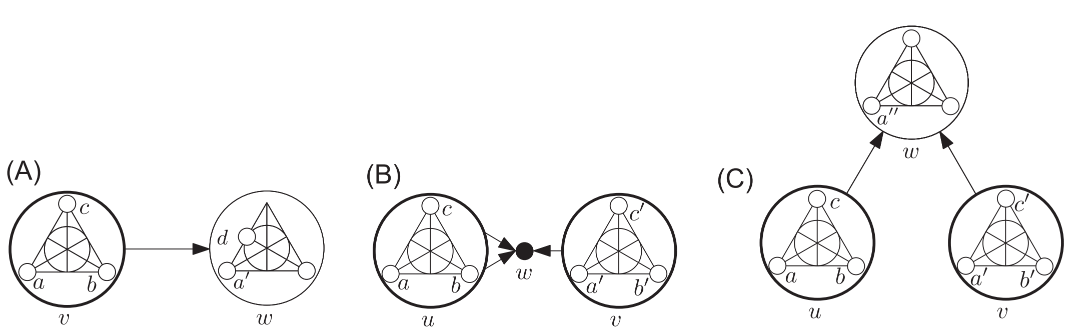

FIGURE 6 The three elementary operations: (A) $FC_{(a,b,c)}(v)$; (B) $ADC(u, v)$; (C) $AF_a(u, v)$

Proof. Consider any possible extension of a feasible packing of $(D, \mathcal{M})$, where we distribute the three arcs entering w among the three arborescences. From the construction, w always gets a' and d from the root. Also, by the assumption of the lemma, v gets $\{a, b, c\}$, and w gets one of them from v. Now, in a feasible extension, w cannot get a from v as $a' \| a$ and cannot get c as $\{a', c, d\}$ is a line. Hence w gets b from v and the packing is feasible as $\{a', b, d\}$ is a base. □

For simplicity, we also use $FC_{(a,b,c)}(v)$ to denote the new vertex w in Operation 4.3.

Operation 4.5. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \| a$, $b' \| b$, and $c' \| c$. Avoid-different-coloring* $\boldsymbol{ADC}(\boldsymbol{u}, \boldsymbol{v})$ *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w to D along with two parallel arcs from u to w and an arc from v to w (see Figure 6B).*

Lemma 4.6. *With the notation as in Operation 4.5, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where all the parallel pairs (a, a'), (b, b'), and (c, c') have different colors.*

Proof. By symmetry, we may assume without loss of generality, that w gets a and b from u in a feasible packing in $(D', \mathcal{M}')$. Then w should get c' from v, which is possible if and only if the color of c' is equal to that of c. Thus the claim follows as any feasible packing in $(D', \mathcal{M}')$ is an extension of that in $(D, \mathcal{M})$. □

For simplicity, we use $ADC(u, v)$ to denote the new vertex w in Operation 4.5.

Operation 4.7. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \| a$, $b' \| b$, and $c' \| c$. Avoid-flip* $\boldsymbol{AF_a}(\boldsymbol{u}, \boldsymbol{v})$ *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w along with an incoming root arc a'', an arc from u to w and an arc from v to w to D (see Figure 6C).*

Lemma 4.8. *With the notation as in Operation 4.7, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where a and a' have the same color and the colors of the pairs (b, b') and (c, c') are different.*

Proof. First we prove that feasible packings in the exceptional case of AF cannot be extended. The vertex w can get either the base $\{a'', b, c'\}$ or the base $\{a'', b', c\}$. However, both contain two elements of the same color, which is impossible.

Next observe that in the nonexceptional case of AF, either b and c' or b' and c are of different colors, say b and c'. Let us color a'' by the color not used by b and c'. Then w can get the base $\{a'', b, c'\}$ that uses the three colors. □

For bases $\{a, b, c\}$ and $\{x, y, z\}$ in the Fano plane with parallel extension, we denote by $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\} \| \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ if each element in $\{a, b, c\}$ is parallel to some element in $\{x, y, z\}$; if the order of parallel elements is important, then we use ordered sets in this notation.

By the previous operations, we can now define the following operation.

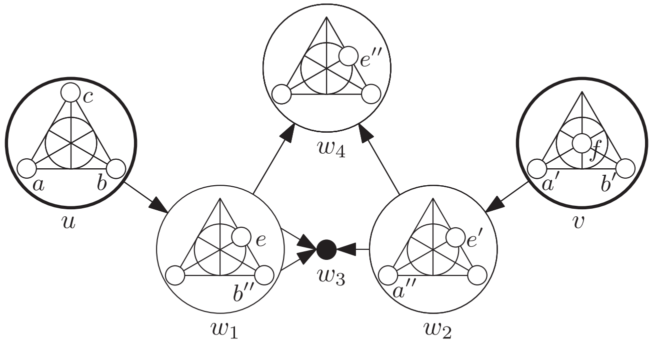

FIGURE 7 An example of $FSC_{(a,b')}(u, v)$, where $(x, y, z) = (b', a', f)$ and $t = e$.

Operation 4.9. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{x, y, z\}$ in every feasible packing, respectively, such that $a \nparallel x$ and $\{a, b, e\} \| \{x, y, k\}$, where $\{b, e, c\}$ and $\{y, k, z\}$ are lines of the Fano plane. Forbid-same-color $\boldsymbol{FSC_{(a,x)}(u, v)}$ extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding four new vertices to D and five new elements to $\mathcal{M}$ as follows. Add $w_1 := FC_{(b,a,c)}(u)$ with new root arcs b'' and e, $w_2 := FC_{(y,x,z)}(v)$, $w_3 := ADC(w_1, w_2)$, and $w_4 := AF_t(w_1, w_2)$, where t denotes the element with $t \in \{b'', e\}$ and $t \nparallel x$ (see Figure 7).*

One can see other examples of FSC in Figure 8, $FSC_{(d,g)}(v, w_2)$ and $FSC_{(f,a''')}(w_1, w_2)$, where (a, b, c, x, y, z, t) in Operation 4.9 corresponds to $(d, a', b', g, a''', c, a^{(4)})$, and $(f, a', b'', a''', g, c, g'')$, respectively.

Lemma 4.10. *With the notation as in Operation 4.9, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where the colors of a and x are the same.*

Proof. By $a \nparallel x$ and $\{x, y, k\} \| \{a, b, e\}$, we have $\{a, b, e\} \| \{x, y, k\} \| \{a, x, t\}$.

First we prove that feasible packings in the exceptional case of FSC cannot be extended. Suppose that a and x are of the same color. By Lemma 4.4, each of w_1 and w_2 gets a base parallel to $\{a, x, t\}$. Since a in w_1 and x in w_2 are of the same color, by Lemma 4.6, the element parallel to t should be of the same color at w_1 and w_2. However, by Lemma 4.8, the element parallel to t should be of different colors at w_1 and w_2, which is a contradiction.

Now in the nonexceptional case a and x have different colors, say red and black. Then, by Lemma 4.4, each of w_1 and w_2 may get a base parallel to (a, x, t) with colors (red, black, blue). By Lemmas 4.6 and 4.8, the packing extends to w_3 and w_4. □

The main operation is the following.

Operation 4.11. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', d\}$ in every feasible packing, respectively, where $a' \| a$, $b' \| b$ and $\{a, d, c\}$ is a line of the Fano plane. Strong-avoid-flip $\boldsymbol{SAF_{(a,b,c)}(u, v)}$ extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding 14 new vertices to D and 19 new elements to $\mathcal{M}$ as follows. First, add two new vertices to D and four new elements to $\mathcal{M}$ by $w_1 := FC_{(b',a',d)}(v)$ (with new root arcs b'' and f) and*

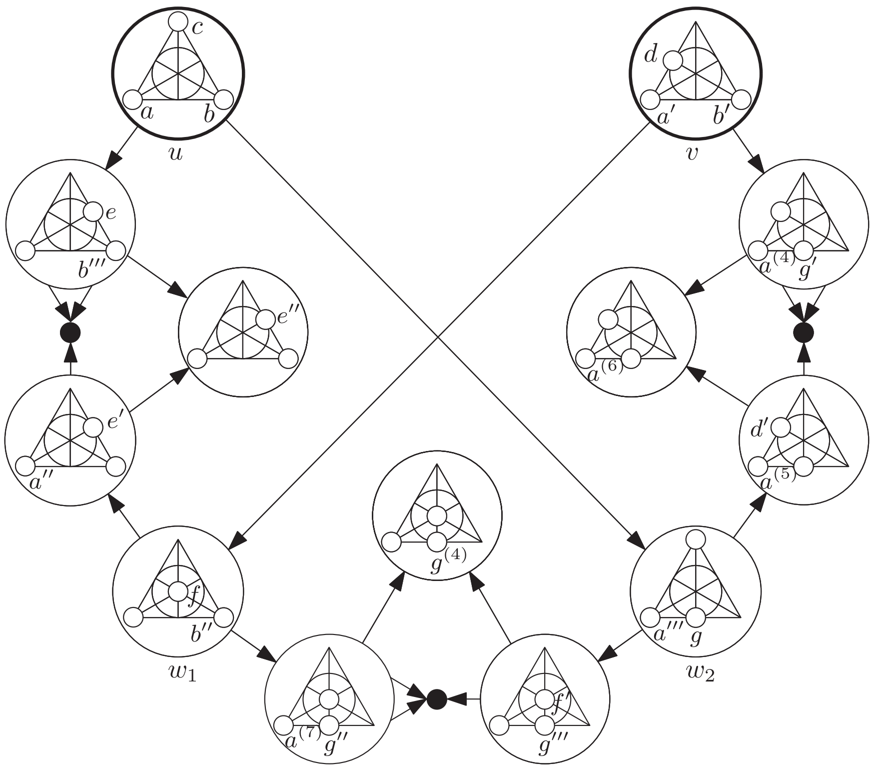

FIGURE 8 The operation $SAF_{(a,b,c)}(u, v)$

$w_2 \coloneqq FC_{(a,c,b)}(u)$ (with new root arcs a''' and g). Then add the remaining new vertices of D' and new elements of $\mathcal{M}'$ by the operations $FSC_{(a,b'')}(u, w_1)$, $FSC_{(d,g)}(v, w_2)$, and $FSC_{(f,a''')}(w_1, w_2)$ (see Figure 8).

Lemma 4.12. *With the notation as in Operation 4.11, a feasible packing in $(D, \mathcal{M})$, where b and b' have the same color, extends to a feasible packing in $(D', \mathcal{M}')$ if and only if the colors of the pairs (a, a') and (c, d) are the same.*

Proof. By relabeling the colors we may assume that the base (a, b, c) that u gets is colored by (red, blue, black).

First, suppose that (a', b', d) is colored by (black, blue, red). w_1 gets the base (a', b'', f) that, by Lemma 4.10 applied for $FSC_{(a,b'')}(u, w_1)$, cannot be colored by (black, red, blue), so, by Lemma 4.4, it is colored by (black, blue, red). Similarly, w_2 gets the base (a''', c, g) that, by Lemma 4.10 applied for $FSC_{(d,g)}(v, w_2)$, cannot be colored by (blue, black, red), so, by Lemma 4.4, it is colored by (red, black, blue). Finally, since f and a''' are red, Lemma 4.10 applied for $FSC_{(f,a''')}(w_1, w_2)$ shows that the packing cannot be extended.

Second, suppose that (a', b', d) is colored by (red, blue, black). w_1 gets the base (a', b'', f) and w_2 gets the base (a''', g, c) and both can be colored, by Lemma 4.4, by (red, blue, black). Lemma 4.10 applied for $FSC_{(a,b'')}(u, w_1)$, $FSC_{(d,g)}(v, w_2)$, and $FSC_{(f,a''')}(w_1, w_2)$ shows that the packing can be extended. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We start with a digraph on two vertices, a root s and the other vertex z_1, along with three parallel arcs a_1, b_1, and c_1 from s to z_1. The underlying matroid is the free matroid on $\partial_s(z_1)$. In the following, the arborescences containing a_1, b_1, and c_1 will be called red, blue, and black, respectively. First, add new vertices $z_2 := FC_{(a_1,b_1,c_1)}(z_1)$ (which gets $\{a_2, b_1, d_1\}$), $z_3 := FC_{(d_1,b_1,a_2)}(z_2)$ (which gets $\{b_1, c_2, d_3\}$). Then apply the operations $SAF_{(a_1,b_1,c_1)}(z_1, z_2)$, $SAF_{(d_1,b_1,a_2)}(z_2, z_3)$, and $SAF_{(c_2,b_1,d_3)}(z_3, z_1)$ (see Figure 9). Applying Lemmas 4.4 and 4.12 twice shows that the base (a_2, b_1, d_1) that z_2 gets is colored by (red, blue, black), the base (b_1, c_2, d_3) that z_3 gets is colored by (blue, red, black). Finally, by Lemma 4.12, no feasible packing exists in the resulting instance. By Lemma 3.13, the resulting instance is rooted $\mathcal{M}$-arc-connected, and hence is a counterexample to Conjecture 1.3. This completes the proof of Theorem 4.1. □

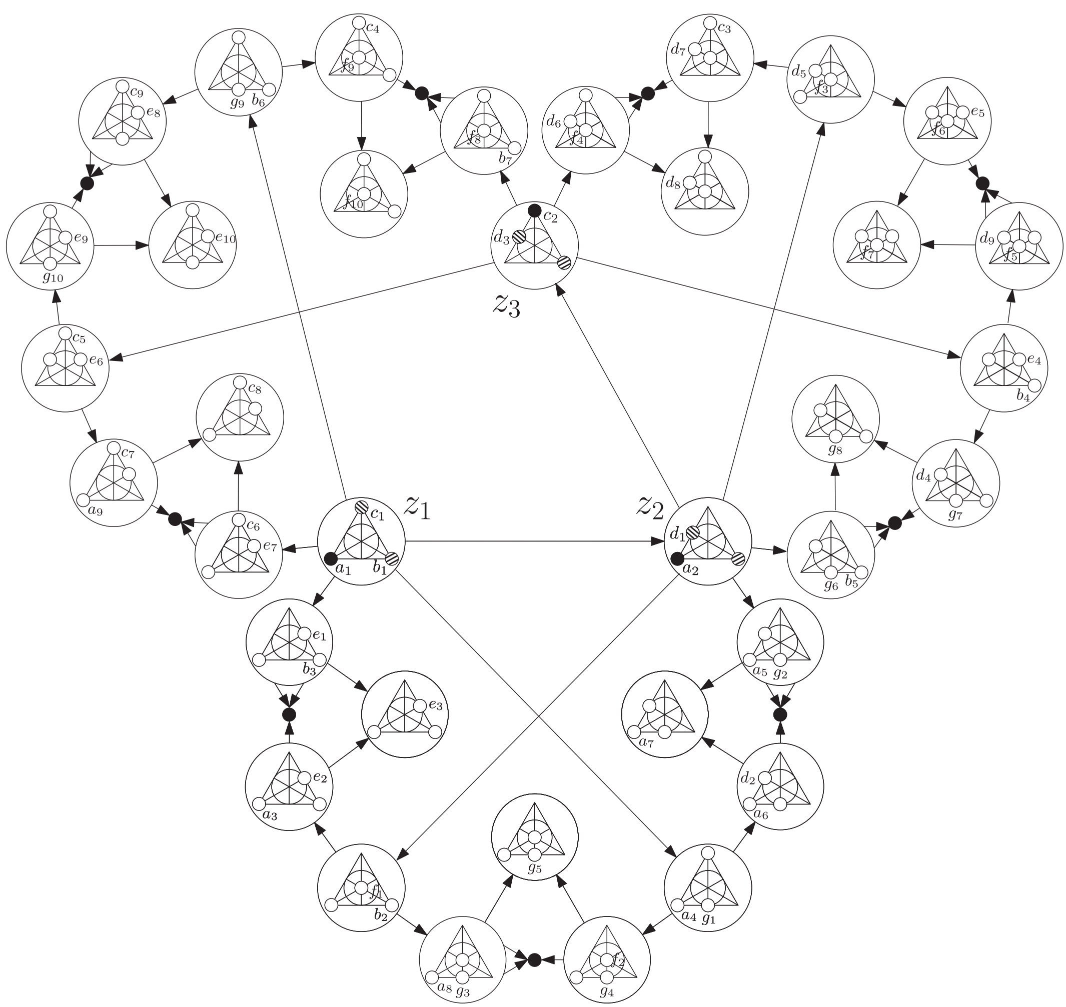

FIGURE 9 The counterexample shown in the proof of Theorem 4.1

Now we turn to the proof of Theorem 4.2. Problem 1.4 is in NP in the case where a linear representation of the matroid is given as input since the packing itself is a witness for the problem that can be checked in polynomial time. We will use the well-known 3-SAT (see [12]) to prove the NP-completeness of our problem.

Let us take a 3-CNF formula. Using the previous operations (and new ones) we will construct a matroid-rooted digraph that has a feasible packing if and only if the formula is satisfiable. To express each clause, our idea is to represent it as a concatenation of majority functions and implement each majority function by using our operations. We first remark the following lemma. Recall that the majority function $\mathsf{maj}(\alpha, \beta, \gamma)$ is a Boolean function that has a value one if and only if at least two among α, β, γ have value one.

Lemma 4.13. *Let* $\alpha, \beta, \gamma \in \{0, 1\}$. *Then*

$$\alpha \vee \beta \vee \gamma = \mathsf{maj}(\mathsf{maj}(\alpha, \beta, 1), \mathsf{maj}(\alpha, \gamma, 1), \mathsf{maj}(\beta, \gamma, 1)). \tag{5}$$

Proof. $\alpha \vee \beta \vee \gamma = 1$ if and only if at least one of α, β, and γ is 1. If, say, $\alpha = 1$, then $\mathsf{maj}(\alpha, \beta, 1) = 1$ and $\mathsf{maj}(\alpha, \gamma, 1) = 1$, hence the right-hand side of (5) is 1. If $\alpha = \beta = \gamma = 0$, then $\mathsf{maj}(\alpha, \beta, 1) = \mathsf{maj}(\alpha, \gamma, 1) = \mathsf{maj}(\beta, \gamma, 1) = 0$, hence the right-hand side of (5) is 0. □

Operation 4.14. *Given* $(D, \mathcal{M})$, *suppose that* $v_1, v_2, v_3 \in V$ *get the bases* $\{a, b, c\}$, $\{a', b', c'\}$, *and* $\{a'', b'', c''\}$, *respectively, in every feasible packing where* $a \| a' \| a''$, $b \| b' \| b''$ *and* $c \| c' \| c''$. *Majority* ***MAJ*** $(\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3})$ *extends* $(D, \mathcal{M})$ *to* $(D', \mathcal{M}')$ *by adding a new vertex* w *with three incoming arcs* v_1w, v_2w, *and* v_3w *(see Figure 10).*

Lemma 4.15. *With the notation as in Operation 4.14, consider a feasible packing of* $(D, \mathcal{M})$ *such that all of* b, b' *and* b'' *are colored by* λ *(and hence there are only two types of possible coloring schemes on each* v_i*). Then the packing extends to a feasible packing of* $(D', \mathcal{M}')$. *Moreover, in every such extension* w *gets a base formed by parallel copies of* a, b, *and* c, *and the coloring of this base is the same as the majority among the three on* v_1, v_2, *and* v_3 *(see Figure 10).*

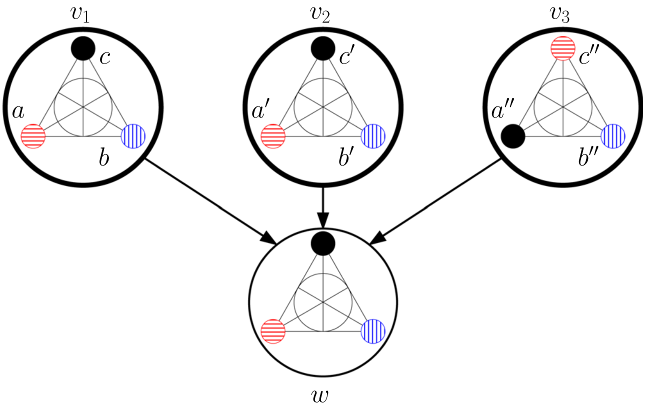

FIGURE 10 *MAJ* (v_1, v_2, v_3) [Color figure can be viewed at wileyonlinelibrary.com]

Proof. Without loss of generality, we can assume that the colorings of (a, b, c) and (a', b', c') coincide, say, they are colored by (red, blue, black). As w has an entering arc from each v_i, w always gets a parallel copy of b colored by blue. Moreover, as w has in-arcs from v_1 and v_2 too, w gets a parallel copy of a or c from v_1 or v_2. Hence w gets a parallel copy of a colored by red or a parallel copy of c colored by black. These two facts already determine the coloring scheme on w as stated in the lemma. □

Two more operations are needed in the NP-completeness proof.

Operation 4.16. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \| a$, $b' \| b$, and $c' \| c$. Copy-one-color* $\boldsymbol{COC_b(u, v)}$ *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding three new vertices to D and two new elements to $\mathcal{M}$ by operations $ADC(u, v)$, $AF_a(u, v)$, and $AF_c(u, v)$.*

Lemma 4.17. *With the notation as in Operation 4.16, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where the colors of b and b' are different.*

Proof. Note that any feasible packing of $(D, \mathcal{M})$ satisfies either one of the following: (i) each pair in (a, a'), (b, b'), and (c, c') has the same color; (ii) all the pairs (a, a'), (b, b'), and (c, c') have different colors; (iii) only (a, a') has the same color; (iv) only (b, b') has the same color; (v) only (c, c') has the same color. By $ADC(u, v)$, $AF_a(u, v)$, and $AF_c(u, v)$, the packing is extendable if and only if (i) or (iv) holds, meaning that (b, b') has the same color. □

Operation 4.18. *Given $(D, \mathcal{M})$, suppose that a vertex $v \in V$ gets the base $\{a, b, c\}$ in every feasible packing. Change-colors* $\boldsymbol{CC_{(a,c)}(v)}$ *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding 45 new vertices to D and 63 new elements to $\mathcal{M}$ as follows. First, add new vertices $w_1 := FC_{(a,b,c)}(v)$ (which gets $\{a', b, d\}$), $w_2 := FC_{(d,b,a')}(w_1)$ (which gets $\{b, c', d'\}$) and $w := FC_{(c',b,d')}(w_2)$ (which gets $\{a'', b, c''\}$). Then add the remaining new vertices of D' and new elements of $\mathcal{M}'$ by the operations $SAF_{(a,b,c)}(v, w_1)$, $SAF_{(d,b,a')}(w_1, w_2)$, and $SAF_{(c',b,d')}(w_2, w)$.*

Lemma 4.19. *With the notation as in Operation 4.18, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$. Moreover, if the base (a, b, c) that v gets is colored by $(\lambda_1, \lambda_2, \lambda_3)$, then the base (a'', b, c'') that w gets is colored by $(\lambda_3, \lambda_2, \lambda_1)$.*

Proof. By relabeling the colors we may assume that the base (a, b, c) that v gets is colored by (red, blue, black). Applying Lemmas 4.4 and 4.12 three times shows that the base (a', b, d) that w_1 gets is colored by (red, blue, black), the base (b, c', d') that w_2 gets is colored by (blue, red, black) and the base (a'', b, c'') that w gets is colored by (black, blue, red). □

For simplicity, we denote $w = CC_{(a,c)}(v)$ if w is as in Operation 4.18.
Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We have seen that the problem is in NP. Hence we only prove that the problem is NP-hard. Note that the Fano matroid and all its parallel extensions are linear as they can be represented by three-dimensional nonzero vectors over the two element field $GF(2)$. Let us take a 3-CNF formula on variables $x_1, x_2, ..., x_n$. First, let $V := \{v_0, ..., v_n\}$ and take a digraph D on $V + s$ whose arc set consists of only root arcs $sv_i (i = 0, ..., n)$, three copies of each. Take a base $\{a, b, c\}$ of the Fano matroid and define a parallel extension of the Fano matroid $\mathcal{M}$ on $\partial_r(V)$ such that, for each $i \in \{0, ..., n\}$, the three arc sv_i form a parallel copy $\{a_i, b_i, c_i\}$ of $\{a, b, c\}$. Next use operation $COC_{b_{i-1}}(v_{i-1}, v_i)$ for $i = 1, ..., n$. This ensures that, in every feasible packing, the parallel copies of b in the bases that $v_0, ..., v_n$ get are colored by the same color, say, blue.

Add $v_1', ..., v_n'$ by using operations $CC_{(a_i, c_i)}(v_i)$ for $i = 1, ..., n$. Hence, in every feasible packing, v_i' gets the colored base (a_i', b_i, c_i') with the same coloring as (c_i, b_i, a_i) for $i = 1, ..., n$. In the following construction, v_i (resp., v_i') will represent the variable x_i (resp., its negate $\bar{x}_i$) for $i = 1, ..., n$. Moreover, v_0 will represent 1.

For each clause ψ of the formula, we first add four new vertices w_1^ψ, w_2^ψ, w_3^ψ, and w_4^ψ using operation MAJ so that it represents ψ according to the equation in Lemma 4.13. (In other words, for a clause, say, for $\psi = x_1 \vee \bar{x}_2 \vee x_3$ we add w_1^ψ with arcs $v_1 w_1^\psi$, $v_2' w_1^\psi$, and $v_0 w_1^\psi$, w_2^ψ with arcs $v_1 w_2^\psi$, $v_3 w_2^\psi$, and $v_0 w_2^\psi$, w_3^ψ with arcs $v_2' w_3^\psi$, $v_3 w_3^\psi$, and $v_0 w_3^\psi$, and w_4^ψ

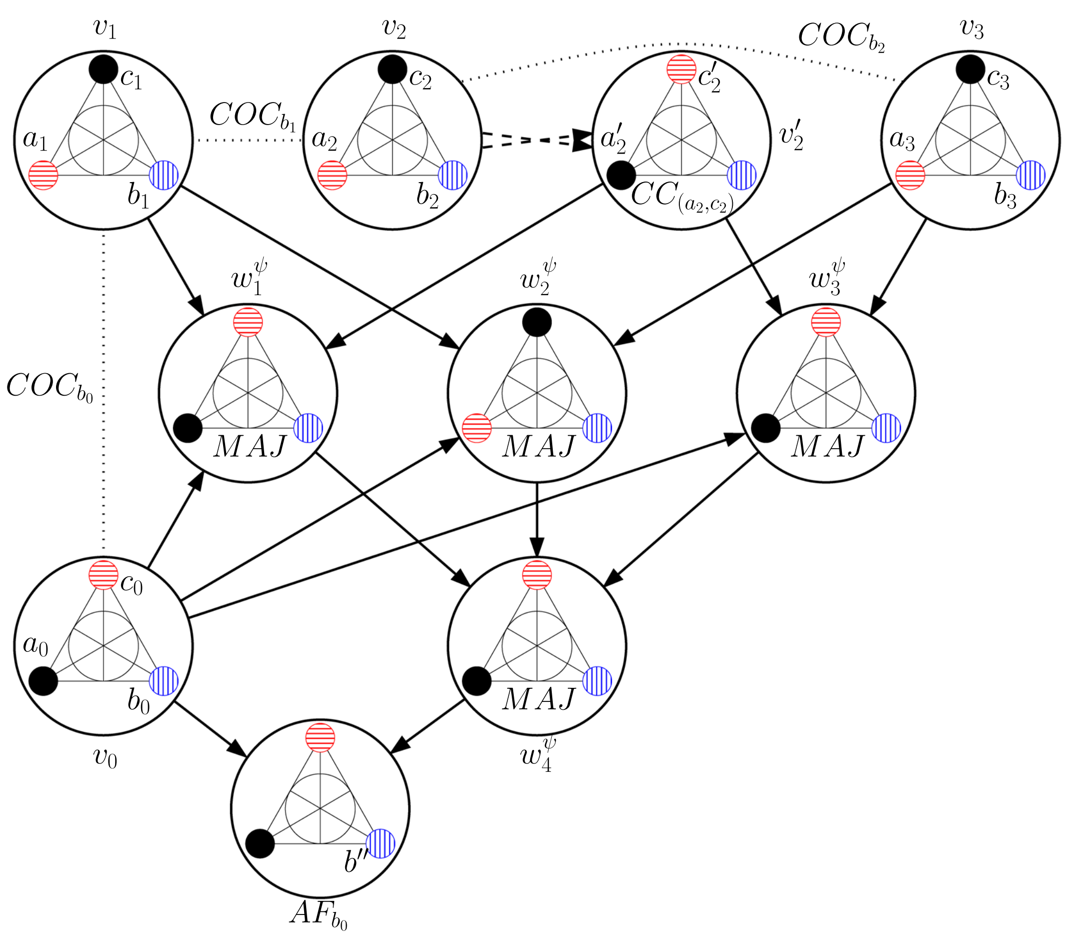

FIGURE 11 A part of the construction in the proof of Theorem 4.2. This demonstrates how the assignment $x_1 = x_2 = x_3 = 0$ makes the clause $\psi = x_1 \vee \bar{x}_2 \vee x_3$ true in the corresponding feasible packing. The crossing dashed arcs represent the operation CC and the dotted edges represent the operation COC [Color figure can be viewed at wileyonlinelibrary.com]

with arcs $w_1^\psi w_4^\psi$, $w_2^\psi w_4^\psi$, and $w_3^\psi w_4^\psi$.) Finally, to ensure the truth of each clause ψ, we further use operation $AF_{b_0}(v_0, w_4^\psi)$ (see Figure 11).

We claim that the formula is satisfiable if and only if $(D, \mathcal{M})$ admits a feasible packing. Note that v_0 always gets the base $\{a_0, b_0, c_0\}$, and without loss of generality we may always suppose that (a_0, b_0, c_0) is colored by (red, blue, black). Then the claim follows by identifying the coloring scheme (red, blue, black) (resp., [black, blue, red]) for a parallel copy of (a, b, c) with a true assignment (resp., a false assignment).

More formally, suppose that the formula has a true assignment. Then, we first construct a feasible packing on $\{s, v_0, v_1,\ldots,v_n\}$ such that v_0 gets the base (a_0, b_0, c_0) colored by (red, blue, black) and each $v_i (1 \le i \le n)$ gets the base (a_i, b_i, c_i) colored by (red, blue, black) if $x_i = 1$ and by (black, blue, red) if $x_i = 0$. By Lemma 4.19, this packing always extends on $\{v_1',\ldots,v_n'\}$ such that each v_i' gets a base formed by parallel copies of a, b, and c colored by black, blue, and red, respectively, if $x_i = 1$ and by red, blue, and black, respectively, if $x_i = 0$. Since the assignment satisfies the formula, Lemmas 4.15 and 4.8 imply that the packing is extendable to a feasible packing on the whole vertex set of D.

Conversely, if $(D, \mathcal{M})$ has a feasible packing, then by $COC_{b_{i-1}}(v_{i-1}, v_i)$, b_i is colored by blue on each v_i. We set x_i in such a way that $x_i = 1$ if and only if (a_i, b_i, c_i) is colored by (red,blue,black) (as in (a_0, b_0, c_0)). By $CC_{(a_i,c_i)}(v_i)$, coloring of (a_i', c_i') is the reverse of the coloring of (a_i, c_i). Moreover, since $AF_{b_0}(v_0, w_4^\psi)$ is used for each clause ψ, the base on w_4^ψ has the same coloring scheme as that of $\{a_0, b_0, c_0\}$ on v_0 by Lemma 4.8. Thus, by Lemma 4.15, the formula is satisfied for this assignment. □

5 | CONCLUDING REMARKS

All the results presented here have undirected and hypergraphic counterparts. To get an undirected counterpart of our positive results for rank-two, graphic, or transversal matroids, one can use [3] and the proof after that. This extends a result of Katoh and Tanigawa [14] on these fundamental matroid classes. Moreover, with the techniques of [5], we also have extensions of these results for dypergraphs (ie, oriented hypergraphs), hypergraphs, and mixed hypergraphs.

On the other hand, Problem 1.3 is NP-complete for dypergraphs as it is NP-complete for digraphs. Also, the proof of the NP-completeness can be applied even for the undirected case. This is because, in the construction of the NP-completeness, we only add vertices with in-degree 3 one by one, and hence the ordering of the vertex addition prescribes the orientation of each edge in a rooted-tree packing.

A challenging open problem is to give a complete characterization of the class of matroids for which Conjecture 1.3 is true. A much easier but still interesting question is whether one can abstract our proof technique for graphic matroids to solve wider classes of matroids such as regular matroids.

ACKNOWLEDGMENTS

The present research was supported by the Project RIME of the laboratory G-SCOP. The authors also would like to acknowledge the support by the Hausdorff Trimester Program on Combinatorial Optimization of the Hausdorff Research Institute, the University of Bonn, where this study was partially carried out. The second author was supported by the ÚNKP-17-4 New National Excellence Program of the Ministry of Human Capacities of Hungary, and by the Hungarian Scientific Research Fund – OTKA, K109240. Project no. NKFI-128673 has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the FK_18 funding scheme. The fourth author was supported by JSPS Postdoctoral Fellowships for Research Abroad and JSPS Grant-in-Aid for Scientific Research (B) 25280004.

ORCID

Csaba Király http://orcid.org/0000-0001-8081-9056

REFERENCES

1. K. Bérczi and A. Frank, *Packing arborescences*, RIMS Kôkyurôku Bessatsu **B23** (2010), 1–31.
2. K. Bérczi, T. Király, and Y. Kobayashi, *Covering intersecting bi-set families under matroid constraints*, SIAM J. Discrete Math. **30** (2016), no. 3, 1758–1774.
3. O. Durand de Gevigney, V. H. Nguyen, and Z. Szigeti, *Matroid-based packing of arborescences*, SIAM J. Discrete Math. **27** (2013), no. 1, 567–574.
4. J. Edmonds, *Edge-disjoint branchings*, Combinatorial Algorithms (B. Rustin, ed.), Academic Press, New York, 1973, pp. 91–96.
5. Q. Fortier, Cs. Király, M. Léonard, Z. Szigeti, and A. Talon, *Old and new results on packing arborescences in directed hypergraphs*, Discrete Appl. Math. **242** (2018), 26–33.
6. A. Frank, *On disjoint trees and arborescences*, Algebr. Methods Graph Theory **25** (1978), 59–169.
7. A. Frank, *On the orientation of graphs*, Theory Ser. B **28** (1980), no. 3, 251–261.
8. A. Frank, Personal communication, 2013.
9. A. Frank, T. Király, and Z. Király, *On the orientation of graphs and hypergraphs*, Discrete Appl. Math. **131** (2003), no. 2, 385–400.
10. A. Frank, T. Király, and M. Kriesell, *On decomposing a hypergraph into k connected sub-hypergraphs*, Discrete Appl. Math. **131** (2003), no. 2, 373–383.
11. S. Fujishige, *A note on disjoint arborescences*, Combinatorica **30** (2010), no. 2, 247–252.
12. M. R. Garey, and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness, Series of books in the mathematical sciences*, W. H. Freeman, New York, NY, USA, 1979.
13. N. Kamiyama, N. Katoh, and A. Takizawa, *Arc-disjoint in-trees in directed graphs*, Combinatorica **29** (2009), no. 2, 197–214.
14. N. Katoh and S. Tanigawa, *Rooted-tree decomposition with matroid constrains and the infinitesimal rigidity of frameworks with boundaries*, SIAM J. Discrete Math. **27** (2013), no. 1, 155–185.
15. Cs. Király, *On maximal independent arborescence packing*, SIAM J. Discrete Math. **30** (2016), no. 4, 2107–2114.
16. Cs. Király, Z. Szigeti, and S. Tanigawa, *Packing of arborescences with matroid constraints via matroid intersection*, Math. Program. (2019). https://doi.org/10.1007/s10107-019-01377-0. Paper in Press.
17. C. St. J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. Lond. Math. Soc. **36** (1961), 445–450.
18. J. G. Oxley, *Matroid theory*, 2nd ed., Oxford University Press, Oxford, 2011.
19. L. Szegő, *On covering intersecting set-systems by digraphs*, Discrete Math. **234** (2001), 187–189.
20. W. T. Tutte, *On the problem of decomposing a graph into n connected factors*, J. Lond. Math. Soc. **36** (1961), 221–230.

How to cite this article: Fortier Q, Király C, Szigeti Z, Tanigawa S-i. On packing spanning arborescences with matroid constraint. *J Graph Theory*. 2019;1–23. https://doi.org/10.1002/jgt.22484