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HIGH-DIMENSIONAL INCIPIENT INFINITE CLUSTERS REVISITED

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ABSTRACT. The incipient infinite cluster (IIC) measure is the percolation measure at criticality conditioned on the cluster of the origin to be infinite. Using the lace expansion, we construct the IIC measure for high-dimensional percolation models in three different ways, extending previous work by the second-named author and Járai. We show that each construction yields the same measure, indicating that the IIC is a robust object. Furthermore, our constructions apply to spread-out versions of both finite-range and long-range percolation models. We also get estimates on structural properties of the IIC, such as the volume of the intersection between the IIC and Euclidean balls.

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1. INTRODUCTION AND MAIN RESULTS

It is a widely believed conjecture for bond percolation on the hypercubic lattice $\mathbb{Z}^d$ with $d \geq 2$ that there are no infinite clusters at the critical point. This conjecture has been verified for $d = 2$ [20],[30], and $d \geq 19$ [5], [18]. Verifying the conjecture for the intermediate values of d, especially the values $d = 3$ to $d = 6$ is arguably one of the most challenging problems in probability today.

While there are no infinite clusters at the critical point, there are typically some very large clusters nearby (see [1] for a more precise statement). Therefore, it is reasonable to believe that one can construct and infinite cluster at the critical point through suitable conditioning and limiting schemes. This cluster is known as an *incipient infinite cluster* (IIC). The first construction of an IIC was carried out by Kesten [31] in two dimensions. In an effort to rigorize physicist's studies of random walk on random fractals (e.g. [4], [36]), Kesten proposed two different limiting schemes, proved their existence, and showed that both limits are equal. Indeed, let $\mathbb{P}_p^{(2d)}$ be the measure of nearest-neighbor bond percolation on $\mathbb{Z}^2$, let $Q_r = \{x : |x| \leq r\}$, let $\mathcal{C}(0)$ be the connected component of the vertex 0 and let p_c denote the critical value of the parameter p. Roughly speaking, Kesten proved that there exists a measure $\mathbb{P}_{\text{IIC}}^{(2d)}$ such that, for any cylinder event F,

$$\lim_{r\to\infty} \mathbb{P}_{p_c}^{(2d)}\left(F \mid 0 \leftrightarrow Q_r^c\right) = \mathbb{P}_{\text{IIC}}^{(2d)}(F) \quad \text{and} \quad \lim_{p\searrow p_c} \mathbb{P}_p^{(2d)}\left(F \mid |\mathcal{C}(0)| = \infty\right) = \mathbb{P}_{\text{IIC}}^{(2d)}(F) \tag{1.1}$$

and that $\mathbb{P}_{\text{IIC}}^{(2d)}$ has the property that

$$\mathbb{P}_{\text{IIC}}^{(2d)}\left(|\mathcal{C}(0)| = \infty\right) = 1. \tag{1.2}$$

We will refer to the left-hand limit in (1.1) as *Kesten's first IIC construction.* Work on the two-dimensional IIC was later continued by Járai [29], who proved that various other natural constructions for the IIC yield the same limiting measure $\mathbb{P}_{\text{IIC}}^{(2d)}$.

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Van der Hofstad, den Hollander and Slade constructed the IIC measure for high-dimensional *oriented percolation* [24]. Later, in [26], van der Hofstad and Járai gave two different constructions for the incipient infinite cluster measure for high-dimensional bond percolation. The first of these constructions is condition critical percolation on the event that $x \in \mathcal{C}(0)$ and then to take the limit $|x| \to \infty$ (see (1.20) below). The second construction is to condition subcritical percolation on the same event, to average over all $x \in \mathbb{Z}^d$, and then to take the limit $p \nearrow p_c$ (see (1.21) below). Van der Hofstad and Járai show that both constructions yield the same measure by using a *lace expansion.* The lace expansion for percolation was developed by Hara and Slade [18] to treat high-dimensional percolation rigorously. It should be noted that in the literature there are no two-dimensional constructions that have been shown to yield the IIC measure for high-dimensional models, or vice-versa.

The main aim of this paper is to expand on the results by van der Hofstad and Járai in the following ways:

(i) We extend all known constructions of the IIC in high dimensions to models of *long-range spread-out percolation*, so they can be dealt with under the same formalism (modulo certain assumptions).
(ii) We prove a new construction of the IIC that uses the asymptotics of the *one-arm probability.* Under certain assumptions we can show that this construction yields the *same* limiting measure as other known constructions. This new construction is the high-dimensional equivalent of Kesten's first IIC construction. It is the first construction that has been shown to work for both two- and high-dimensional models.
(iii) We prove structural properties of the IIC, such as bounds on the volume of the intersection of the IIC with Euclidean balls centered at the origin, and the density of pivotal edges for the backbone of the IIC in such balls. In a sequel to this paper, [21], we analyse the extrinsic properties of random walk on the IIC. The structural properties we prove here will form the cornerstone for that analysis.
(iv) We introduce several new techniques for bounding probabilities and expectations in terms of the asymptotics of the two-point function in Fourier space.
(v) We prove a lower bound on the extrinsic one-arm probability for long-range spread-out percolation.

We now start by formally introducing the models.

Bond percolation on $\mathbb{Z}^d$. We consider the graph $\mathbb{Z}^d$ as a complete graph, i.e., the set of edges (or bonds) is $\mathbb{B} = \big\{\{x,y\} \mid x,y \in \mathbb{Z}^d\big\}$. We study bond percolation on this graph: we make the edges of the graph *open* in a random way and study the resulting subgraph of open edges. For every $x,y \in \mathbb{Z}^d$, let the edge $\{x,y\}$ be open independently with probability $p_{xy} = pD(x,y)$, where D is a probability distribution on $\mathbb{Z}^d$. Thus p is the average number of open edges per vertex. In this paper, the function $D(\cdot\,,\cdot)$ is considered to be invariant under lattice symmetries and rotations by 90°. As a result $D(u,v) = D(0,v-u)$. We often abbreviate $D(x) = D(0,x)$. We assume that $p \in [0, \|D\|_\infty^{-1}]$, so that $pD(x,y) \le 1$ for all $x,y \in \mathbb{Z}^d$. In our choice of D we consider the following three important families:

The first family is the well-studied case of *nearest-neighbor* percolation, where an edge $\{x,y\}$ is open with probability $q \in [0,1]$ whenever $|x-y| = 1$, and closed otherwise. Here $|x|$ denotes the Euclidean norm of $x \in \mathbb{Z}^d$. In terms of the above general setting, this corresponds to letting $D(x) = (2d)^{-1}\mathbb{1}_{\{|x|=1\}}$ and $p = 2dq$.

The second family is *finite-range spread-out* percolation. Let h be a nonnegative bounded function on $\mathbb{R}^d$ that is piecewise continuous, has the symmetries described above, is supported in $[-1,1]^d$, and is normalized, $\int_{[-1,1]^d} h(x)\mathrm{d}x = 1$. For $L \in \mathbb{N}$ we define

$$D(x) = \frac{h(x/L)}{\sum_{x\in\mathbb{Z}^d} h(x/L)}. \tag{1.3}$$

We call L the *spread-out* parameter. Our proofs typically require that L is sufficiently large. We will elaborate on this below.

The standard example of a finite-range spread-out distribution is

$$pD(x) = \frac{p}{(2L+1)^d - 1} \mathbb{1}_{\{0<\|x\|_\infty \le L\}}. \tag{1.4}$$

The third family is that of *long-range spread-out* percolation. Again we define D in terms of a parameter L and a function h through (1.3). The function h is assumed to have all the same properties as for finite-range spread-out percolation, except that we do not assume that h has bounded support. Instead, we assume that there exists $\alpha, c_1, c_2 > 0$ and $\ell < \infty$ such that

$$c_1|x|^{-d-\alpha} \le h(x) \le c_2|x|^{-d-\alpha} \qquad \text{for all } |x| \ge \ell. \tag{1.5}$$

The exponent α can be any positive real number, although we get the most interesting results for $\alpha \in (0,2]$. In such cases, the spatial variance of D is infinite: $\sum_x |x|^2 D(x) = \infty$.

The standard example of a long-range spread-out distribution is

$$pD(x) = p\frac{\mathcal{N}_L}{\max\{|x|/L, 1\}^{d+\alpha}}, \tag{1.6}$$

where $\mathcal{N}_L$ is a normalizing constant.

Throughout the rest of this paper we consider $\alpha \in (0,2)\cup(2,\infty)$, that is, we consider all allowed values except $\alpha = 2$. When $\alpha = 2$ we get logarithmic corrections to many of the bounds, and although these do not complicate any of the proofs, writing them down everywhere would make our results more cumbersome to read.

Here and throughout the rest of the paper, we write $(2\wedge\alpha)$ as a shorthand for $\min\{2,\alpha\}$ when considering long-range spread-out percolation with parameter α. To simplify notation, we will sometimes write a general result in terms of $(2\wedge\alpha)$. When the model under consideration does not depend on α we will assume that either the parameter α is redundant (e.g. when defining a constant $K = K(d,\alpha)$), or we set $\alpha = \infty$ (e.g. when the result depends on $(2\wedge\alpha)$).

Rather than stating our results for the three above families of distributions D, we state our results in terms of the Fourier transform of D, i.e.,

$$\hat{D}(k) = \sum_{x\in\mathbb{Z}^d} D(x)\mathrm{e}^{ik\cdot x} \qquad \text{for} \quad k \in [-\pi,\pi)^d. \tag{1.7}$$

Our results only depend on the choice of model through the properties of $\hat{D}(k)$ and the choice of d and L, so to state our results with the greatest generality we work under the following assumption:

Assumption D [Bounds on $\hat{D}$]. *Consider a d-dimensional percolation model. Let $L = 1$ for nearest-neighbor models. Otherwise, let L be the spread-out parameter. The model satisfies the following bounds: There exist constants c_1 and c_2 such that*

$$1-\hat{D}(k) \ge c_1 L^{(2\wedge\alpha)}|k|^{(2\wedge\alpha)} \qquad \text{if} \quad \|k\|_\infty \le L^{-1}; \tag{1.8}$$

$$1-\hat{D}(k) > c_2 \qquad \text{if} \quad \|k\|_\infty \ge L^{-1}. \tag{1.9}$$

Furthermore, there exists a constant w with $0 < w = O(L^{(2\wedge\alpha)})$ such that, for $\varepsilon > 0$ sufficiently small,

$$1-\hat{D}(k) \le w|k|^{(2\wedge\alpha)} \qquad \text{if} \quad |k| \le \varepsilon. \tag{1.10}$$

It follows by direct computation that Assumption D holds for nearest-neighbor models. The bounds (1.8) and (1.9) were proved for finite-range spread-out models in [27, Appendix A]. Assumption D is proved for long-range spread-out models in [9, Proposition 1.1]. This proof can be modified to prove (1.10) for finite-range spread-out models.

In all percolation models we discuss, p is the parameter of the model, and it is well known that percolation undergoes a phase transition at the critical threshold

$$p_c = \sup\{p \mid \chi(p) < \infty\}, \tag{1.11}$$

where

$$\chi(p) = \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow x) \tag{1.12}$$

is the 'expected cluster size' (or 'susceptibility'), $\mathbb{P}_p$ is the product measure with parameter p, and $\{x \leftrightarrow y\}$ denotes the event that the vertices x and y are connected by a path of open edges. Note that our definition of p_c differs from the standard definition

$$p_c = \inf\{p : \theta(p) > 0\} \tag{1.13}$$

where $\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty)$ and $\mathcal{C}(0)$ is the connected component of the origin. Nevertheless, both definitions have been proved to be equivalent in our context, cf. [2], [37].

Mean-field behavior in high dimensions. Understanding percolation at the critical point p_c is in general a difficult (and in many cases unsolved) problem. In the high-dimensional case some significant advances have been made. In the context of percolation, 'high-dimensional' has the rather precise meaning that the triangle diagram

$$\triangle_p(0) \equiv \sum_{x,y\in\mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow x)\,\mathbb{P}_p(x \leftrightarrow y)\,\mathbb{P}_p(y \leftrightarrow 0) \tag{1.14}$$

is finite whenever $p \le p_c$. We call this the *triangle condition.*

Define the *mean-field parameter*

$$\beta = \beta(d,L) = \begin{cases} L^{-d} & \text{for spread-out models,} \\ d^{-1} & \text{for nearest-neighbor models.} \end{cases} \tag{1.15}$$

We say that a model has mean-field parameter β if the values of d and L for this model yield the value β as defined above.

A stronger version of the triangle condition is aptly called the *strong triangle condition*: Let β_0 and K be model-dependent constants that only depend on d and α when the model is spread-out, and that are independent of d for the nearest-neighbor model. A model with mean-field parameter β satisfies the strong triangle condition if

$$\triangle_{p_c}(0) \le 1 + K\beta. \tag{1.16}$$

This bound is proved in [19] for nearest-neighbor percolation with $d \ge 19$ (but it is generally believed that it holds for all $d > 6$), it is proved in [18] for finite-range spread-out models with $d > 6$ and $L \ge L_0$, where $L_0 = L_0(d)$ is a large constant. In [23] the bound is proved for long-range spread-out models with $d > 3(2\wedge\alpha)$ and $L \ge L_0' = L_0'(d,\alpha)$.

All of these proofs use a lace expansion: a method invented by Brydges and Spencer to study weakly self-avoiding walk [8] that was first applied to percolation by Hara and Slade [18]. In fact, with the exceptions of [32] and [38], for any model for which the triangle condition has been proved, a lace expansion was used to prove the strong triangle condition.

Under the triangle condition (i.e., if $\triangle_{p_c}(0) < \infty$), various critical exponents exist and take on the same value as for percolation on an infinite tree, see e.g. Aizenman and Newman [3] and Barsky and Aizenman [5]. Based on an analogy with the Ising model, these values are called 'mean-field values'.

In this paper we will always assume that the strong triangle condition (1.16) holds. This implies that our models have a sufficiently large dimension. To save space in the statement of the theorems below, we will state this requirement as an assumption:

Assumption C [Sufficiently large dimension]. *The dimension of the model satisfies:*

(i) $d > 6$ *if the model belongs to the nearest-neighbor family;*
(ii) $d > 6$ *if the model belongs to the finite-range spread-out family;*
(iii) $d > 3(2 \wedge \alpha)$ *if the model belongs to the long-range spread-out family.*

Note that this is a necessary, but not a sufficient condition on the dimension for the results in our paper, because β depends on d and even though we need that β is smaller than β_0 (so that the strong triangle condition holds), we also need that β is smaller than some other constant that may be smaller than β_0.

Here and throughout the paper, $f = o(g)$ denotes that $\lim_{n\to\infty} f(n)/g(n) = 0$ (or some other appropriate limit), $f \simeq g$ denotes that $f = cg(1+o(1))$ for some constant c and $f \asymp g$ denotes that both $f \le Cg$ and $f \ge cg$ hold asymptotically for some constants $c, C > 0$.

We define the *two-point function*

$$\mathbb{P}_p(x \leftrightarrow y) = \tau_p(x-y). \tag{1.17}$$

For nearest-neighbor percolation in dimension $d \ge 19$ and for finite-range spread-out percolation in dimension $d > 6$, Hara [16] and Hara, van der Hofstad and Slade [17], respectively, prove the two-point function estimate

$$\tau_{p_c}(x-y) \simeq |x-y|^{2-d}. \tag{1.18}$$

The asymptotic relation (1.18) is *not* true for the long-range model with $\alpha < 2$ as this would imply that $\sum_{|x|\le r} \tau(x) \asymp r^2$. But we prove later on that $\sum_{|x|\le r} \tau(x) \asymp r^{(2\wedge\alpha)}$, so the two-point function cannot possibly scale as $|x|^{2-d}$ when $\alpha \in (0,2)$. More importantly, Chen and Sakai prove that there exists a class of long-range models such that

$$\mathbb{P}_{p_c}(x \leftrightarrow y) \simeq |x-y|^{(2\wedge\alpha)-d}. \tag{1.19}$$

This asymptotic behavior is generally conjectured to hold for all models in the long-range spread-out family when $d > 3(2\wedge\alpha)$. (In fact, Chen and Sakai prove stronger bounds, where they identify the constants, but note that the assumptions they make on D are stronger than the assumptions we make in this paper, see [11, Assumption 1.1].)

The incipient infinite cluster. Van der Hofstad and Járai [26] consider the following two constructions of the IIC: Write $\mathfrak{F}_0$ for the algebra of *cylinder events* (i.e., events that are determined by finitely many bonds), and $\mathfrak{F}$ for the σ-algebra of *events* (i.e., the σ-algebra generated by $\mathfrak{F}_0$). The first construction is

$$\mathbb{P}_{\mathrm{IIC}}(F) \equiv \lim_{|x|\to\infty} \mathbb{P}_x(F) \equiv \lim_{|x|\to\infty} \mathbb{P}_{p_c}(F \mid 0 \leftrightarrow x), \qquad F \in \mathfrak{F}_0, \tag{1.20}$$

whenever the limit exists. The second construction is

$$\mathbb{Q}_{\mathrm{IIC}}(F) \equiv \lim_{p\nearrow p_c} \mathbb{Q}_p(F) \equiv \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(F \cap \{0 \leftrightarrow x\}), \qquad F \in \mathfrak{F}_0, \tag{1.21}$$

whenever the limit exists. Here $\mathbb{P}_{\mathrm{IIC}}$ and $\mathbb{Q}_{\mathrm{IIC}}$ are understood as limits in the space of probability measures on $\{0,1\}^{\mathbb{B}}$ in the weak topology. It is a priori not clear that these limits exist. This is a topic that we will discuss in more detail later on.

We call $\mathbb{Q}_p$ the *susceptibility measure* because of the presence of the susceptibility $\chi(p)$. It will play an important role in our analysis.

In [26] it is proved that subject to (1.18), the measures $\mathbb{P}_{\mathrm{IIC}}$ and $\mathbb{Q}_{\mathrm{IIC}}$ exist and that they are the same measure. We conjecture that this is the case in all dimensions. But the proof depends crucially on (1.18), so it only applies to models where such asymptotics are known. For the class of long-range spread-out percolation as described above, we have no useful bounds on the two-point function $\mathbb{P}_{p_c}(x \leftrightarrow y)$. This means that we cannot use such a relation to bound the IIC measure for high-dimensional percolation. The following theorem circumvents this problem by using the (weaker) 'strong triangle condition' instead of bounds on the two-point function.

Theorem 1.1 [Existence of the IIC measure under the strong triangle condition]. *Consider a model that satisfies Assumptions C and D with mean-field parameter β. There exists a constant β_1 that only depends on d and α when the model is spread-out, and that is independent of d for the nearest-neighbor model such that, if the model satisfies the strong triangle condition (1.16) with $\beta \le \beta_1$, then the limit (1.21) exists for this model and for any cylinder event F. Consequently, $\mathbb{Q}_{\text{IIC}}$ can be extended to the σ-algebra of events $\sigma(\mathfrak{F}_0) = \mathfrak{F}$.*

Although this is only a minor improvement on [26, Theorem 1.2], it will turn out to be a very useful one, because this lets us deal with the three model families at once. In Section 3 we give an outline of the changes that need to be made to the proof in [26] to prove Theorem 1.1.

The statement of the theorem in terms of the mean-field parameter β is a bit unusual. Typically, papers about high-dimensional percolation state results for sufficiently large d and L. We have chosen to use β instead, because this allows us to only require that β is sufficiently small, rather than give three different conditions for the three different families of models we discuss in this paper.

We show that there exist two more constructions that both give the same IIC measure as in Theorem 1.1. These constructions are based on assumptions that we make about the properties of critical percolation. These properties are not proved for long-range percolation, but are in the spirit of some results from [11] and [34]. The first assumption that we make is that the two-point function bounds (1.19) hold for the family of long-range percolation models.

To state the second assumption we need a few definitions. The vertex set Q_r is defined to be the Euclidean ball of radius r around the origin, that is,

$$Q_r = \{x \in \mathbb{Z}^d : |x| \le r\}. \tag{1.22}$$

It is generally conjectured that at criticality, the probability of having a path from 0 to Q_r^c (the outside of a ball of radius r) asymptotically behaves as a power of r,

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \asymp r^{-1/\varrho} \tag{1.23}$$

where ϱ is the *one-arm exponent* (cf. [15, Section 9.1]). Using Smirnov's work [40], Lawler, Schramm, and Werner [35] proved that $\varrho = 48/5$ for site percolation on the two-dimensional triangular lattice. They also conjectured that this is the value of the exponent for any planar lattice.

Kozma and Nachmias [34] proved the following one-arm exponent for high-dimensional percolation when $\tau_{p_c}(x) \asymp |x|^{2-d}$:

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \asymp r^{-2}. \tag{1.24}$$

As mentioned before, the condition on the x-space asymptotics of τ_{p_c} has been proved for nearest-neighbor percolation and finite-range spread-out percolation, but not for long-range spread-out percolation. We will assume that the one-arm exponent also exists for long-range percolation, but since we do not know its value in this case, we will write $1/\varrho$. Our conjecture is that for long-range percolation the correct value for ϱ is $2/(4 \wedge \alpha)$. Although Theorem 1.5 below establishes that this is a valid lower bound, we will not assume this. Instead we will use the weaker assumption that ϱ is well defined (in the sense of (1.23)) and $\varrho \in [1/(2 \wedge \alpha), \infty)$. Furthermore, in point (ii) of the theorem below, we also assume that the asymptotics are stronger than upper and lower bound, that is, the relation is "$\simeq$" instead of "$\asymp$". Note that we only use these assumptions in the statement and proof of Theorem 1.2, the statement and proof of Theorems 1.1, 1.5 and 1.6 do not require these assumptions.

Theorem 1.2 [Conditional IIC measure existence].

(i) Consider a long-range spread-out model with $\alpha \in (0,2)\cup(2,\infty)$, $d > 3(2\wedge\alpha)$ and mean-field parameter β. Assume (1.19). There exists a model-dependent constant β_2 that only depends on d and α such that, if $\beta \le \beta_2$, the limit

$$\mathbb{P}_{\mathrm{IIC}}(F) \equiv \lim_{|x|\to\infty} \mathbb{P}_x(F) \equiv \lim_{|x|\to\infty} \mathbb{P}_{p_c}(F \mid 0 \leftrightarrow x) \tag{1.25}$$

exists for all cylinder events. Moreover, $\mathbb{P}_{\mathrm{IIC}} = \mathbb{Q}_{\mathrm{IIC}}$.

(ii) Consider a model that satisfies Assumptions C and D and mean-field parameter β. Assume that there exists $\varrho \in [1/(2\wedge\alpha),\infty)$ such that

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \simeq r^{-1/\varrho}. \tag{1.26}$$

There exists a model-dependent constant β_3 that only depends on d and α in the case of spread-out models, and is independent of d in the case of nearest-neighbor models such that, if $\beta \le \beta_3$, the limit

$$\mathbb{R}_{\mathrm{IIC}}(F) \equiv \lim_{r\to\infty} \mathbb{R}_r(F) \equiv \lim_{r\to\infty} \mathbb{P}_{p_c}(F \mid 0 \leftrightarrow Q_r^c) \tag{1.27}$$

exists for all cylinder events F. Moreover, $\mathbb{R}_{\mathrm{IIC}} = \mathbb{Q}_{\mathrm{IIC}}$.

(iii) Consider a model that satisfies Assumptions C and D and mean-field parameter $\beta \le \beta_3$. Assume that there exists $\varrho \in [1/(2\wedge\alpha),\infty)$ such that

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \simeq r^{-1/\varrho}. \tag{1.28}$$

Then, for this model there exists a sequence $r_1(\beta) < r_2(\beta) < \dots$ such that for all cylinder events F one has

$$\lim_{n\to\infty} \mathbb{R}_{r_n(\beta)}(F) = \lim_{n\to\infty} \mathbb{P}_{p_c}(F \mid 0 \leftrightarrow Q^c_{r_n(\beta)}) = \mathbb{Q}_{\mathrm{IIC}}(F). \tag{1.29}$$

Theorem 1.2(ii) and (iii) yield versions of the IIC as in Kesten's first IIC construction in (1.1).

By the results of [11] and [34] the assumptions (1.19) and (1.28) hold unconditionally for certain classes of models, so that we have the following corollary:

Corollary 1.3 [Unconditional IIC measure existence].

(i) The limit (1.25) holds for all cylinder events F and for all models that satisfy Assumption 1.1 in [11] *with mean-field parameter $\beta \le \beta_2$.*

(ii) For nearest-neighbor models and for finite-range spread-out models that satisfy Assumption C with mean-field parameter $\beta \le \beta_3$ there exists a sequence $r_1(\beta) < r_2(\beta) < \dots$ such that (1.29) holds for all cylinder events F.

Remark 1.4. Theorem 1.2(i) has already been proved for nearest-neighbor models and for finite-range spread-out models that satisfy Assumption C with mean-field parameter $\beta \le \beta_4$ by van der Hofstad and Járai [26].

In Theorem 1.2(ii) and (iii) we assume that for long-range percolation the one-arm critical exponent ϱ exists and $\varrho \in [1/(2\wedge\alpha),\infty)$. We prove that if ϱ exists, then $\varrho \ge 2/(4\wedge\alpha)$:

Theorem 1.5 [A lower bound on the one-arm probability for long-range percolation]. *When $d > 3(2\wedge\alpha)$, there exists $c > 0$ such that for critical long-range spread-out percolation models with parameter α,*

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \ge \frac{c}{r^{(4\wedge\alpha)/2}}. \tag{1.30}$$

We prove this theorem in Section 5.

We conjecture that $2/(4\wedge\alpha)$ is indeed the correct value for ϱ. Supporting evidence for this comes from Janson and Marckert's analysis of the one-dimensional discrete snake with long-range step distribution [28]. Indeed, their results indicate that the probability that the maximal displacement of critical branching random walk exceeds r is proportional to $r^{-(4\wedge\alpha)/2}$. Since we can consider branching random walk to be a mean-field model for high-dimensional percolation, we expect that the behavior of the maximal displacement of branching random walk is similar that of the one-arm probability of high-dimensional percolation. It would be interesting to see if $\varrho = 2/(4 \wedge \alpha)$ indeed holds for percolation in high dimensions.

Euclidean distance. Using properties of $\mathbb{Q}_{\mathsf{IIC}}$ allows us to estimate the expected volume of the intersection between Euclidean balls and the cluster at the origin.

Let $\mathbb{E}_{\mathsf{IIC}}$ be the expectation with respect to $\mathbb{Q}_{\mathsf{IIC}}$, and let $\mathsf{IIC} = \mathsf{IIC}(\omega)$ be the (infinite) connected component of 0. Let $N_{\mathrm{Bb}}(r)$ be the number of edges in the *backbone* of IIC at Euclidean distance at most r from 0, that is, all 'directed' edges $b = (\underline{b}, \overline{b})$ with $\underline{b} \in Q_r \cap \mathsf{IIC}$ such that $\{0 \leftrightarrow \underline{b}\}$ and $\{\overline{b} \leftrightarrow \infty\}$ occur disjointly and b is open.

Theorem 1.6 [Cluster and backbone volume bounds]. *For any percolation model that satisfies the assumptions of Theorem 1.1,*

$$\mathbb{E}_{p_c}[|Q_r \cap \mathcal{C}(0)|] \asymp r^{(2\wedge\alpha)}; \tag{1.31}$$
$$\mathbb{E}_{\mathsf{IIC}}[|Q_r \cap \mathsf{IIC}|] \asymp r^{2(2\wedge\alpha)}; \tag{1.32}$$
$$\mathbb{E}_{\mathsf{IIC}}[N_{\mathrm{Bb}}(r)] \asymp r^{(2\wedge\alpha)}. \tag{1.33}$$

Let $B_r(0;\mathcal{G})$ be the *graph-metric ball* of radius r around 0, where the graph-metric $d_{\mathcal{G}}(x,y)$, for all $x, y \in \mathbb{Z}^d$, is given by the number of edges on a shortest path between x and y in the graph $\mathcal{G}$. The graph-metric is also referred to as the *intrinsic distance*, because it only depends on the intrinsic structure of the graph. Theorem 1.6 can be contrasted with [33, Theorems 1.3, 1.4] where $\mathbb{E}_{p_c}[|B_r(0;\mathcal{C}(0))|]$ is proved to be of order r, regardless of the range of the model (i.e., the value of α does not influence the asymptotics).

This paper is organized as follows:

- (i) In Section 2 we perform a lace expansion for the measure $\mathbb{R}_{\mathsf{IIC}}$.
- (ii) In Section 3 we use this lace expansion to prove Theorem 1.2, subject to Proposition 2.5 and Lemma 2.7. Theorem 1.2(ii) and (iii) are proved in full detail, whereas we only present a rough outline of the proof of Theorem 1.2(i). We also give an outline of the proof of Theorem 1.1.
- (iii) In Section 4 we prove Theorem 1.6 using Fourier space techniques. We also prove a useful lemma that establishes a way of 'reversing the limit' for $\mathbb{Q}_{\mathsf{IIC}}$. Both results are important ingredients in the analyses of [21] and [22].
- (iv) In Section 5 we prove Theorem 1.5.
- (v) In Sections 6 and 7 we prove Proposition 2.5 and Lemma 2.7.

2. THE LACE EXPANSION

Lace expansions for percolation have been presented in numerous papers, cf. [7], [18], [39]. In particular, van der Hofstad and Járai [26] performed it with limiting schemes for the IIC in mind. Our approach is quite similar to theirs, and given that our three limiting schemes require only slightly different lace expansions, we refer the reader to the expansions for $\mathbb{P}_x$ and $\mathbb{Q}_p$ in [26] and focus mainly on the lace expansion of $\mathbb{R}_r$. This expansion is the most involved of the three, and it actually contains almost all of the elements that are required for the expansion of the other two measures. At the end of Section 3 we explain how the other two lace expansions are done. In the sections that follow we show how the limiting behavior of the terms in the expansion can be used to show that all three constructions yield the same measure.

Before we start the expansion we restate an important lemma that is at the heart of every lace expansion, namely the *Factorization Lemma* (Lemma 2.2 below).

2.1. The Factorization Lemma

Parts of this subsection are taken almost verbatim from [25, Section 2], where also the proof of Lemma 2.2 appears. We start with a few definitions.

Definition 2.1.

(i) *For any pair $x, y \in \mathbb{Z}^d$, we write $\{x, y\}$ to signify the* undirected edge *between x and y, and we write (x, y) to signify the* directed edge *from x to y. When dealing with directed edges $b = (\underline{b}, \overline{b})$, we call $\underline{b}$ the 'bottom' vertex, and $\overline{b}$ the 'top' vertex. We define $\mathcal{E}_r = \{(\underline{b}, \overline{b}) : \underline{b} \in Q_r, \overline{b} \in \mathbb{Z}^d\}$, the set of directed edges with the bottom vertex inside Q_r and the top vertex in $\mathbb{Z}^d$.*

(ii) *Let ω be an edge configuration and b an (open or closed) edge. Let ω^b be the same edge configuration with the status of the edge b changed. We say an edge b is a* pivotal edge *for the configuration ω and the event E, if $\omega \in E$ and $\omega^b \notin E$, or if $\omega \notin E$ and $\omega^b \in E$. An edge b that is pivotal for a configuration ω and a connection event $\{A \leftrightarrow B\}$ will always be assumed to be directed, i.e., $b = (\underline{b}, \overline{b})$, in such a way that $\omega, \omega^b \in \{A \leftrightarrow \underline{b}\} \cap \{\overline{b} \leftrightarrow B\}$. When we say that an edge is pivotal for an event this should be taken to mean that it is pivotal for that event in some fixed but unspecified configuration.*

(iii) *Given a set of vertices A and an edge configuration ω, we define ω_A, the restriction of ω to A, to be*

$$\omega_A(\{x, y\}) = \begin{cases} \omega(\{x, y\}) & \text{if } x, y \in A, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

for every x, y such that $\{x, y\}$ is an edge. In other words, we get ω_A from ω by making every edge that does not have both endpoints in A closed.

(iv) *Given a (deterministic or random) set of vertices A and an event E, we say that E* occurs on *A, and write $\{E \text{ on } A\}$, if $\omega_A \in E$. In other words, $\{E \text{ on } A\}$ means that E occurs on the (possibly modified) configuration in which every edge that does not have both endpoints in A is made closed. We adopt the convention that $\{x \leftrightarrow x \text{ on } A\}$ occurs if and only if $x \in A$.*

Similarly, we say that E occurs off *A, and write $\{E \text{ off } A\}$, if $\{E \text{ on } A^c\}$, where A^c is the complement of A.*

We say that E occurs through *A, and write $\{E \text{ through } A\}$ for the event that E occurs, but E does* not *occur if all the edges with at least one endpoint in A are made closed, that is, $\{E \text{ through } A\} = E \setminus \{E \text{ off } A\}$. For a two-point event $\{x \leftrightarrow y \text{ through } A\}$ we write $\{x \xleftrightarrow{A} y\}$.*

(v) *Given a (deterministic or random) set of vertices A, we define the* restricted percolation measure *for any event E:*

$$\mathbb{P}_p^A(E) = \mathbb{P}_p(E \text{ off } A). \tag{2.2}$$

Given two vertices, x and y, we define the restricted two-point function:

$$\tau_p^A(x, y) = \mathbb{P}_p(\{x \leftrightarrow y\} \text{ off } A) = \mathbb{P}_p^A(x \leftrightarrow y). \tag{2.3}$$

(vi) *Given an edge configuration and a set $A \subseteq \mathbb{Z}^d$, we define $\mathcal{C}(A)$ to be the set of vertices to which A is connected, i.e., $\mathcal{C}(A) = \{y \in \mathbb{Z}^d : A \leftrightarrow y\}$. Given an edge configuration and an edge b, we define the restricted cluster $\tilde{\mathcal{C}}^b(A)$ to be the set of vertices $y \in \mathcal{C}(A)$ to which A is connected in the (possibly modified) configuration in which b is made closed. When $A = \{x\}$ for some $x \in \mathbb{Z}^d$, as will often occur, we write $\mathcal{C}(\{x\}) = \mathcal{C}(x)$.*

The statement of the Factorization Lemma is in terms of *two* independent percolation configurations, whose laws are indicated by subscripts 0 and 1. We use the same subscripts for random variables, to indicate which law describes their distribution. Thus, the law of $\tilde{\mathcal{C}}_0^{(u,v)}(y)$ is described by $\mathbb{P}_0$, with corresponding expectation $\mathbb{E}_0$.

Lemma 2.2 [Factorization Lemma, [25]]. *Fix $p \in [0, \|D\|_\infty^{-1}]$, a directed edge (u,v), a vertex y, and events E, F. Assume that p is such that $\theta(p) = 0$. Then,*

$$\mathbb{E}\Big(\mathbb{1}_{\{E \text{ on } \tilde{\mathcal{C}}^{(u,v)}(y),\, F \text{ off } \tilde{\mathcal{C}}^{(u,v)}(y)\}}\Big) = \mathbb{E}_0\Big(\mathbb{1}_{\{E \text{ on } \tilde{\mathcal{C}}_0^{(u,v)}(y)\}}\mathbb{E}_1\big(\mathbb{1}_{\{F \text{ off } \tilde{\mathcal{C}}_0^{(u,v)}(y)\}}\big)\Big). \tag{2.4}$$

Moreover, when $E \subseteq \{u \in \tilde{\mathcal{C}}^{(u,v)}(y), v \notin \tilde{\mathcal{C}}^{(u,v)}(y)\}$, the event on the left-hand side of (2.4) is independent of the occupation status of (u,v).

2.2. The lace expansion of the one-arm IIC measure

In this section we give the lace expansion for the measure $\mathbb{R}_{\text{IIC}}$ as defined in Theorem 1.2. This lace-expansion is similar to the expansion derived in [25].

The measure $\mathbb{R}_{\text{IIC}}$ is defined for cylinder events and two-point events. The aim is to show that for some increasing subsequence (r_n), the measure

$$\mathbb{R}_{\text{IIC}}(F) = \lim_{n\to\infty} \mathbb{R}_{r_n}(F) = \lim_{n\to\infty} \mathbb{P}_{p_c}(F \mid 0 \leftrightarrow Q_{r_n}^c) = \lim_{r\to\infty} \frac{\mathbb{P}_{p_c}(F, 0 \leftrightarrow Q_{r_n}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n}^c)} \tag{2.5}$$

equals $\mathbb{Q}_{\text{IIC}}(F)$. We assume $F \in \mathfrak{F}_0$ to be determined by the edges in Q_m, for some $1 \le m \le r_1$.

Repeatedly using the inclusion-exclusion principle, we will chip away at the event $\{F, 0 \leftrightarrow Q_r^c\}$, separating out increasingly improbable events, until we end up with a complicated but manageable expression for the right-hand side of (2.5). In Section 3 we show that the limit $\mathbb{R}_{\text{IIC}}$ equals $\mathbb{Q}_{\text{IIC}}$.

When the event $\{0 \leftrightarrow Q_r^c\}$ occurs, this implies that $\{Q_m \leftrightarrow Q_r^c\}$ also occurs for any $m \le r$. Now there are two cases: The first case is that there are no pivotal edges for $\{Q_m \leftrightarrow Q_r^c\}$. This implies that both $\{0 \leftrightarrow Q_r^c\}$ and $\{Q_m \Longleftrightarrow Q_r^c\}$ occur (where, for $A, B \subset \mathbb{Z}^d$, $\{A \Longleftrightarrow B\}$ denotes the event that there are at least two disjoint paths of open edges between A and B). The second case is that there is a pivotal edge for $\{Q_m \leftrightarrow Q_r^c\}$. In this case, we write (u,v) for the first pivotal edge for $\{Q_m \leftrightarrow Q_r^c\}$. Since $\{0 \leftrightarrow Q_r^c\}$, the edge (u,v) is also pivotal for $\{0 \leftrightarrow Q_r^c\}$. We can therefore write

$$\begin{aligned}
\mathbb{P}_{p_c}(F, 0 \leftrightarrow Q_r^c) =& \mathbb{P}_{p_c}(F \cap \{0 \leftrightarrow Q_r^c\} \cap \{Q_m \Longleftrightarrow Q_r^c\}) \\
&+ \sum_{(u,v)\in\mathcal{E}_r} \mathbb{P}_{p_c}\big(F \cap \{0 \leftrightarrow u\} \cap \{Q_m \Longleftrightarrow u\} \\
&\qquad\qquad \cap \{(u,v) \text{ is open and pivotal for } Q_m \leftrightarrow Q_r^c\}\big) \\
=& \mathbb{P}_{p_c}(F, 0 \leftrightarrow Q_r^c, Q_m \Longleftrightarrow Q_r^c) \\
&+ \sum_{(u,v)\in\mathcal{E}_r} \mathbb{P}_{p_c}\big(\{F \cap \{0 \leftrightarrow u\} \cap \{Q_m \Longleftrightarrow u\} \cap \{Q_m \leftrightarrow Q_r^c\}^c \text{ on } \tilde{\mathcal{C}}^{(u,v)}(Q_m)\} \\
&\qquad\qquad \cap \{(u,v) \text{ open}\} \cap \{v \leftrightarrow Q_r^c \text{ off } \tilde{\mathcal{C}}^{(u,v)}(Q_m)\}\big).
\end{aligned} \tag{2.6}$$

In the last step we used the standard partition of an event involving a fixed pivotal edge into a part that occurs *before* the edge (i.e., on $\tilde{\mathcal{C}}^{(u,v)}(Q_m)$) and a part occurring *after* the edge (i.e., off $\tilde{\mathcal{C}}^{(u,v)}(Q_m)$). The extra event $\{Q_m \leftrightarrow Q_r^c\}^c$ that occurs on $\tilde{\mathcal{C}}^{(u,v)}(Q_m)$ on the right-hand side of (2.6) is there to ensure that the edge (u,v) is still pivotal after the partition.

We define

$$\xi^{(0)}(r;F) = \mathbb{P}_{p_c}(F \cap \{0 \leftrightarrow Q_r^c, Q_m \Longleftrightarrow Q_r^c\}) \tag{2.7}$$

and

$$\gamma^{(0)}(r;F) = \sum_{(u,v)\in\mathcal{E}_r} p_{uv}\mathbb{E}_0[\mathbb{1}_{\{F\cap\{0\leftrightarrow u, Q_m \Longleftrightarrow u, Q_m \leftrightarrow Q_r^c\} \text{ on } \tilde{\mathcal{C}}^{(u,v)}(Q_m)\}}\mathbb{P}_1^{\tilde{\mathcal{C}}^{(u,v)}(Q_m)}(v \leftrightarrow Q_r^c)]. \tag{2.8}$$

Using that for any event E, $\mathbb{1}_{E^c} = 1 - \mathbb{1}_E$, and applying this and the Factorization Lemma to the right-hand side of (2.6) yields

$$\mathbb{P}_{p_c}(F, 0 \leftrightarrow Q_r^c) = \xi^{(0)}(r;F) - \gamma^{(0)}(r;F) + \sum_{(u,v)\in\mathcal{E}_r} p_{uv}\mathbb{E}_{p_c}[\mathbb{1}_{F\cap\{0\leftrightarrow u, Q_m \Longleftrightarrow u\}}\mathbb{P}_{p_c}^{\tilde{\mathcal{C}}^{(u,v)}(Q_m)}(v \leftrightarrow Q_r^c)]. \quad (2.9)$$

For $x \in \mathbb{Z}^d$, define

$$\pi^{(0)}(x, r; F) = \mathbb{P}_{p_c}(F \cap \{0 \leftrightarrow x, Q_m \Longleftrightarrow x\}). \quad (2.10)$$

Although $\pi^{(0)}(x, r; F)$ is independent of r, the higher order terms $\pi^{(n)}(x, r; F)$ do depend on r, so we write the redundant argument r here for compatibility later on. For $v \in \mathbb{Z}^d$ define

$$\psi^{(0)}(v, r; F) = \sum_{u\in Q_r} p_{uv}\pi^{(0)}(u, r; F) \quad (2.11)$$

and

$$R^{(0)}(r;F) = \sum_{(u,v)\in\mathcal{E}_r} p_{uv}\mathbb{E}_{p_c}[\mathbb{1}_{F\cap\{0\leftrightarrow u, Q_m \Longleftrightarrow u\}}(\mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) - \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}^{(u,v)}(Q_m)}(v \leftrightarrow Q_r^c))]. \quad (2.12)$$

Then,

$$\sum_{(u,v)\in\mathcal{E}_r} p_{uv}\mathbb{P}_{p_c}(F \cap \{0 \leftrightarrow u, Q_m \Longleftrightarrow u\})\mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) = \sum_{v\in\mathbb{Z}^d}\psi^{(0)}(v, r; F)\mathbb{P}_{p_c}(v \leftrightarrow Q_r^c). \quad (2.13)$$

and using the identity

$$\mathbb{P}_{p_c}^A(v \leftrightarrow Q_r^c) = \mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) - [\mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) - \mathbb{P}_{p_c}^A(v \leftrightarrow Q_r^c)] \quad (2.14)$$

with $A = \tilde{\mathcal{C}}^{(u,v)}(Q_m)$, we can write

$$\mathbb{P}_{p_c}(F, 0 \leftrightarrow Q_r^c) = \xi^{(0)}(r;F) - \gamma^{(0)}(r;F) + \sum_{v\in\mathbb{Z}^d}\psi^{(0)}(v, r; F)\mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) - R^{(0)}(r;F). \quad (2.15)$$

Now the aim is expand $R^{(0)}(r;F)$.

For $A \subseteq \mathbb{Z}^d$, define the events

$$E'(v, x; A) = \left\{ \begin{array}{l} v \xleftrightarrow{A} x \text{ and there is no pivotal edge } (u_1, v_1) \\ \text{for the connection } v \leftrightarrow x \text{ such that } v \xleftrightarrow{A} u_1 \end{array} \right\}, \quad (2.16)$$

and

$$E''(v, r; A) = \left\{ \begin{array}{l} v \xleftrightarrow{A} Q_r^c \text{ and there is no pivotal edge } (u_1, v_1) \\ \text{for the connection } v \leftrightarrow Q_r^c \text{ such that } v \xleftrightarrow{A} u_1 \end{array} \right\}. \quad (2.17)$$

Write $A \dot\cup B$ for the *disjoint union* of A and B: the union of the events A and B that have no elements in common (i.e., $A \cap B = \varnothing$). We can partition the event $\{v \xleftrightarrow{A} Q_r^c\}$ into a disjoint union of events E' and E'':

Lemma 2.3. *For any $v \in \mathbb{Z}^d$, $A \subseteq \mathbb{Z}^d$ and $r \in \mathbb{N}$:*

$$\{v \xleftrightarrow{A} Q_r^c\} = E''(v, r; A) \dot\cup \dot{\bigcup_{b=(\underline{b},\overline{b})\in\mathcal{E}_r}} [E'(v, \underline{b}; A) \cap \{b \text{ is open and pivotal for } v \leftrightarrow Q_r^c\}]. \quad (2.18)$$

Proof. We decompose the event $\{v \xleftrightarrow{A} Q_r^c\}$ according to whether or not there is an open pivotal edge $b = (\underline{b}, \overline{b})$ such that (1) $\{v \xleftrightarrow{A} \underline{b}\}$ and (2) $b \in \mathcal{E}_r$ is the first such edge along the path from v to Q_r^c that has this property. When such an edge does not exist, the event $E''(v, r; A)$ occurs. If an edge b with these properties does exist, then since it is the first edge that is pivotal for $\{v \leftrightarrow Q_r^c\}$ and $\{v \xleftrightarrow{A} \underline{b}\}$ occurs, there can be no other edge $b' \in \mathcal{E}_r$ that is open and pivotal for $\{v \leftrightarrow \underline{b}\}$ such that $\{v \xleftrightarrow{A} \underline{b}'\}$. Therefore, $E'(v, \underline{b}; A)$ holds. □

By the Factorization Lemma, for any $v \in Q_r$, $r \in \mathbb{N}$, $b = (\underline{b}, \overline{b}) \in \mathcal{E}_r$ and $A \subseteq \mathbb{Z}^d$,

$$\mathbb{E}_{p_c}[\mathbb{1}_{\{E'(v,\underline{b};A)\cap\{b \text{ open \& piv. for } v\leftrightarrow Q_r^c\}\}}] = p_b \mathbb{E}_{p_c}[\mathbb{1}_{\{E'(v,\underline{b};A)\cap\{v\leftrightarrow Q_r^c\}^c \text{ on } \tilde{\mathcal{C}}^b(v)\}} \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}^b(v)}(\overline{b} \leftrightarrow Q_r^c)] \tag{2.19}$$

where $p_b = p_{\underline{b}\overline{b}}$.

Using $\{v \xleftrightarrow{A} Q_r^c\} = \{v \leftrightarrow Q_r^c\} \setminus \{v \leftrightarrow Q_r^c \text{ off } A\}$, Definition 2.1 and Lemma 2.3 and (2.19), we can write

$$\begin{aligned}\mathbb{P}_{p_c}(v \xleftrightarrow{A} Q_r^c) = \mathbb{P}_{p_c}(E''(v,r;A)) &+ \sum_{b\in\mathcal{E}_r} p_b \mathbb{E}_{p_c}[\mathbb{1}_{E'(v,\underline{b};A)} \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}^b(v)}(\overline{b} \leftrightarrow Q_r^c)] \\ &- \sum_{b\in\mathcal{E}_r} p_b \mathbb{E}_{p_c}[\mathbb{1}_{\{E'(v,\underline{b};A)\cap\{v\leftrightarrow Q_r^c\} \text{ on } \tilde{\mathcal{C}}^b(v)\}} \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}^b(v)}(\overline{b} \leftrightarrow Q_r^c)].\end{aligned} \tag{2.20}$$

Given edges $(u_0, v_0), (u_1, v_1), \ldots$, we denote

$$\tilde{\mathcal{C}}_0 = \tilde{\mathcal{C}}^{(u_0,v_0)}(Q_m) \quad \text{and} \quad \tilde{\mathcal{C}}_j = \tilde{\mathcal{C}}^{(u_j,v_j)}(v_{j-1}) \quad \text{for } j \geq 1 \tag{2.21}$$

and write

$$\mathbb{1}_j = \mathbb{1}_{E'(v_{j-1},u_j;\tilde{\mathcal{C}}_{j-1})}, \quad \text{for } j \geq 1. \tag{2.22}$$

Inserting (2.20) with $v = 0$ and $A = \tilde{\mathcal{C}}_0$ into (2.12), yields

$$\begin{aligned}R^{(0)}(r;F) = &\sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0} \mathbb{E}_0[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \mathbb{E}_1[\mathbb{1}_{E''(v_0,r;\tilde{\mathcal{C}}_0)}]] \\ &+ \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0} \sum_{(u_1,v_1)\in\mathcal{E}_r} p_{u_1v_1} \mathbb{E}_0[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \mathbb{E}_1[\mathbb{1}_1 \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}_1}(v_1 \leftrightarrow Q_r^c)]] \\ &- \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0} \sum_{(u_1,v_1)\in\mathcal{E}_r} p_{u_1v_1} \mathbb{E}_0[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \\ &\qquad \times \mathbb{E}_1[\mathbb{1}_{\{E'(v_0,u_1;\tilde{\mathcal{C}}_0)\cap\{v_0\leftrightarrow Q_r^c\} \text{ on } \tilde{\mathcal{C}}_1\}} \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}_1}(v_1 \leftrightarrow Q_r^c)]].\end{aligned} \tag{2.23}$$

We define the first term on the right-hand side as $\xi^{(1)}(r;F)$ and the last term as $\gamma^{(1)}(r;F)$. We define

$$\pi^{(1)}(u,r;F) = \sum_{(u_0 v_0)\in\mathcal{E}_r} p_{u_0v_0} \mathbb{E}_0\left[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \mathbb{E}_1\left[\mathbb{1}_{E'(v_0,u;\tilde{\mathcal{C}}_0)}\right]\right], \tag{2.24}$$

and

$$\psi^{(1)}(v,r;F) = \sum_{u\in Q_r} p_{uv} \pi^{(1)}(u,r;F). \tag{2.25}$$

We define $R^{(1)}(r;F)$ such that

$$\begin{aligned}\sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0} \sum_{(u_1 v_1)\in\mathcal{E}_r} p_{u_1v_1} \mathbb{E}_0[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \mathbb{E}_1[\mathbb{1}_1 \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}_1}(v_1 \leftrightarrow Q_r^c)]] \\ = \sum_{v\in Q_r} \psi^{(1)}(v,r;F) \mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) - R^{(1)}(r;F)\end{aligned} \tag{2.26}$$

where we used that $\mathbb{P}_{p_c}^{\tilde{\mathcal{C}}_1}(v \leftrightarrow Q_r^c) = \mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) - \mathbb{P}_{p_c}(v \xleftrightarrow{\tilde{\mathcal{C}}_1} Q_r^c)$.

Hence, we can write $R^{(0)}(r;F)$ as

$$R^{(0)}(r;F) = \xi^{(1)}(r;F) - \gamma^{(1)}(r;F) + \sum_{v\in\mathbb{Z}^d} \psi^{(1)}(v,r;F) \mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) - R^{(1)}(r;F). \tag{2.27}$$

From here we continue to extract terms $\xi^{(2)}(r;F)$, $\gamma^{(2)}(r;F)$, $\sum_{v\in Q_r} \psi^{(2)}(v,r;F)\mathbb{P}_{p_c}(v \leftrightarrow Q_r^c)$ and $R^{(2)}(r;F)$ from $R^{(1)}(r;F)$, and so forth. We end up with the following:

Proposition 2.4 [The lace expansion]. *For $N \geq 0$ and $0 < m \leq r$,*

$$\begin{aligned}\mathbb{P}_{p_c}(F, 0 \leftrightarrow Q_r^c) = \sum_{n=0}^{N} (-1)^n \xi^{(n)}(r;F) - \sum_{n=0}^{N} (-1)^n \gamma^{(n)}(r;F) \\ + \sum_{n=0}^{N} (-1)^n \sum_{v \in \mathbb{Z}^d} \psi^{(n)}(v,r;F) \mathbb{P}_{p_c}(v \leftrightarrow Q_r^c) + (-1)^{N+1} R^{(N)}(r;F).\end{aligned} \tag{2.28}$$

Here, $\xi^{(0)}(r;F)$ is given by (2.7), $\gamma^{(0)}(r;F)$ is given by (2.8), $\pi^{(0)}(x,r;F)$ is given by (2.10), and for $n \geq 1$,

$$\begin{aligned}\xi^{(n)}(r;F) &= \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0} \cdots \sum_{(u_{n-1},v_{n-1})\in\mathcal{E}_r} p_{u_{n-1}v_{n-1}} \mathbb{E}_0\Big[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \\ &\quad \times \mathbb{E}_1\Big[\mathbb{1}_1 \mathbb{E}_2\big[\mathbb{1}_2 \cdots \mathbb{E}_{n-1}[\mathbb{1}_{n-1}\mathbb{E}_n[\mathbb{1}_{E''(v_{n-1},r;\tilde{\mathcal{C}}_{n-1})}]] \cdots \big]\Big]\Big];\end{aligned} \tag{2.29}$$

$$\begin{aligned}\gamma^{(n)}(r;F) &= \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0} \cdots \sum_{(u_n,v_n)\in\mathcal{E}_r} p_{u_nv_n} \mathbb{E}_0\Big[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \\ &\quad \times \mathbb{E}_1\Big[\mathbb{1}_1 \mathbb{E}_2\big[\mathbb{1}_2 \cdots \mathbb{E}_{n-1}[\mathbb{1}_{\{E'(v_{n-1},u_n;\tilde{\mathcal{C}}_{n-1})\cap\{v_{n-1}\leftrightarrow Q_r^c\} \text{ on } \tilde{\mathcal{C}}_n\}} \mathbb{P}_n^{\tilde{\mathcal{C}}_n}(v_n \leftrightarrow Q_r^c)] \cdots \big]\Big]\Big];\end{aligned} \tag{2.30}$$

$$\begin{aligned}\pi^{(n)}(x,r;F) &= \sum_{(u_0v_0)\in\mathcal{E}_r} p_{u_0,v_0} \cdots \sum_{(u_{n-1},v_{n-1})\in\mathcal{E}_r} p_{u_{n-1}v_{n-1}} \mathbb{E}_0\Big[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \\ &\quad \times \mathbb{E}_1\Big[\mathbb{1}_1 \mathbb{E}_2\big[\mathbb{1}_2 \cdots \mathbb{E}_{n-1}[\mathbb{1}_{n-1}\mathbb{E}_n[\mathbb{1}_{E'(v_{n-1},x;\tilde{\mathcal{C}}_{n-1})}]] \cdots \big]\Big]\Big].\end{aligned} \tag{2.31}$$

Also, for $n \geq 0$,

$$\psi^{(n)}(v,r;F) = \sum_{u \in Q_r} p_{uv} \pi^{(n)}(u,r;F), \tag{2.32}$$

and

$$\begin{aligned}R^{(N)}(r;F) = \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0} \cdots \sum_{(u_N,v_N)\in\mathcal{E}_r} p_{u_Nv_N} \mathbb{E}_0\Big[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}} \\ \times \mathbb{E}_1\Big[\mathbb{1}_1 \mathbb{E}_2\big[\mathbb{1}_2 \cdots \mathbb{E}_N[\mathbb{1}_N(\mathbb{P}_{p_c}(v_N \leftrightarrow Q_r^c) - \mathbb{P}_{p_c}^{\tilde{\mathcal{C}}_N}(v_N \leftrightarrow Q_r^c))] \cdots \big]\Big]\Big].\end{aligned} \tag{2.33}$$

2.3. Bounds on the expansion terms

To prove Theorem 1.2(ii) and (iii), we have to give bounds on the terms of the expansion. For this we use the following proposition:

Proposition 2.5 [Fundamental bound on expansion terms]. *Under the assumptions of both Theorem 1.2(ii) and (iii), the following holds: For some $\delta > 0$, any $r \in \mathbb{N}$ and any $F \in \mathfrak{F}_0$ there is exists a constant $K = K(F, \beta, d, \alpha, \delta)$ such that*

$$\sum_{x\in\mathbb{Z}^d} \sum_{n=0}^{\infty} |x|^{(2\wedge\alpha)+\delta} \pi^{(n)}(x,r;F) \leq K \qquad \text{and} \qquad \sum_{x\in\mathbb{Z}^d} \sum_{n=0}^{\infty} |x|^{(2\wedge\alpha)+\delta} \psi^{(n)}(x,r;F) \leq K. \tag{2.34}$$

We prove this proposition in Sections 6 and 7.

Remark 2.6. The assumptions under which this result holds can be modified to apply to more general cases. In particular, setting $r = \infty$ and $F = \Omega$ in (2.31), we can define

$$\Pi^{\text{classical}}(x) \equiv \sum_{n=0}^{\infty} (-1)^n \pi^{(n)}(x, \infty; \Omega). \tag{2.35}$$

It is a well-known result (see e.g. [39, Chapter 10]) that the 'classical' inclusion-exclusion lace expansion for the percolation two-point function yields the convolution equation

$$\tau_{p_c}(x) = \Pi^{\text{classical}}(x) + (p_c D * \tau_{p_c} * \Pi^{\text{classical}})(x). \tag{2.36}$$

With minor modifications to the proof of Proposition 2.5 one can show that for some $\delta' > 0$ and $\beta \le \beta_4 = \beta_4(d,\alpha)$, there is a $K' = K'(\beta, d, \alpha, \delta')$ such that

$$\sum_{x\in\mathbb{Z}^d} |x|^{(2\wedge\alpha)+\delta'} \pi^{(n)}(x,\infty;\Omega) \le K'. \tag{2.37}$$

Two other bounds that we will use in the upcoming section are stated in the following Lemma:

Lemma 2.7 [Bounds on expansion terms]. *Under the assumptions of both Theorem 1.2(ii) and (iii), the following holds:*

(i) For any $r \in \mathbb{N}$ and any $F \in \mathfrak{F}_0$, there is a constant $K'' = K''(F, \beta, d, \alpha)$ and an $\varepsilon > 0$ such that

$$\sum_{n=0}^{\infty} \xi^{(n)}(r;F) \le \frac{K''}{r^{1/\varrho+\varepsilon}} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \gamma(r;F) \le \frac{K''}{r^{1/\varrho+\varepsilon}}; \tag{2.38}$$

(ii) For any $r \in \mathbb{N}$

$$\lim_{N\to\infty} R^{(N)}(r;F) = 0. \tag{2.39}$$

We prove this lemma in Section 6 using Proposition 2.5 and Lemma 6.1 below.

3. Existence of the IIC in various constructions

3.1. Existence of the one-arm IIC measure

Proof of Theorem 1.2(ii) and (iii) subject to Proposition 2.5 and Lemma 2.7. We start by defining

$$\Xi(r;F) = \sum_{n=0}^{\infty} (-1)^n \xi^{(n)}(r;F); \tag{3.1}$$

$$\Gamma(r;F) = \sum_{n=0}^{\infty} (-1)^n \gamma^{(n)}(r;F); \tag{3.2}$$

$$\Pi(x,r;F) = \sum_{n=0}^{\infty} (-1)^n \pi^{(n)}(x,r;F); \tag{3.3}$$

$$\Psi(x,r;F) = \sum_{n=0}^{\infty} (-1)^n \psi^{(n)}(x,r;F). \tag{3.4}$$

By Proposition 2.5, Lemma 2.7, and (2.32), it follows that the sums on the right-hand sides of (3.1)–(3.4) converge. Therefore we may take the limit $N \to \infty$ in (2.28) to get

$$\mathbb{P}_{p_c}(F, 0 \leftrightarrow Q_r^c) = \Xi(r;F) - \Gamma(r;F) + \sum_{y\in\mathbb{Z}^d} \Psi(y,r;F)\mathbb{P}_{p_c}(y \leftrightarrow Q_r^c). \tag{3.5}$$

Dividing (3.5) by $\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)$ gives

$$\mathbb{R}_r(F) = \frac{\Xi(r;F) - \Gamma(r;F)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)} + \sum_{x\in\mathbb{Z}^d} \Psi(x,r;F) \frac{\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)}. \tag{3.6}$$

Let $\Psi(x;F)$ denote the function $\Psi(x,r;F)$ with all the summations over edges extended to the set $\mathbb{Z}^d \times \mathbb{Z}^d$, and let $\Pi(x;F)$ be defined similarly. Then $\Psi(y;F) = \lim_{r\to\infty} \Psi(y,r;F)$. The aim now is to show that $\lim_{r\to\infty} \mathbb{R}_r(F) = \sum_{y\in\mathbb{Z}^d} \Psi(y;F)$.

By (1.28) and Proposition Lemma 2.7(i),

$$\lim_{r\to\infty} \frac{\Xi(r;F) - \Gamma(r;F)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)} = 0. \tag{3.7}$$

We are left to deal with the second term of (3.5). To evaluate the right-hand side, we split up the sum over x into three parts. For $a \in (0,1)$ we evaluate separately the contributions to the sum from $|x| \le r^a$, $r^a < |x| \le r/4$ and $|x| > r/4$. We first prove that the latter two parts make only a small contribution to the whole, i.e.,

$$\sum_{|x|>r^a} \Psi(x,r;F)\frac{\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)} = o(1), \tag{3.8}$$

so the dominant contributions to the sum arise from $x \in Q_{r^a}$.

We start by observing that for all x with $|x| < r/4$,

$$\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) \le \mathbb{P}_{p_c}(0 \leftrightarrow Q_{r/2}^c) \le Cr^{-1/\varrho}, \tag{3.9}$$

so from (1.28) it follows that

$$\frac{\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)} \le C. \tag{3.10}$$

Therefore,

$$\sum_{r^a<|x|\le r/4} \Psi(x,r;F)\frac{\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)} \le C \sum_{r^a<|x|\le r/4} |\Psi(x,r;F)|. \tag{3.11}$$

For all x such that $r^a < |x|$, we have $|x|/r^a > 1$, so by Proposition 2.5

$$C\sum_{r^a<|x|\le r/4} |\Psi(x,r;F)| \le \frac{C}{r^{a((2\wedge\alpha)+\delta)}} \sum_{r^a<|x|<r/4} |x|^{(2\wedge\alpha)+\delta}|\Psi(x,r;F)| \le \frac{C}{r^{a((2\wedge\alpha)+\delta)}} = o(1). \tag{3.12}$$

Hence, the contributions to (3.8) that come from $r^a < |x| \le r/4$ is $o(1)$.

To bound the contributions to (3.8) that come from $|x| > r/4$ we also use Proposition 2.5: now we have by (1.28) that $\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \ge cr^{-1/\varrho} \ge c|4x|^{-(2\wedge\alpha)}$. Furthermore, $(4|x|)^\delta/r^\delta \ge 1$ so it follows that

$$\begin{aligned}\sum_{|x|\ge r/4} \frac{|\Psi(x,r;F)|}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)} &\le \sum_{|x|\ge r/4} C|x|^{(2\wedge\alpha)}|\Psi(x,r;F)| \\ &\le \frac{4^\delta C}{r^\delta}\sum_{x\in\mathbb{Z}^d} |x|^{(2\wedge\alpha)+\delta}|\Psi(x,r;F)| = O(r^{-\delta}) = o(1).\end{aligned} \tag{3.13}$$

This proves (3.8).

In Theorem 1.2(ii) we assumed that $\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \simeq r^{-1/\varrho}$, which implies by monotonicity of the one-arm probability in r that the ratio of the one-arm probabilities converges to 1 whenever $|x|$ is sufficiently small, i.e.,

$$\lim_{r\to\infty} \frac{\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)} = \lim_{r\to\infty} \frac{r^{1/\varrho}}{(r-r^a)^{1/\varrho}}(1+o(1)) = 1. \tag{3.14}$$

Furthermore, $\psi^{(n)}(x,r;F)$ is monotonically increasing as r increases. Hence, taking the limit $r \to \infty$ in (3.6), it follows that

$$\mathbb{R}_{\mathrm{IIC}}(F) = \lim_{r\to\infty} \mathbb{R}_r(F) = \sum_{x\in\mathbb{Z}^d} \Psi(x;F) = p_c \sum_{y\in\mathbb{Z}^d} \Pi(y;F), \tag{3.15}$$

exists by summability and dominated convergence. The last step follows from $\sum_v D(v-u) = 1$ and (2.32). Note that the right-hand side of (3.15) is the same expression as was obtained through the other known limiting schemes for construction of the IIC (as given in [26]) so $\mathbb{R}_{\mathrm{IIC}}$ is in fact the same measure as $\mathbb{P}_{\mathrm{IIC}}$ and $\mathbb{Q}_{\mathrm{IIC}}$. This completes the proof of Theorem 1.2(ii).

To prove Theorem 1.2(iii), we can follow the same steps as above, except that now a more involved analysis of the limit ratio on the left-hand side of (3.14) is required. The important contributions still come from the vertices near the origin, i.e., $|x| \le r^a$. We show that for such x, the ratio of the probabilities converges to 1 along some subsequence of (r).

Lemma 3.1 [Convergence of the ratio]. *Under the assumptions of Theorem 1.2(iii) there exists a sequence* (r_n) *with* $r_n \to \infty$ *as* $n \to \infty$*, such that for any* $a \in (0,1)$

$$\limsup_{n\to\infty} \max_{|x|\le r_n^a} \left| \frac{\mathbb{P}_{p_c}(x \leftrightarrow Q_{r_n}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n}^c)} - 1 \right| = 0. \tag{3.16}$$

Proof. Let $\mathcal{A}$ be the set of accumulation points of

$$\{r^{1/\varrho}\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) | r \in \mathbb{R}\}. \tag{3.17}$$

The set $\mathcal{A}$ is closed. In the finite-range setting, Kozma and Nachmias proved that $\mathcal{A}$ is bounded and positive [34], and for long-range percolation this is our assumption. Hence, it is compact and contains a positive minimum:

$$A = \min \mathcal{A} \in (0,\infty). \tag{3.18}$$

Since A is an accumulation point, there exists a subsequence $(\tilde{r}_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \tilde{r}_n^{1/\varrho}\mathbb{P}_{p_c}(0 \leftrightarrow Q_{\varrho_n}^c) = A. \tag{3.19}$$

Choose the sequence (r_n) such that $r_n - r_n^a = \tilde{r}_n$.

Take $N \in \mathbb{N}$ such that $|x| \le r_n^a$ for all $n \ge N$. By translation invariance of the measure $\mathbb{P}_{p_c}$ and monotonicity of the event $\{0 \leftrightarrow Q_r^c\}$ as r increases, we have for $x \in Q_{r^a}$ the bounds

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r+r^a}^c) \le \mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) \le \mathbb{P}_{p_c}(0 \leftrightarrow Q_{r-r^a}^c). \tag{3.20}$$

This implies

$$\max_{|x|\le r_n^a} \left| \frac{\mathbb{P}_{p_c}(x \leftrightarrow Q_{r_n}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n}^c)} - 1 \right| \le \max\left(1 - \frac{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n+r_n^a}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n}^c)}, \frac{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n-r_n^a}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n}^c)} - 1\right) \tag{3.21}$$

so that it suffices to show that there exists a subsequence $(r_n)_{n\in\mathbb{N}}$ so that both

$$\lim_{n\to\infty} \frac{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n+r_n^a}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n}^c)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n-r_n^a}^c)} = 1. \tag{3.22}$$

Since $\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c)$ is monotonically decreasing in r, both ratios are at most equal to 1. Monotonicity also implies that $\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n+r_n^a}^c) \le \mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n-r_n^a}^c)$, so (3.22) follows once we show that

$$\liminf_{n\to\infty} \frac{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n+r_n^a}^c)}{\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n-r_n^a}^c)} = 1. \tag{3.23}$$

It is obvious that the left-hand side is at most 1, so we will focus on proving that it also is at least 1.

We give a proof by contradiction. Suppose that there exists an $0 < \varepsilon < 1$ such that, for the sequence $(r_n)_{n\in\mathbb{N}}$,

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n+r_n^a}^c \mid 0 \leftrightarrow Q_{r_n-r_n^a}^c) \le 1 - \varepsilon. \tag{3.24}$$

Then also

$$\frac{(r_n - r_n^a)^{1/\varrho}}{(r_n + r_n^a)^{1/\varrho}} \frac{(r_n + r_n^a)^{1/\varrho}\,\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n+r_n^a}^c)}{(r_n - r_n^a)^{1/\varrho}\,\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n-r_n^a}^c)} \le 1 - \varepsilon. \tag{3.25}$$

There exists an $N' \in \mathbb{N}$, such that for $n \ge N'$,

$$\frac{(r_n - r_n^a)^{1/\varrho}}{(r_n + r_n^a)^{1/\varrho}} \ge \sqrt{1-\varepsilon}, \tag{3.26}$$

so it follows that

$$\sqrt{1-\varepsilon}\,(r_n + r_n^a)^{1/\varrho}\,\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n+r_n^a}^c) \le (1-\varepsilon)\,(r_n - r_n^a)^{1/\varrho}\,\mathbb{P}_{p_c}(0 \leftrightarrow Q_{r_n-r_n^a}^c). \tag{3.27}$$

Recall that $\tilde{r}_n = r_n - r_n^a$. Taking lim inf on both sides of (3.27) and using the fact that A is the minimum of $\mathcal{A}$, we get

$$\begin{aligned}\sqrt{1-\varepsilon}A &\le \sqrt{1-\varepsilon}\liminf_n (r_n + r_n^a)^{1/\varrho}\mathbb{P}_{p_c}(0 \leftrightarrow Q^c_{r_n+r_n^a}) \\ &\le (1-\varepsilon)\liminf_n \tilde{r}_n^{1/\varrho}\mathbb{P}_{p_c}(0 \leftrightarrow Q^c_{\tilde{r}_n}) = (1-\varepsilon)A,\end{aligned} \tag{3.28}$$

which yields a contradiction. This proves (3.22) and hence the claim of the lemma follows. □

Applying Lemma 3.1, it follows that

$$\lim_{n\to\infty}\mathbb{P}_{p_c}(F \mid 0 \leftrightarrow Q^c_{r_n}) = \sum_{x\in\mathbb{Z}^d}\Psi(x;F) = p_c\sum_{y\in\mathbb{Z}^d}\Pi(y;F), \tag{3.29}$$

along a subsequence (r_n). Observe that this limit is the same as the one we got in (3.15), and furthermore, following the proofs that are outlined in the next subsection, we can easily check that this is also equal to $\lim_{p\nearrow p_c}\mathbb{Q}_p(F)$, proving Theorem 1.2(iii). □

3.2. Existence of the IIC susceptibility and two-point constructions

The IIC susceptibility construction is similar to the one given in [26] and the proof of Theorem 1.1 can be given along the same lines as that of the original construction (i.e., [26, Theorem 1.2]). The single modification of the argument is that the x-space bounds on the two-point function used to bound the lace expansion diagrams in [26] are replaced by bounds on the triangle diagram in Fourier space to achieve the same effect. The main reasoning remains unchanged. Hence, we will not give the proof.

The proof of Theorem 1.2(i), in turn, is very similar to [26, Theorem 1.1]. The only modification that we need to make here is that we must replace every instance of the nearest-neighbor and finite-range two-point function by the long-range two-point function, and subsequently, we must apply our assumed bound (1.19) to every instance $\tau_{p_c}(x)$ instead of the bound (1.18) that is used in [26].

Going through the steps of the proof with this replacement, it is not hard to see that the only thing we need to show to complete the proof is that for $d > 3(2\wedge\alpha)$ and $L > L_1$, there exists a constant $C' = C'(d,\alpha,L,F)$ such that

$$|\Pi(x;F)| \le \frac{C'}{(|x|+1)^{2(d-(2\wedge\alpha))}} \tag{3.30}$$

where $\Pi(x;F)$ is now defined as in [26]. That this bound indeed holds follows immediately from (1.19) and [17, Proposition 1.8(c)].

4. Volume estimates of the IIC: proof of Theorem 1.6

In this section we calculate upper and lower bounds for the expectations of the volume of critical percolation clusters and IICs inside Euclidean balls. We also calculate bounds on the expected volume of the IIC backbone.

4.1. Review of important results

Before we start with the proof of Theorem 1.6 we discuss a few useful results.

4.1.1. *The BK-inequality.* An important tool in the coming analysis is the *van den Berg-Kesten inequality* (BK for short) [6],[15]. We call an event A *increasing* if for any two configurations ω and ω' such that $\omega \le \omega'$ (that is, any edge that is open in ω is also open in ω'), $\omega \in A$ implies $\omega' \in A$. Hence, by a standard coupling argument, if A is increasing, then $\mathbb{P}_p(A) \le \mathbb{P}_{p'}(A)$ whenever $p < p'$. For two increasing events A and B we write $A \circ B$ to indicate the disjoint occurrence of A

and B. This is the event that the set of edges can be split in two parts, say K and K^c, such that A occurs on K (i.e., $\omega_K \in A$) and B occurs on K^c (i.e., $\omega_{K^c} \in B$). The BK-inequality states

$$\mathbb{P}_p(A \circ B) \le \mathbb{P}_p(A)\mathbb{P}_p(B). \tag{4.1}$$

We use the BK-inequality to bound the probability of complicated events, such as those in the various lace expansion terms, by a product of the probabilities of the disjointly occurring events that make up its parts.

4.1.2. *Bounds on two triangle diagrams.* It is often useful to reduce events to triplets of disjointly occurring path event. Taking the probability of these triplets results in the so-called *triangle diagrams:*

$$\triangle_p(x) = (\tau_p * \tau_p * \tau_p)(x), \qquad \bar{\triangle}_p = \sup_{x\in\mathbb{Z}^d} \triangle_p(x), \tag{4.2}$$

$$T_p(x) = (\tau_p * \tau_p * D * \tau_p)(x), \qquad T_p = \sup_{x\in\mathbb{Z}^d} T_p(x). \tag{4.3}$$

We can give bounds on the two triangle diagrams for the percolation models that we study in terms of the parameter β (as defined in (1.15)):

Lemma 4.1 [An upper bound on the open triangle]. *Consider a percolation model that satisfies Assumptions C and D with mean-field parameter β. We have $\bar{\triangle}_p \le 1 + O(\beta)$ and $T_p \le O(\beta)$ whenever $\beta \le \beta_0$ and $p \le p_c$.*

Variants of this lemma have been proved numerous times in the lace expansion literature. The bound on $\bar{\triangle}_p$ was proved in [18] for finite-range models, and in [23] for long-range models. The bound on T_p follows from a similar argument.

4.2. Bounds on the expected volume of critical clusters in a ball: proof of (1.31)

The aim of this section is to bound the volume of a critical percolation configuration inside a Euclidean ball, that is, we prove bounds on

$$\mathbb{E}_{p_c}[|Q_r \cap \mathcal{C}(0)|] = \sum_{x\in Q_r} \tau_{p_c}(x) = \sum_{x\in\mathbb{Z}^d} \tau_{p_c}(x)\mathbb{1}_{Q_r}(-x) = (\tau_{p_c} * \mathbb{1}_{Q_r})(0). \tag{4.4}$$

We start with the upper bound. The proof makes use Fourier-space bounds on τ_{p_c}. The Fourier transform of the indicator function may be negative, making it difficult to use. To get around this issue, we introduce the function $g_r(x)$: let $\Lambda_r = \{x \in \mathbb{Z}^d : \|x\|_\infty \le r\}$ and define

$$g_r(x) = (2r+1)^d (p_r * p_r)(x) \qquad \text{with} \qquad p_r(x) = \frac{\mathbb{1}_{\Lambda_r}(x)}{(2\lfloor r \rfloor + 1)^d}. \tag{4.5}$$

Note that p_r is in fact a probability distribution. The precise definition of g_r is not very important. What is important is that g_r satisfies the following criteria:

(i) $g_r(x) \ge \mathbb{1}_{Q_r}(x)$ for all $x \in \mathbb{Z}^d$;
(ii) $\hat{g}_r(k) \ge 0$ for all $k \in [-\pi,\pi]^d$;
(iii) $\sum_{x\in\mathbb{Z}^d} \tau_{p_c}(x) g_r(-x) \le C r^{(2\wedge\alpha)}$.

When all three criteria are satisfied the result is the desired upper bound $\mathbb{E}_{p_c}[|Q_r \cap \mathcal{C}(0)|] \le C r^{(2\wedge\alpha)}$.

It is easy to check criterion (i), and (ii) follows from the fact that $\hat{g}_r(k) = (2r+1)^d \hat{p}_r(k)^2$. All that is left is to check criterion (iii): We start by mentioning the bounds for $p \le p_c$,

$$0 \le \hat{\tau}_p(k) \le \frac{1+O(\beta)}{1-\hat{D}(k)}. \tag{4.6}$$

The lower bound was established in [3] for all percolation two-point functions. In [18], the upper bound is proved for finite-range percolation in dimension $d > 6$ and for nearest-neighbor percolation in dimension $d \ge 19$ and in [23] the same bound is proved for long-range spread-out models in dimensions $d > 3(2\wedge\alpha)$. Note that it is not a-priori clear that $\hat\tau_{p_c}(k)$ is well-defined, since $\hat\tau_{p_c}(0) = \chi(p_c) = \infty$. Hara, in [16, Appendix A] proves that $\hat\tau_{p_c}(k)$ is well-defined nonetheless.

The Fourier transform of $(p_r * p_r)(x)$ is

$$\widehat{(p_r * p_r)}(k) = \frac{\widehat{\mathbb{1}_{\Lambda_r}}(k)^2}{(2\lfloor r\rfloor+1)^{2d}} = \frac{1}{(2\lfloor r\rfloor+1)^{2d}}\prod_{i=1}^{d}\left(\frac{\sin([2\lfloor r\rfloor+1]k_i/2)}{\sin(k_i/2)}\right)^2, \tag{4.7}$$

that is, it is the square of the d-dimensional Dirichlet kernel. Since p_r is a probability distribution, its Fourier transform has a maximum value of 1. Using the above bounds and Assumption D,

$$\begin{aligned}(\tau_{p_c} * g_r)(0) &= (2r+1)^d \int_{[-\pi,\pi]^d} \hat\tau_{p_c}(k)\hat p_r(k)^2 \frac{\mathrm{d}^d k}{(2\pi)^d}\\ &\le C_A r^d \int_{|k|\le 1/r} \frac{1}{|k|^{(2\wedge\alpha)}}\frac{\mathrm{d}^d k}{(2\pi)^d} + C_B r^{d+(2\wedge\alpha)}\int_{|k|\ge 1/r}\hat p_r(k)^2\frac{\mathrm{d}^d k}{(2\pi)^d}.\end{aligned} \tag{4.8}$$

Here, C_A and C_B are both positive constants that depend only on α, L and d. We bound the two terms separately. For the first term on the right-hand side we have

$$C_A r^d \int_{|k|\le 1/r} \frac{1}{|k|^{(2\wedge\alpha)}}\frac{\mathrm{d}^d k}{(2\pi)^d} = C_A' r^d \int_0^{1/r} k^{d-1-(2\wedge\alpha)}\,\mathrm{d}k \le C r^{(2\wedge\alpha)}. \tag{4.9}$$

To bound the second term on the right-hand side we extend the integration over k to $[-\pi,\pi]^d$ and get

$$\begin{aligned}C_B r^{d+(2\wedge\alpha)}\int_{|k|\ge 1/r}\hat p_r(k)^2\frac{\mathrm{d}^d k}{(2\pi)^d} &\le C_B r^{d+(2\wedge\alpha)}\int_{[-\pi,\pi]^d}\hat p_r(k)^2\frac{\mathrm{d}^d k}{(2\pi)^d}\\ &= \frac{C_B r^{d+(2\wedge\alpha)}}{(2\lfloor r\rfloor+1)^{2d}}(\mathbb{1}_{\Lambda_r} * \mathbb{1}_{\Lambda_r})(0) = \frac{C_B r^{d+(2\wedge\alpha)}}{(2\lfloor r\rfloor+1)^{d}} \le C r^{(2\wedge\alpha)}.\end{aligned} \tag{4.10}$$

Combining the bounds for (4.9) and (4.10) gives the desired upper bound.

In much the same way we can determine a lower bound for $\mathbb{E}_{p_c}[|Q_r\cap\mathcal{C}(0)|]$. Define the function

$$h_r(x) = \frac{r^d}{d^{d/2}}(p_q * p_q)(x) \qquad \text{where} \qquad q = \left\lfloor \frac{r}{2\sqrt{d}}\right\rfloor. \tag{4.11}$$

Again, the precise choice of h_r is not very important. It is important that h_r satisfies the following three criteria:

(i) $h_r(x) \le \mathbb{1}_{Q_r}(x)$ for all $x\in\mathbb{Z}^d$;
(ii) $\hat h_r(k) \ge 0$ for all $k\in[-\pi,\pi]^d$;
(iii) $\sum_{x\in\mathbb{Z}^d}\tau_{p_c}(x)h_r(-x) \ge c r^{(2\wedge\alpha)}$.

When these criteria are satisfied, the result is a lower bound $\mathbb{E}_{p_c}[|Q_r\cap\mathcal{C}(0)|] \ge c r^{(2\wedge\alpha)}$.

That criterion (i) holds follows since (1) $h_r(x)$ is maximized at $x=0$ and by our choice of q

$$h_r(0) = \frac{r^d}{d^{d/2}}\frac{|\Lambda_q|}{(2q+1)^{2d}} \le 1 \tag{4.12}$$

and (2) $h_r(x) = 0$ for all x such that $|x| > r$. Just as in the case of the upper bound, criterion (ii) is easily checked. All that is left is to check the last criterion. Write

$$\begin{aligned}(\tau_{p_c} * h_r)(0) &= \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)\hat{h}_r(k)\frac{\mathrm{d}^d k}{(2\pi)^d} \\ &= \frac{r^d}{d^{d/2}(2q+1)^{2d}} \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)\prod_{i=1}^d \left(\frac{\sin([2q+1]k_i/2)}{\sin(k_i/2)}\right)^2 \frac{\mathrm{d}^d k}{(2\pi)^d}.\end{aligned} \tag{4.13}$$

Since $x^2/2 \le \sin(x)^2 \le x^2$ for $|x| \le 1$, we bound, for $(2q+1)|k_i| \le 1$,

$$\frac{1}{(2q+1)^{2d}}\prod_{i=1}^d \left(\frac{\sin([2q+1]k_i/2)}{\sin(k_i/2)}\right)^2 \ge \frac{1}{(2q+1)^{2d}}\prod_{i=1}^d \frac{(\frac{2q+1}{2})^2 k_i^2}{k_i^2/8} \ge 2^{-d}, \tag{4.14}$$

Using this bound and (1.10), we get

$$\int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)\hat{h}_r(k)\frac{\mathrm{d}^d k}{(2\pi)^d} \ge \frac{cr^d}{d^{d/2}2^d}\int_{|k|\le\varepsilon/r} \frac{1}{|k|^{(2\wedge\alpha)}}\frac{\mathrm{d}^d k}{(2\pi)^d} \ge \frac{cr^d}{d^{d/2}2^d r^{d-(2\wedge\alpha)}} \ge c(d)r^{(2\wedge\alpha)} \tag{4.15}$$

for $\varepsilon > 0$ sufficiently small and $c(d)$ a constant depending on d. This completes the proof of (1.31). □

4.3. Bounds on the expectation of the backbone volume: proof of (1.33)

Backbone edges are those edges that have a path from 0 to one end of the edge, disjointly from a path from the other end of the edge to infinity. Therefore

$$\mathbb{E}_{\mathrm{IIC}}[N_{\mathrm{Bb}}(r)] = \sum_{b\in\mathcal{E}_r} \mathbb{E}_{\mathrm{IIC}}[\mathbb{1}_{\{b \text{ is a backbone edge}\}}] = \sum_{b\in\mathcal{E}_r} \mathbb{E}_{\mathrm{IIC}}[\mathbb{1}_{\{0\leftrightarrow \underline{b}\}\circ\{b \text{ open}\}\circ\{\overline{b}\leftrightarrow\infty\}}]. \tag{4.16}$$

Backbone events are by definition *not* cylinder events, and hence it is a priori not clear whether the limiting scheme that gives $\mathbb{Q}_{\mathrm{IIC}}$ can be reversed. The aim of this section is to show that we can.

We call an open edge $b = \{x, y\} \in \mathbb{Z}^d$ *backbone-pivotal* when every infinite self-avoiding walk in the IIC starting at the origin uses this edge.

It is not difficult to show that there is an infinite number of backbone-pivotal edges $\mathbb{Q}_{\mathrm{IIC}}$-a.s. Indeed, having a finite number of backbone-pivotal edges implies that there exist at least two disjoint infinite paths from the top of the last backbone pivotal. In Theorem 1.4(ii) of [26] it is proved that in the finite-range setting this does not happen $\mathbb{Q}_{\mathrm{IIC}}$-a.s. Their proof is easily modified to our setting.

The backbone-pivotal edges can be ordered as $(b_i)_{i=1}^\infty$. We can order them so that every infinite self-avoiding walk starting at 0 passes through b_i before passing through b_{i+1}. Also, we can think of the backbone-pivotal edges as being *directed* edges $b = (x, y)$, where the direction is such that $\{0 \leftrightarrow x\}$ uses different edges than $\{y \leftrightarrow \infty\}$. For a directed edge $b = (x, y)$, we let $\underline{b} = x$ denote its *bottom*, and $\overline{b} = y$ its *top*. Writing b_m for the mth backbone-pivotal edge, we define

$$\mathsf{S}_m^\infty \equiv \tilde{\mathcal{C}}^{b_m}(0) \setminus \tilde{\mathcal{C}}^{b_{m-1}}(0) \tag{4.17}$$

to be the subgraph of the mth "backbone sausage" (where, by convention $\tilde{\mathcal{C}}^{b_0}(0) = \varnothing$).

If 0 is connected to Q_r^c and there are precisely n open pivotal edges for this connection, we can again impose an ordering on the open pivotal edges $(b_i)_{i=1}^n$ in such a way that any self-avoiding path from 0 to Q_r^c passes through b_i before passing through b_{i+1}. If $n \ge m$, we let $\mathsf{S}_m^{(r)} \equiv \tilde{\mathcal{C}}^{b_m}(0) \setminus \tilde{\mathcal{C}}^{b_{m-1}}(0)$ and we let $\mathsf{S}_m^{(r)} = \varnothing$ whenever $0 \not\leftrightarrow Q_r^c$ or $n < m$.

In the same way, we let $\mathsf{S}_m^x \equiv \tilde{\mathcal{C}}^{b_m}(0) \setminus \tilde{\mathcal{C}}^{b_{m-1}}(0)$ where b_m now is the mth open pivotal edge for $\{0 \leftrightarrow x\}$, and $\mathsf{S}_m^x = \varnothing$ if no mth pivotal bond exists for the connection $\{0 \leftrightarrow x\}$.

We are interested in events that take place on the first m backbone sausages. To this end, define

$$\mathsf{Z}_m^\infty \equiv \bigcup_{i=1}^m \mathsf{S}_i^\infty, \qquad \mathsf{Z}_m^{(r)} \equiv \bigcup_{i=1}^m \mathsf{S}_i^{(r)}, \qquad \text{and} \qquad \mathsf{Z}_m^x \equiv \bigcup_{i=1}^m \mathsf{S}_i^x. \tag{4.18}$$

Note that $\mathsf{Z}_m^{(r)}$ and Z_m^x may contain fewer than m backbone sausages. Even though events occurring on Z_m^∞ are not necessarily cylinder events, it is still possible to reverse the IIC-limit for such events, as the following lemma demonstrates.

Lemma 4.2 [Backbone limit reversal lemma]. *Consider a model such that for all cylinder events F, $\mathbb{Q}_{\mathrm{IIC}}(F) = \lim_{p\nearrow p_c} \mathbb{Q}_p(F)$. Then, for any event E and any $m \in \mathbb{N}$,*

$$\mathbb{Q}_{\mathrm{IIC}}\left(E \text{ on } \mathsf{Z}_m^\infty\right) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\left(\{E \text{ on } \mathsf{Z}_m^x\} \cap \{0 \leftrightarrow x\}\right). \tag{4.19}$$

Proof. Fix m throughout the proof. We prove the lemma via comparison of Z_m^∞, $\mathsf{Z}_m^{(r)}$ and Z_m^x. To this end, we define the events

$$\Lambda_{(R)}^\infty \equiv \left\{\omega\colon \mathsf{Z}_m^\infty = \mathsf{Z}_m^{(R)}\right\}; \tag{4.20}$$

$$\Lambda_{(R)}^{(r)} \equiv \left\{\omega\colon \mathsf{Z}_m^{(r)} = \mathsf{Z}_m^{(R)} \text{ and } \mathsf{Z}_m^{(R)} \text{ contains at least } m \text{ pivotals}\right\}; \tag{4.21}$$

$$\Lambda_x^{(r)} \equiv \left\{\omega\colon \mathsf{Z}_m^{(r)} = \mathsf{Z}_m^x \text{ and } \mathsf{Z}_m^x \text{ contains at least } m \text{ pivotals}\right\}. \tag{4.22}$$

We show that it is improbable that these sets are different when we compare them on the same configuration and near the origin. Therefore, we may replace one with the other once we take a suitable limit.

We start by observing that for any R, $\Lambda_{(R)}^\infty \subseteq \Lambda_{(R+1)}^\infty$ and that for all $r < R$ and $x \in Q_R^c$, $\Lambda_{(R)}^{(r)} \subseteq \Lambda_{(R)}^{(r+1)}$ and $\Lambda_x^{(r)} \subseteq \Lambda_x^{(r+1)}$.

For any R we can write

$$\left\{E \text{ on } \mathsf{Z}_m^\infty\right\} = \left(\left\{E \text{ on } \mathsf{Z}_m^\infty\right\} \cap \Lambda_{(R)}^\infty\right) \dot\cup \left(\left\{E \text{ on } \mathsf{Z}_m^\infty\right\} \cap (\Lambda_{(R)}^\infty)^c\right) \equiv F_m^1(R) \dot\cup F_m^2(R). \tag{4.23}$$

At the end of the proof we take the limit $R \to \infty$. In this limit, the event $(\Lambda_{(R)}^\infty)^c$ has probability 0 under $\mathbb{Q}_{\mathrm{IIC}}$ for the following reasons: The occurrence of $(\Lambda_{(R)}^\infty)^c$ implies that there exists a path from one of the first m sausages to Q_R^c that is disjoint of the backbone. In the limit $R \to \infty$ this implies that there exist *two* disjoint connections to ∞ and this event does not occur $\mathbb{Q}_{\mathrm{IIC}}$-almost surely. Indeed, since $\Lambda_{(R)}^\infty \subseteq \Lambda_{(R+1)}^\infty$, we have by monotone convergence that

$$\lim_{R\to\infty} \mathbb{Q}_{\mathrm{IIC}}\left((\Lambda_{(R)}^\infty)^c\right) = \mathbb{Q}_{\mathrm{IIC}}\left(\lim_{R\to\infty} (\Lambda_{(R)}^\infty)^c\right) = 0. \tag{4.24}$$

For $F_m^1(R)$, the occurrence of $\Lambda_{(R)}^\infty$ implies $\{E \text{ on } \mathsf{Z}_m^\infty\} = \{E \text{ on } \mathsf{Z}_m^{(R)}\}$. Furthermore, for any r such that $0 < r < R$ we can write

$$F_m^1(R) = \left(\left\{E \text{ on } \mathsf{Z}_m^{(R)}\right\} \cap \Lambda_{(R)}^\infty \cap \Lambda_{(R)}^{(r)}\right) \dot\cup \left(\left\{E \text{ on } \mathsf{Z}_m^{(R)}\right\} \cap \Lambda_{(R)}^\infty \cap (\Lambda_{(R)}^{(r)})^c\right) \equiv G_m^1(R,r) \dot\cup G_m^2(R,r). \tag{4.25}$$

In the double limit where first $R \to \infty$ and then $r \to \infty$, the probability of $G_m^2(R,r)$ vanishes as

$$\lim_{r\to\infty}\lim_{R\to\infty} \mathbb{Q}_{\mathrm{IIC}}(G_m^2(R,r)) \le \lim_{r\to\infty}\lim_{R\to\infty} \mathbb{Q}_{\mathrm{IIC}}((\Lambda_{(R)}^{(r)})^c) = \lim_{r\to\infty} \mathbb{Q}_{\mathrm{IIC}}((\Lambda_{(r)}^\infty)^c) = 0. \tag{4.26}$$

Here we again used the argument that in the limit there must exist two disjoint paths to ∞.

We can rewrite $G_m^1(R,r)$ as follows:

$$G_m^1(R,r) = \left(\left\{E \text{ on } \mathsf{Z}_m^{(R)}\right\} \cap \Lambda_{(R)}^{(r)}\right) \setminus \left(\left\{E \text{ on } \mathsf{Z}_m^{(R)}\right\} \cap \Lambda_{(R)}^{(r)} \cap (\Lambda_{(R)}^\infty)^c\right) \equiv H_m^1(R,r) \setminus H_m^2(R,r). \tag{4.27}$$

Since $H_m^2(R,r) \subseteq (\Lambda_{(R)}^\infty)^c$ we again have that $\mathbb{Q}_{\mathrm{IIC}}(H_m^2(R,r)) \to 0$ as $R \to \infty$.

Now, $H_m^1(R,r)$ is a cylinder event, so that (1.21) applies,

$$\mathbb{Q}_{\mathrm{IIC}}(H_m^1(R,r)) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in Q_R^c} \mathbb{P}_p\left(H_m^1(R,r) \cap \{0 \leftrightarrow x\}\right), \tag{4.28}$$

(where the sum over $x \in Q_R$ vanishes in the $p \nearrow p_c$ limit).

The crucial observation is that for $r < R$ and $x \in Q_R^c$ we have $\Lambda_{(R)}^{(r)} \cap \{0 \leftrightarrow x\} = \Lambda_x^{(r)} \cap \{0 \leftrightarrow x\}$, so that

$$\{E \text{ on } \mathsf{Z}_m^{(R)}\} \cap \Lambda_{(R)}^{(r)} \cap \{0 \leftrightarrow x\} = \{E \text{ on } \mathsf{Z}_m^{x}\} \cap \Lambda_{(R)}^{(r)} \cap \{0 \leftrightarrow x\}. \tag{4.29}$$

It follows that

$$\begin{aligned} H_m^1(R,r) \cap \{0 \leftrightarrow x\} &= \{E \text{ on } \mathsf{Z}_m^{(R)}\} \cap \Lambda_{(R)}^{(r)} \cap \{0 \leftrightarrow x\} = \{E \text{ on } \mathsf{Z}_m^{x}\} \cap \Lambda_{(R)}^{(r)} \cap \{0 \leftrightarrow x\} \\ &= \big(\{E \text{ on } \mathsf{Z}_m^{x}\} \cap \{0 \leftrightarrow x\}\big) \setminus \big(\{E \text{ on } \mathsf{Z}_m^{x}\} \cap (\Lambda_{(R)}^{(r)})^c \cap \{0 \leftrightarrow x\}\big) \\ &\equiv M_m^1(x) \setminus M_m^2(R,r,x). \end{aligned} \tag{4.30}$$

For $M_m^2(R,r,x)$ we note that $(\Lambda_{(R)}^{(r)})^c$ is a cylinder event, so that (1.21) implies

$$\begin{aligned} \lim_{r\to\infty} \lim_{R\to\infty} \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in Q_R^c} \mathbb{P}_p(M_m^2(R,r,x)) &\le \lim_{r\to\infty} \lim_{R\to\infty} \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in Q_R^c} \mathbb{P}_p((\Lambda_{(R)}^{(r)})^c, 0 \leftrightarrow x) \\ &\le \lim_{r\to\infty} \lim_{R\to\infty} \mathbb{Q}_{\mathrm{IIC}}((\Lambda_{(R)}^{(r)})^c) = \lim_{r\to\infty} \mathbb{Q}_{\mathrm{IIC}}((\Lambda_{(r)}^{\infty})^c) = 0. \end{aligned} \tag{4.31}$$

Combining (4.23)–(4.30),

$$\begin{aligned} \mathbb{Q}_{\mathrm{IIC}}\big(E \text{ on } \mathsf{Z}_m^{\infty}\big) \quad &= \quad \mathbb{Q}_{\mathrm{IIC}}(F_m^2(R)) + \mathbb{Q}_{\mathrm{IIC}}(G_m^2(R,r)) - \mathbb{Q}_{\mathrm{IIC}}(H_m^2(R,r)) \\ &\quad + \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in Q_R^c} \big(\mathbb{P}_p(M_m^1(x)) - \mathbb{P}_p(M_m^2(R,r,x))\big). \end{aligned} \tag{4.32}$$

Now we add $0 = \lim_{p\nearrow p_c} \chi(p)^{-1} \sum_{x\in Q_R} \mathbb{P}_p(M_m^1(x))$ to the right-hand side, so that the term involving $M_m^1(x)$ is independent of r and R. Then we let $R \to \infty$, so that $\mathbb{Q}_{\mathrm{IIC}}(F_m^2(R))$ and $\mathbb{Q}_{\mathrm{IIC}}(H_m^2(R,r))$ vanish. After this we let $r \to \infty$, so that the terms involving $G_m^2(R,r)$ and $M_m^2(R,r,x)$ also disappear, by (4.26) and (4.31). The result is

$$\mathbb{Q}_{\mathrm{IIC}}\big(E \text{ on } \mathsf{Z}_m^{\infty}\big) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(M_m^1(x)) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\big(\{E \text{ on } \mathsf{Z}_m^{x}\} \cap \{0 \leftrightarrow x\}\big), \tag{4.33}$$

completing the proof. □

Let $\mathsf{Bb}(\omega)$ denote the backbone edge set of a configuration ω, and let $\mathsf{S}[A,B](\omega)$ denote the set of open edges between the sets A and B, that is, $\{u,v\} \in \mathsf{S}[A,B](\omega)$ whenever $\{u,v\}$ is open and $\{a \leftrightarrow u\} \circ \{v \leftrightarrow b\}$ for some $a \in A, b \in B$. Similarly, write $\mathsf{Bb}_{\mathsf{piv}}(\omega)$ for the set of backbone pivotal edges, and $\mathsf{S}_{\mathsf{piv}}[A,B](\omega)$ for the set of open pivotal edges for the event that there exists a connection between the sets A and B.

In this paper we use two specific cases of the above lemma.

Corollary 4.3 [Backbone limit reversal lemma for sets of edges]. *Consider a model such that* $\mathbb{Q}_{\mathrm{IIC}}(F) = \lim_{p\nearrow p_c} \mathbb{Q}_p(F)$ *for all cylinder events F. Let $\{b_i\}_{i=1}^n$ be a fixed and finite set of edges. Then,*

(i)

$$\mathbb{Q}_{\mathrm{IIC}}\big(\{b_i\}_{i=1}^n \subseteq \mathsf{Bb}\big) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\big(\{b_i\}_{i=1}^n \subseteq \mathsf{S}[0,x]\big); \tag{4.34}$$

(ii)

$$\mathbb{Q}_{\mathrm{IIC}}\big(\{b\}_{i=1}^n \subseteq \mathsf{Bb}_{\mathsf{piv}}\big) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p\big(\{b_i\}_{i=1}^n \subseteq \mathsf{S}_{\mathsf{piv}}[0,x]\big). \tag{4.35}$$

Proof. The proof for both cases follows by the same argument, so we only prove it for (i). Define

$$A_m \equiv \big\{\{b_i\}_{i=1}^n \subseteq \mathsf{Z}_m^{\infty}\big\}, \qquad A_\infty \equiv \big\{\{b_i\}_{i=1}^n \subseteq \mathsf{Bb}\big\} = \bigcup_{m=1}^{\infty} A_m, \tag{4.36}$$

$$B_m(x) \equiv \big\{\{b_i\}_{i=1}^n \subseteq \mathsf{Z}_m^{x}\big\} \cap \{0 \leftrightarrow x\}, \qquad B_\infty(x) \equiv \big\{\{b_i\}_{i=1}^n \subseteq \mathsf{S}[0,x]\big\} \cap \{0 \leftrightarrow x\} = \bigcup_{m=1}^{\infty} B_m. \tag{4.37}$$

Since $A_m \subseteq A_{m+1}$ for $m \geq 1$, we may partition A_∞ as $A_\infty = A_1 \cup \bigcup_{m\geq 1}(A_{m+1} \setminus A_m)$. Note that this is a union over disjoint subsets. We can write a similar partition for $B_\infty(x)$, for every $x \in \mathbb{Z}^d$. Next we apply Lemma 4.2 to each term,

$$\begin{aligned}\mathbb{Q}_{\text{IIC}}(A_\infty) =& \mathbb{Q}_{\text{IIC}}(A_1) + \sum_{m=1}^{\infty} (\mathbb{Q}_{\text{IIC}}(A_{m+1}) - \mathbb{Q}_{\text{IIC}}(A_m)) \\ =& \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(B_1(x)) \\ &+ \sum_{m=1}^{\infty}\left(\lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(B_{m+1}(x)) - \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(B_m(x))\right).\end{aligned} \tag{4.38}$$

Observe that for all $m \geq 1$, $\mathbb{P}_p(B_m(x)) \leq \mathbb{P}_{p_c}(B_m(x))$ and $\sum_x \mathbb{P}_{p_c}(B_m(x)) < \infty$, so we may use dominated convergence to deduce

$$\mathbb{Q}_{\text{IIC}}(A_\infty) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \left(\mathbb{P}_p(B_1(x)) + \sum_{m=1}^{\infty} \big(\mathbb{P}_p(B_{m+1}(x)) - \mathbb{P}_p(B_m(x))\big)\right). \tag{4.39}$$

(Note that the dominating function $\mathbb{P}_{p_c}(B_m(x))$ can also be used in the proof of (ii).) Since clearly $\mathbb{P}_p(B_m(x)) \to \mathbb{P}_p(B_\infty(x))$ as $m \to \infty$, the telescoping sum on the right-hand side is equal to $\mathbb{P}_p(B_\infty(x)) - \mathbb{P}_p(B_1(x))$, so

$$\mathbb{Q}_{\text{IIC}}(A_\infty) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(B_\infty(x)), \tag{4.40}$$

as we set out to prove. □

4.3.1. *Upper bound on the expectation of the backbone volume.* Applying Corollary 4.3(i) to (4.16), we get

$$\begin{aligned}\mathbb{E}_{\text{IIC}}[N_{\text{Bb}}(r)] &= \sum_{b\in\mathcal{E}_r} \mathbb{Q}_{\text{IIC}}(b \in \text{Bb}) = \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{b\in\mathcal{E}_r} \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(b \text{ open and pivotal for } 0 \leftrightarrow x) \\ &\leq \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{b\in\mathcal{E}_r} \sum_{x\in\mathbb{Z}^d} \mathbb{E}_p[\mathbb{1}_{\{0\leftrightarrow \underline{b}\}\circ\{b \text{ open}\}\circ\{\overline{b}\leftrightarrow x\}}].\end{aligned} \tag{4.41}$$

Applying the BK-inequality to (4.41) gives

$$\mathbb{E}_{\text{IIC}}[N_{\text{Bb}}(r)] \leq \lim_{p\nearrow p_c} \frac{1}{\chi(p)} \sum_{b\in\mathcal{E}_r} \sum_{x\in\mathbb{Z}^d} \tau_p(\underline{b}) p D(b) \tau_p(x - \overline{b}). \tag{4.42}$$

Summing over x and then $\overline{b}$ and bounding the sum over $b \in \mathcal{E}_r$ by the sum over $\underline{b} \in Q_r$, we get a factor $\chi(p)$ and a factor p, respectively. After this we take the limit $p \nearrow p_c$:

$$\mathbb{E}_{\text{IIC}}[N_{\text{Bb}}(r)] \leq p_c \sum_{\underline{b}\in Q_r} \tau_{p_c}(\underline{b}). \tag{4.43}$$

The upper bound in (1.31) that we proved in the previous section completes the proof of upper bound in (1.33). □

4.3.2. *Lower bound on the expectation of the backbone volume.* To get a lower bound on $\mathbb{E}_{\text{IIC}}[N_{\text{Bb}}(r)]$ we count only the backbone-pivotal edges. Recall the definition of h_r given in (4.11). We bound

$$\mathbb{E}_{\text{IIC}}[N_{\text{Bb}}(r)] = \sum_{b\in\mathcal{E}_r} \mathbb{Q}_{\text{IIC}}(b \in \text{Bb}) \geq \sum_{b\in\mathcal{E}_r} \mathbb{Q}_{\text{IIC}}(b \in \text{Bb}_{\text{piv}}) \geq \sum_{b\in\mathbb{Z}^d\times\mathbb{Z}^d} h_r(\underline{b}) \mathbb{Q}_{\text{IIC}}(b \in \text{Bb}_{\text{piv}}). \tag{4.44}$$

That the second inequality is necessary is not immediately obvious, but it will turn out to be crucial for getting a good bound in the case of nearest-neighbor percolation. Now we apply Corollary

4.3(ii) to get

$$\sum_{b\in\mathbb{Z}^d\times\mathbb{Z}^d} h_r(\underline{b})\mathbb{Q}_{\mathrm{IIC}}(b\in\mathsf{Bb}_{\mathrm{piv}}) = \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} h_r(\underline{b})\mathbb{P}_p(b\in\mathsf{S}_{\mathrm{piv}}[0,x]). \tag{4.45}$$

By the definition of $\mathsf{S}_{\mathrm{piv}}[0,x]$ we have

$$\{b\in\mathsf{S}_{\mathrm{piv}}[0,x]\} = \{0\leftrightarrow\underline{b}\text{ on }\tilde{\mathcal{C}}^b(0)\}\circ\{b\text{ open}\}\circ\{\overline{b}\leftrightarrow x\text{ off }\tilde{\mathcal{C}}^b(0)\}, \tag{4.46}$$

so we can apply the Factorization Lemma to the right-hand side of (4.45) to get

$$\begin{aligned}\mathbb{E}_{\mathrm{IIC}}[N_{\mathrm{Bb}}(r)] &\geq \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)h_r(\underline{b})\mathbb{E}_0[\mathbb{1}_{\{0\leftrightarrow\underline{b}\text{ on }\tilde{\mathcal{C}}^b(0)\}}\mathbb{E}_1[\mathbb{1}_{\{\overline{b}\leftrightarrow x\text{ off }\tilde{\mathcal{C}}^b(0)\}}]]\\ &= \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)h_r(\underline{b})\mathbb{E}_0[\mathbb{1}_{\{0\leftrightarrow\underline{b}\}}\mathbb{E}_1[\mathbb{1}_{\{\overline{b}\leftrightarrow x\text{ off }\tilde{\mathcal{C}}^b(0)\}}]].\end{aligned} \tag{4.47}$$

In the second line we left out the condition "on $\tilde{\mathcal{C}}^b(0)$" because

$$\{0\leftrightarrow\underline{b}\text{ on }\tilde{\mathcal{C}}^b(0)\} = \{0\leftrightarrow\underline{b}\}\setminus\{0\leftrightarrow\underline{b}\text{ through }\mathbb{Z}^d\setminus\tilde{\mathcal{C}}^b(0)\}, \tag{4.48}$$

but $\{0\leftrightarrow\underline{b}\text{ through }\mathbb{Z}^d\setminus\tilde{\mathcal{C}}^b(0)\}$ implies that b is pivotal for the connection $\{0\leftrightarrow\underline{b}\}$, which means that the event $\{0\leftrightarrow\overline{b}\}$ has to occur. But the indicator $\mathbb{1}_{\{\overline{b}\leftrightarrow x\text{ off }\tilde{\mathcal{C}}^b(0)\}}$ is always 0 for such events, so the change from $\{0\leftrightarrow\underline{b}\text{ on }\tilde{\mathcal{C}}^b(0)\}$ to $\{0\leftrightarrow\underline{b}\}$ has no effect on the expectation.

We write $\tilde{\mathcal{C}}^b_0(0)$ to remind ourselves that the cluster is random with respect to $\mathbb{E}_0$, but fixed with respect to $\mathbb{E}_1$. We bound the expectations in (4.47) from the inside out:

$$\begin{aligned}\mathbb{E}_{\mathrm{IIC}}[N_{\mathrm{Bb}}(r)] \geq& \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)h_r(\underline{b})\Big(\tau_p(\underline{b})\tau_p(x-\overline{b}) - \mathbb{E}_0\Big[\mathbb{1}_{\{0\leftrightarrow\underline{b}\}}\mathbb{P}_p\Big(\overline{b}\xleftrightarrow{\tilde{\mathcal{C}}^b_0(0)} x\Big)\Big]\Big)\\ \equiv& N_1 - N_2.\end{aligned} \tag{4.49}$$

For the inequality we used the identity $\{E\text{ off }A\} = E\setminus\{E\text{ through }A\}$. We give separate bounds for N_1 and N_2.

Consider N_1 first:

$$\begin{aligned}N_1 &= \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} h_r(\underline{b})\tau_p(\underline{b})pD(b)\tau_p(x-\overline{b})\\ &= p_c\sum_{\underline{b}\in\mathbb{Z}^d} h_r(\underline{b})\tau_{p_c}(\underline{b}) = p_c\int_{[-\pi,\pi]^d}\hat{\tau}_{p_c}(k)\hat{h}_r(k)\frac{\mathrm{d}^dk}{(2\pi)^d}.\end{aligned} \tag{4.50}$$

To get the second equality we summed over x and then $\overline{b}$, as we did for the upper bound.

The bound on N_2 is harder:

$$N_2 = \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)h_r(\underline{b})\mathbb{E}_0\Big[\mathbb{1}_{\{0\leftrightarrow\underline{b}\}}\mathbb{P}_p\Big(\overline{b}\xleftrightarrow{\tilde{\mathcal{C}}^b_0(0)} x\Big)\Big] \tag{4.51}$$

and note that here we need an upper bound. The dependence in the second two-point function implies that there is a path from some vertex along the path $\overline{b}\leftrightarrow x$ to another vertex on the path $0\leftrightarrow\underline{b}$, and that this path does not use the edge b. Consider a fixed set of vertices $A\subset\mathbb{Z}^d$. Then,

$$\{\overline{b}\leftrightarrow x\text{ through }A\}\subseteq\bigcup_{a\in A}\{\overline{b}\leftrightarrow a\}\circ\{a\leftrightarrow x\}. \tag{4.52}$$

Therefore,

$$\begin{aligned}\mathbb{P}_p(\overline{b} \xleftrightarrow{A} x) &\le \mathbb{P}_p\Big(\bigcup_{a\in A}\{\overline{b}\leftrightarrow a\}\circ\{a\leftrightarrow x\}\Big) \le \sum_{a\in\mathbb{Z}^d}\mathbb{1}_{\{a\in A\}}\mathbb{P}_p(\{\overline{b}\leftrightarrow a\}\circ\{a\leftrightarrow x\})\\ &\le \sum_{a\in\mathbb{Z}^d}\mathbb{1}_{\{a\in A\}}\tau_p(a-\overline{b})\tau_p(x-a).\end{aligned}\tag{4.53}$$

Since the set $\tilde{\mathcal{C}}_0^b(0)$ is fixed with respect to the expectation $\mathbb{E}_1$ we may apply (4.53) to the expectation on the right-hand side of (4.51) with $A=\mathcal{C}_0(0)\supset\tilde{\mathcal{C}}_0^b(0)$:

$$\begin{aligned}N_2 &\le \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b},a\in\mathbb{Z}^d} pD(b)h_r(\underline{b})\mathbb{E}_0\left[\mathbb{1}_{\{0\leftrightarrow\underline{b}\}}\mathbb{1}_{\{0\leftrightarrow a\}}\tau_p(a-\overline{b})\tau_p(x-a)\right]\\ &= \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b},a\in\mathbb{Z}^d} pD(b)h_r(\underline{b})\mathbb{P}_p(0\leftrightarrow\underline{b},0\leftrightarrow a)\tau_p(a-\overline{b})\tau_p(x-a).\end{aligned}\tag{4.54}$$

We use the tree-graph bound [3]:

$$\mathbb{P}_p(0\leftrightarrow\underline{b},0\leftrightarrow a)\le\sum_{z\in\mathbb{Z}^d}\tau_p(z)\tau_p(\underline{b}-z)\tau_p(a-z)\tag{4.55}$$

and insert the above inequality into (4.54) to get

$$\begin{aligned}N_2 &\le \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b},a,z\in\mathbb{Z}^d} h_r(\underline{b})\tau_p(z)\tau_p(\underline{b}-z)pD(b)\tau_p(a-\overline{b})\tau_p(a-z)\tau_p(x-a)\\ &= p_c\sum_{\underline{b},\overline{b},a,z\in\mathbb{Z}^d} h_r(\underline{b})\tau_{p_c}(z)\tau_{p_c}(\underline{b}-z)D(b)\tau_{p_c}(a-\overline{b})\tau_{p_c}(a-z).\end{aligned}\tag{4.56}$$

Define

$$T'_{p_c}(x)=\tau_{p_c}(x)(D*\tau_{p_c}*\tau_{p_c})(x).\tag{4.57}$$

An upper bound on its Fourier transform is

$$\begin{aligned}|\hat{T}'_{p_c}(k)| &= \left|\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{ik\cdot x}T'_{p_c}(x)\right|\le|\hat{T}'_{p_c}(0)|\\ &\le\sum_{v,w,y\in\mathbb{Z}^d}D(v)\tau_{p_c}(w-v)\tau_{p_c}(y-w)\tau_{p_c}(y)=T_{p_c}(0)\le C\beta,\end{aligned}\tag{4.58}$$

with $T_p(x)$ as given by (4.3). The bound on $T_{p_c}(0)$ follows from Lemma 4.1.

With this definition we can write

$$N_2\le p_c\sum_{\underline{b},z\in\mathbb{Z}^d}h_r(\underline{b})T'_{p_c}(\underline{b}-z)\tau_{p_c}(z)=p_c(\tau_{p_c}*T'_{p_c}*h_r)(0).\tag{4.59}$$

We can bound N_2 by expressing the right-hand side in terms of its Fourier transform:

$$\begin{aligned}N_2 &\le p_c(\tau_{p_c}*T'_{p_c}*h_r)(0)=p_c\int_{[-\pi,\pi]^d}\hat{\tau}_{p_c}(k)\hat{T}'_{p_c}(k)\hat{h}_r(k)\frac{\mathrm{d}^dk}{(2\pi)^d}\\ &\le C\beta p_c\int_{[-\pi,\pi]^d}\hat{\tau}_{p_c}(k)\hat{h}_r(k)\frac{\mathrm{d}^dk}{(2\pi)^d},\end{aligned}\tag{4.60}$$

where the second inequality follows from (4.58).

With bounds on both N_1 and N_2 we can conclude that, when β is small enough,

$$\mathbb{E}_{\mathrm{IIC}}[N_{\mathrm{Bb}}(r)]\ge N_1-N_2\ge p_c(1-C\beta)\int_{[-\pi,\pi]^d}\hat{\tau}_{p_c}(k)\hat{h}_r(k)\frac{\mathrm{d}^dk}{(2\pi)^d}\ge c'(d)r^{(2\wedge\alpha)}.\tag{4.61}$$

for some constant $c'(d)$ that only depends on d. The final inequality follows from (4.15). This concludes the proof of the lower bound. Combining the upper and lower bound completes the proof of (1.33). □

4.4. Bounds on the expected IIC volume in a ball: proof of (1.32)

Define the IIC two-point function

$$\varrho(y) \equiv \mathbb{Q}_{\mathrm{IIC}}(0 \leftrightarrow y) = \lim_{p \nearrow p_c} \frac{1}{\chi(p)} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow y, 0 \leftrightarrow x). \tag{4.62}$$

Since the event $\{0 \leftrightarrow y\}$ is not a cylinder event, it is not immediately obvious that we can write it as a limit. Nevertheless, in [26] it is proved that this is allowed.

Using the techniques of the previous paragraphs, we can easily find an upper bound. The lower bound requires more work.

4.4.1. *IIC volume expectation upper bound.* We start by bounding (4.62) using the tree-graph bound (4.55):

$$\varrho(y) \le \lim_{p \nearrow p_c} \frac{1}{\chi(p)} \sum_{x,z \in \mathbb{Z}^d} \tau_p(z)\tau_p(x-z)\tau_p(y-z). \tag{4.63}$$

Keeping z fixed and summing over x we get a factor $\chi(p)$. Then, with the divergence of the susceptibility canceled, we can take the limit $p \nearrow p_c$:

$$\varrho(y) \le \sum_{z \in \mathbb{Z}^d} \tau_{p_c}(z)\tau_{p_c}(y-z) = (\tau_{p_c} * \tau_{p_c})(y). \tag{4.64}$$

The expected volume of the IIC in a Euclidean ball is given by

$$\mathbb{E}_{\mathrm{IIC}}[|Q_r \cap \mathrm{IIC}|] = \sum_{y \in Q_r} \varrho(y). \tag{4.65}$$

Using the same techniques as in Section 4.2, we get

$$\begin{aligned}
\mathbb{E}_{\mathrm{IIC}}[|Q_r \cap \mathrm{IIC}|] &\le \sum_{x,y \in \mathbb{Z}^d} \tau_{p_c}(x)\tau_{p_c}(y-x) g_r(-y) \\
&\le C' r^d \int_{[-\pi,\pi]^d} \frac{\hat{p}_r(k)^2}{[1-\hat{D}(k)]^2} \frac{\mathrm{d}^d k}{(2\pi)^d} \\
&\le C_A r^d \int_{|k| \le 1/r} \frac{1}{|k|^{2(2\wedge\alpha)}} \frac{\mathrm{d}^d k}{(2\pi)^d} \\
&\quad + C_B r^{d+2(2\wedge\alpha)} \int_{|k| \ge 1/r} \hat{p}_r(k)^2 \frac{\mathrm{d}^d k}{(2\pi)^d} \le C r^{2(2\wedge\alpha)}.
\end{aligned} \tag{4.66}$$

□

4.4.2. *IIC volume expectation lower bound.* This bound is the most involved, because we need to use the Factorization Lemma twice. We bound (4.62) from below by

$$\varrho(y) \ge \lim_{p \nearrow p_c} \frac{1}{\chi(p)} \sum_{x,\underline{b},\overline{b} \in \mathbb{Z}^d} \mathbb{P}_p \begin{pmatrix} 0 \leftrightarrow y, 0 \leftrightarrow x, b = (\underline{b}, \overline{b}) \text{ is the first edge that is} \\ \text{open and pivotal for } 0 \leftrightarrow x \text{ but not for } 0 \leftrightarrow y \end{pmatrix} \tag{4.67}$$

Observe that

$$\begin{aligned}
&\{0 \leftrightarrow y, 0 \leftrightarrow x, b \text{ is the first edge that is open and pivotal for } 0 \leftrightarrow x \text{ but not for } 0 \leftrightarrow y\} \\
&\quad = \{b \text{ open}\} \circ \{\{0 \leftrightarrow \underline{b}\} \circ \{\underline{b} \leftrightarrow y\} \text{ on } \tilde{\mathcal{C}}^b(0)\} \circ \{\overline{b} \leftrightarrow x \text{ on } \mathbb{Z}^d \setminus \tilde{\mathcal{C}}^b(0)\}.
\end{aligned} \tag{4.68}$$

Indeed, if there exist connections from 0 to x and y, and if there exists an open edge $b=(\underline{b},\overline{b})$ that is pivotal for $0\leftrightarrow x$ but not for $0\leftrightarrow y$, then the connection $\overline{b}\leftrightarrow x$ occurs off $\tilde{\mathcal{C}}^b(0)$. If this was not the case, then b would not be pivotal for $0\leftrightarrow x$. Similarly, the event $0\leftrightarrow y$ occurs on $\tilde{\mathcal{C}}^b(0)$, since otherwise b would be pivotal for $0\leftrightarrow y$. By requiring that b is the first edge that has these properties we also get disjoint occurrence of the events, i.e. $\{0\leftrightarrow \underline{b}\}\circ\{\underline{b}\leftrightarrow y\}$ on $\tilde{\mathcal{C}}^b(0)$.

Applying the Factorization Lemma gives

$$\varrho(y)\geq \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)\mathbb{E}_0[\mathbb{1}_{\{0\leftrightarrow\underline{b}\}\circ\{\underline{b}\leftrightarrow y\}}\mathbb{E}_1[\mathbb{1}_{\{\overline{b}\leftrightarrow x \text{ off } \tilde{\mathcal{C}}_0^b(0)\}}]], \tag{4.69}$$

where we left out the condition "on $\tilde{\mathcal{C}}_0^b(0)$" again, for the same reason that we were allowed to leave it out in (4.47). For a fixed set of vertices A,

$$\mathbb{P}_p(x\leftrightarrow y \text{ off } A)=\tau_p(y-x)-\mathbb{E}_p[\mathbb{1}_{\{x\leftrightarrow y \text{ through } A\}}]. \tag{4.70}$$

Since $\tilde{\mathcal{C}}_0^b(0)$ is fixed with respect to $\mathbb{E}_1$ we may apply this identity to (4.69) and sum over $y\in Q_r$ to get

$$\begin{aligned}\sum_{y\in Q_r}\varrho(y)&\geq \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{y\in Q_r}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)\mathbb{E}_0\big[\mathbb{1}_{\{0\leftrightarrow\underline{b}\}\circ\{\underline{b}\leftrightarrow y\}}(\tau_p(x-\overline{b})\\ &\qquad -\mathbb{E}_1[\mathbb{1}_{\{\overline{b}\leftrightarrow x \text{ through } \tilde{\mathcal{C}}_0^b(0)\}}])\big]\\ &\geq \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{y\in \mathbb{Z}^d}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)h_r(y)\mathbb{E}_0\big[\mathbb{1}_{\{0\leftrightarrow\underline{b}\}\circ\{\underline{b}\leftrightarrow y\}}(\tau_p(x-\overline{b})\\ &\qquad -\mathbb{E}_1[\mathbb{1}_{\{\overline{b}\leftrightarrow x \text{ through } \tilde{\mathcal{C}}_0^b(0)\}}])\big]\equiv S_1-S_2.\end{aligned} \tag{4.71}$$

In the second inequality we have again replaced the sum over $y\in Q_r$ by the sum over $y\in\mathbb{Z}^d$ and inserted a factor $h_r(y)$. This is a necessary step for getting a good bound on S_1.

We first give an upper bound on S_2, and then establish a lower bound on S_1. As mentioned, the set $\tilde{\mathcal{C}}_0^b(0)$ is fixed with respect to $\mathbb{E}_1$. We start by applying the BK-inequality to $\mathbb{E}_1$,

$$\mathbb{E}_1[\mathbb{1}_{\{\overline{b}\leftrightarrow x \text{ through } \tilde{\mathcal{C}}_0^b(0)\}}]\leq \sum_{a\in\mathbb{Z}^d}\mathbb{1}_{\{a\in\tilde{\mathcal{C}}_0^b(0)\}}\tau_p(\overline{b}-a)\tau_p(x-a). \tag{4.72}$$

To bound $\mathbb{E}_0$, we observe that

$$\begin{aligned}\big(\{0\leftrightarrow\underline{b}\}\circ\{\underline{b}\leftrightarrow y\}\big)\cap\{a\in\tilde{\mathcal{C}}_0^b(0)\}\subseteq&\left(\bigcup_{z\in\mathbb{Z}^d}\{0\leftrightarrow z\}\circ\{z\leftrightarrow\underline{b}\}\circ\{\underline{b}\leftrightarrow y\}\circ\{z\leftrightarrow a\}\right)\\ &\cup\left(\bigcup_{z\in\mathbb{Z}^d}\{0\leftrightarrow\underline{b}\}\circ\{\underline{b}\leftrightarrow z\}\circ\{z\leftrightarrow y\}\circ\{z\leftrightarrow a\}\right).\end{aligned} \tag{4.73}$$

Applying (4.72) and (4.73) to S_2, and applying the BK-inequality again, we get the upper bound

$$\begin{aligned}S_2\leq \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{y\in\mathbb{Z}^d}\sum_{x,\underline{b},\overline{b}\in\mathbb{Z}^d} pD(b)h_r(y)\big[&\tau_p(z)\tau_p(\underline{b}-z)\tau_p(y-\underline{b})\tau_p(a-z)\tau_p(\overline{b}-a)\tau_p(x-a)\\ &+\tau_p(\underline{b})\tau_p(z-\underline{b})\tau_p(y-z)\tau_p(a-z)\tau_p(\overline{b}-a)\tau_p(x-a)\big].\end{aligned} \tag{4.74}$$

Summing over x, taking the limit $p\nearrow p_c$ and applying the definition of T'_{p_c}, (4.57), we get

$$S_2\leq 2p_c\sum_{y\in\mathbb{Z}^d}h_r(y)(\tau_{p_c}*T'_{p_c}*\tau_{p_c})(y)=2p_c(\tau_{p_c}*T'_{p_c}*\tau_{p_c}*h_r)(0) \tag{4.75}$$

We end up with a bound that is very similar to N_2 in the previous section. Hence, modifying (4.60) to include an extra factor τ_{p_c}, we get

$$S_2\leq C\beta r^{2(2\wedge\alpha)}. \tag{4.76}$$

We now establish a lower bound on S_1. Immediately we can sum over x and $\overline{b}$ to get factors $\chi(p)$ and p and take the limit $p \nearrow p_c$:

$$S_1 = \sum_{y,\underline{b},\overline{b}\in\mathbb{Z}^d} p_c h_r(y)\mathbb{P}_{p_c}(\{0 \leftrightarrow \underline{b}\} \circ \{\underline{b} \leftrightarrow y\}). \tag{4.77}$$

Observe that

$$\{0 \leftrightarrow \underline{b}\} \circ \{\underline{b} \leftrightarrow y\} \supseteq \bigcup_{e:\underline{e}=\underline{b}} \{e \text{ is open and pivotal for } 0 \leftrightarrow y\}. \tag{4.78}$$

and

$$\{e \text{ is open and pivotal for } 0 \leftrightarrow y\} = \{0 \leftrightarrow \underline{e} \text{ on } \tilde{\mathcal{C}}_0^e(0)\} \cap \{e \text{ open}\} \cap \{\overline{e} \leftrightarrow y \text{ off } \tilde{\mathcal{C}}^e(0)\}. \tag{4.79}$$

Making this replacement, applying the Factorization Lemma again, and applying (4.70) we get the lower bound

$$S_1 \geq \sum_{y,\underline{b}\in\mathbb{Z}^d} \sum_{e:\underline{e}=\underline{b}} p_c^2 h_r(y) D(e) \mathbb{E}_0[\mathbb{1}_{\{0\leftrightarrow\underline{e}\}}(\tau_{p_c}(y-\overline{e}) - \mathbb{E}_1[\mathbb{1}_{\{\overline{e}\leftrightarrow y \text{ through } \tilde{\mathcal{C}}_0^e(0)\}}])] \equiv S_{1,1} - S_{1,2} \tag{4.80}$$

where we again left out the condition "on $\tilde{\mathcal{C}}_0^e(0)$" for the same reason that we were allowed to leave it out in (4.47).

Writing $S_{1,1}$ in terms of its Fourier transform, we get

$$\begin{aligned} S_{1,1} &= \sum_{y,\underline{b}\in\mathbb{Z}^d} \sum_{e:\underline{e}=\underline{b}} p_c^2 D(e) h_r(y) \tau_{p_c}(\underline{e}) \tau_{p_c}(y-\overline{e}) = p_c^2 (\tau_{p_c} * D * \tau_{p_c} * h_r)(0) \\ &= p_c^2 \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)^2 \hat{D}(k) \hat{h}_r(k) \frac{\mathrm{d}^d k}{(2\pi)^d}. \end{aligned} \tag{4.81}$$

Rewriting the right-hand side gives

$$\begin{aligned} S_{1,1} &= p_c^2 \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)^2 (1 - [1-\hat{D}(k)]) \hat{h}_r(k) \frac{\mathrm{d}^d k}{(2\pi)^d} \\ &= p_c^2 \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)^2 \hat{h}_r(k) \frac{\mathrm{d}^d k}{(2\pi)^d} - p_c^2 \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)^2 [1-\hat{D}(k)] \hat{h}_r(k) \frac{\mathrm{d}^d k}{(2\pi)^d}. \end{aligned} \tag{4.82}$$

For the second integral we get an upper bound:

$$p_c^2 \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)^2 [1-\hat{D}(k)] \hat{h}_r(k) \frac{\mathrm{d}^d k}{(2\pi)^d} \leq C \int_{[-\pi,\pi]^d} \frac{\hat{h}_r(k)[1-\hat{D}(k)]}{[1-\hat{D}(k)]^2} \frac{\mathrm{d}^d k}{(2\pi)^d}. \tag{4.83}$$

We split up the integral and bound:

$$\begin{aligned} C \int_{[-\pi,\pi]^d} \frac{\hat{h}_r(k)}{[1-\hat{D}(k)]} \frac{\mathrm{d}^d k}{(2\pi)^d} &\leq C_A \int_{|k|\leq 1/r} \frac{1}{|k|^{(2\wedge\alpha)}} \frac{\mathrm{d}^d k}{(2\pi)^d} + C_B r^{(2\wedge\alpha)-d} \int_{[-\pi,\pi]^d} \hat{h}_r(k) \frac{\mathrm{d}^d k}{(2\pi)^d} \\ &\leq C_A r^{(2\wedge\alpha)} + C_B r^{(2\wedge\alpha)}. \end{aligned} \tag{4.84}$$

Here the first integral has been bounded in the same way as (4.9) and the second one in the same way as (4.10). Combining both bounds, we can conclude that

$$S_{1,1} \geq p_c^2 \int_{[-\pi,\pi]^d} \hat{\tau}_{p_c}(k)^2 \hat{h}_r(k) \frac{\mathrm{d}^d k}{(2\pi)^d} - C r^{(2\wedge\alpha)}. \tag{4.85}$$

For $S_{1,2}$ we need an upper bound. Using (4.53) and the tree-graph bound (4.55), we get

$$\begin{aligned} S_{1,2} &= \sum_{y,\underline{b}\in\mathbb{Z}^d}\sum_{e:\underline{e}=\underline{b}} p_c^2 h_r(y)D(e)\mathbb{E}_0[\mathbb{1}_{\{0\leftrightarrow\underline{e}\}}\mathbb{E}_1[\mathbb{1}_{\{\overline{e}\leftrightarrow y \text{ through } \tilde{\mathcal{C}}^e(0)\}}]] \\ &\le \sum_{y,\underline{b},v,v'\in\mathbb{Z}^d}\sum_{e:\underline{e}=\underline{b}} p_c^2 h_r(y)D(e)\tau_{p_c}(v)\tau_{p_c}(\underline{e}-v)\tau_{p_c}(v'-\overline{e})\tau_{p_c}(v'-v)\tau_{p_c}(y-v'). \end{aligned} \tag{4.86}$$

Observe that

$$\tau_{p_c}(v'-v)\left[\sum_{\underline{b}\in\mathbb{Z}^d}\sum_{e:\underline{e}=\underline{b}}\tau_{p_c}(\underline{e}-v)D(\overline{e}-\underline{e})\tau_{p_c}(v'-\overline{e})\right] \le T'_{p_c}(v'-v). \tag{4.87}$$

This implies

$$S_{1,2} \le p_c^2 \sum_{y,v,v'\in\mathbb{Z}^d} h_r(y)\tau_{p_c}(v)T'_{p_c}(v'-v)\tau_{p_c}(y-v') = p_c^2(\tau_{p_c} * T'_{p_c} * \tau_{p_c} * h_r)(0). \tag{4.88}$$

We rewrite the right-hand side in terms of its Fourier transform and apply (4.58):

$$S_{1,2} \le p_c^2 \int_{[-\pi,\pi]^d} \hat\tau_{p_c}(k)^2\hat T'_{p_c}(k)\hat h_r(k)\frac{\mathrm{d}^d k}{(2\pi)^d} \le p_c^2 C'\beta \int_{[-\pi,\pi]^d} \hat\tau_{p_c}(k)^2\hat h_r(k)\frac{\mathrm{d}^d k}{(2\pi)^d}. \tag{4.89}$$

Finally, combining the bounds (4.82), (4.89) and (4.76), we get, for β small enough,

$$\begin{aligned} \mathbb{E}_{\mathrm{IIC}}[|Q_r\cap\mathcal{C}(0)|] &\ge S_{1,1}-S_{1,2}-S_2 \\ &\ge p_c^2(1-C'\beta)\int_{[-\pi,\pi]^d}\hat\tau_{p_c}(k)^2\hat h_r(k)\frac{\mathrm{d}^d k}{(2\pi)^d} - Cr^{(2\wedge\alpha)} \ge c''(d)r^{2(2\wedge\alpha)} \end{aligned} \tag{4.90}$$

for some constant $c''(d)$. The last inequality follows from a similar bound as (4.15). This completes the proof of Theorem 1.6. □

5. A LOWER BOUND ON THE LONG-RANGE ONE-ARM PROBABILITY: PROOF OF THEOREM 1.5

In this section we restrict ourselves to models of long-range spread-out percolation only.

The heuristics of the proof of Theorem 1.5 are simple: if the cluster reaches distance r, then either the cluster contains many vertices, or the cluster contains an edge that is very long (of order r). To bound the probability that the cluster is large, we use a simple second moment estimate. This contributes the dominant term to the lower bound when $\alpha \ge 4$, as the probability of finding a long edge is negligible in this regime. But when $\alpha < 4$ this is not the case anymore, and the dominant contribution will be due to the existence of long edges. To establish this, we show that the existence of long edges is only weakly dependent on the size of the cluster, and vice versa.

Proof of Theorem 1.5. We start by proving $\mathbb{P}_{p_c}(0\leftrightarrow Q_r^c) \ge c/r^{\alpha/2}$. Let $\mathcal{C}_r(0)$ be the *r-truncated cluster* of 0, that is, the percolation cluster of 0 generated by using the edge probability

$$p_c D_r(x) = p_c D(x)\mathbb{1}_{\{|x|\le r\}}$$

instead of $D(x)$. Note that by the definition of the long-range family (in particular, by (1.5)) there exist $\zeta,\xi>0$ such that

$$\sum_{x\in\mathbb{Z}^d} p_c D_r(x) = \sum_{x\in Q_r} p_c D(x) \begin{cases} \le 1-\zeta r^{-\alpha} \\ \ge 1-\xi r^{-\alpha} \end{cases} \tag{5.1}$$

when r is sufficiently large.

One way for a path from 0 to reach Q_r^c is if $\mathcal{C}_r(0)$ is at least of size k (we will fix the value of k later), and at least one of the vertices, say v, of $\mathcal{C}_r(0)$ is an endpoint of an open edge e that has length at least $2r$. Then either $v \in Q_r^c$ and so there exists a path, or, perhaps more likely, $v \in Q_r$, but then the other endpoint of e is in Q_r^c. Hence, we have

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \geq \mathbb{P}_{p_c}(|\mathcal{C}_r(0)| \geq k, \exists e = (\underline{e}, \overline{e}) \text{ such that } \underline{e} \in \mathcal{C}_r(0), |e| > 2r). \tag{5.2}$$

Edge probabilities are translation invariant and independent, and there are at least k vertices in $\mathcal{C}_r(0)$, so we have a lower bound on the right-hand side,

$$\mathbb{P}_{p_c}(|\mathcal{C}_r(0)| \geq k)\left(1 - \mathbb{P}_{p_c}(\nexists\, \overline{e} \in Q_{2r}^c \text{ such that } \{0, \overline{e}\} \text{ is open})^k\right). \tag{5.3}$$

Observe that

$$\begin{aligned}\mathbb{P}_{p_c}(\nexists\, \overline{e} \in Q_{2r}^c \text{ such that } \{0, \overline{e}\} \text{ is open}) =& \mathbb{P}_{p_c}\left(\bigcap_{x \in Q_{2r}^c} \{\{0, x\} \text{ is closed}\}\right) \\ &= \prod_{x \in Q_{2r}^c} \left(1 - p_c D(x)\right).\end{aligned} \tag{5.4}$$

Now, since $1 - z \leq \mathrm{e}^{-z}$, we have

$$\begin{aligned}1 - \mathbb{P}_{p_c}(\nexists\, \overline{e} \in Q_{2r}^c \text{ such that } \{0, \overline{e}\} \text{ is open})^k &\geq 1 - \exp\left(-k p_c \sum_{x \in Q_{2r}^c} D(x)\right) \\ &\geq 1 - \mathrm{e}^{-k p_c \zeta / (2r)^\alpha} \geq \frac{k p_c \zeta}{2(2r)^\alpha},\end{aligned} \tag{5.5}$$

for all $k < 2(2r)^\alpha / p_c \zeta$.

Thus,

$$\mathbb{P}_{p_c}(0 \leftrightarrow Q_r^c) \geq \frac{\zeta k}{(2r)^\alpha} \mathbb{P}_{p_c}(|\mathcal{C}_r(0)| \geq k) \qquad \text{for all} \quad k < \frac{2(2r)^\alpha}{p_c \zeta}. \tag{5.6}$$

All we need now is a lower bound on $\mathbb{P}_{p_c}(|\mathcal{C}_r(0)| \geq k)$.

Combining results of [5] and [23], it follows that there exists a constants $C_1 \geq c_1 > 0$ so that

$$\frac{c_1}{\sqrt{s}} \leq \mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq s) \leq \frac{C_1}{\sqrt{s}} \tag{5.7}$$

holds for long-range percolation when $d > d_c$. Furthermore, we have

$$\mathbb{P}_{p_c}(|\mathcal{C}_r(0)| \geq k) \geq \mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq k) - \mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq k, |\mathcal{C}_r(0)| < k). \tag{5.8}$$

To bound the first term on the right-hand side we use (5.7). For the second term we need an upper bound. Given that $|\mathcal{C}_r(0)| < k$, for $|\mathcal{C}(0)| \geq k$ to hold as well, it is necessary that there exist at least one open edge that is longer than r with at least one endpoint in $\mathcal{C}_r(0)$. Thus,

$$\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq k, |\mathcal{C}_r(0)| < k) \leq \mathbb{P}_{p_c}(|\mathcal{C}_r(0)| < k, \exists e = \{\underline{e}, \overline{e}\} \text{ with } \underline{e} \in \mathcal{C}_r(0) \text{ s.t. } |e| > r, e \text{ open}). \tag{5.9}$$

The probability of having such an edge only depends on $|\mathcal{C}_r(0)|$, the number of possible endpoints for this edge. Hence, we can condition on the size of $\mathcal{C}_r(0)$ and use translation invariance and independence of edges for an upper bound:

$$\begin{aligned}&\mathbb{P}_{p_c}\big(\exists e = \{\underline{e}, \overline{e}\} \text{ with } \underline{e} \in \mathcal{C}_r(0) \text{ s.t. } |e| > r, e \text{ open} \,\big|\, |\mathcal{C}_r(0)| < k\big) \mathbb{P}_{p_c}(|\mathcal{C}_r(0)| < k) \\ &\qquad \leq \sum_{s=1}^{k-1} s\, \mathbb{P}_{p_c}(\exists v \in Q_r^c \text{ s.t. } \{0, v\} \text{ open}) \mathbb{P}_{p_c}(|\mathcal{C}_r(0)| = s).\end{aligned} \tag{5.10}$$

Similar to (5.4) we have that

$$\mathbb{P}_{p_c}(\exists v \in Q_r^c \text{ s.t. } \{0, v\} \text{ open}) = 1 - \prod_{x \in Q_{2r}^c} (1 - p_c D(x)) \leq \sum_{x \in Q_{2r}^c} p_c D(x) \leq \xi r^{-\alpha}, \tag{5.11}$$

where the first bound follows by a simple induction argument and the second bound follows by (5.1). We apply the lower bound in (5.1) to (5.10) to get the upper bound

$$\xi r^{-\alpha}\sum_{s=1}^{k-1}\mathbb{P}_{p_c}(|\mathcal{C}_r(0)|\geq s)\leq \xi r^{-\alpha}\sum_{s=1}^{k-1}\mathbb{P}_{p_c}(|\mathcal{C}(0)|\geq s)\leq \xi r^{-\alpha}\sum_{s=1}^{k-1}\frac{C_1}{\sqrt{s}}\leq C_2\sqrt{k}r^{-\alpha} \tag{5.12}$$

where we used (5.7) in the second-to-last step. Applying the above bound to (5.6) with $k=\varepsilon^2 r^\alpha$ and some suitably small constant ε thus gives

$$\mathbb{P}_{p_c}(0\leftrightarrow Q_r^c)\geq \frac{\zeta\varepsilon^2 r^\alpha}{(2r)^\alpha}\mathbb{P}_{p_c}(|\mathcal{C}_r(0)|\geq \varepsilon^2 r^\alpha)\geq \frac{\zeta\varepsilon^2}{2^\alpha}\left(\frac{c_1}{\varepsilon r^{\alpha/2}}-\frac{C_2\varepsilon}{r^{\alpha/2}}\right)\geq \frac{c'}{r^{\alpha/2}}, \tag{5.13}$$

completing the proof for $\alpha\in(0,4]$.

To prove the theorem for $\alpha>4$, that is, to establish $\mathbb{P}_{p_c}(0\leftrightarrow Q_r^c)\geq c/r^2$, we use the second moment method. Fix n large, and define

$$N_{r,nr}=\#\{x: x\in Q_{nr}\setminus Q_r \text{ and } 0\leftrightarrow x\}. \tag{5.14}$$

Then,

$$\mathbb{P}_{p_c}(0\leftrightarrow Q_r^c)\geq \mathbb{P}_{p_c}(N_{r,nr}\geq 1). \tag{5.15}$$

By the second moment method, we have

$$\mathbb{P}_{p_c}(N_{r,nr}\geq 1)\geq \frac{\mathbb{E}_{p_c}[N_{r,nr}]^2}{\mathbb{E}_{p_c}[N_{r,nr}^2]}. \tag{5.16}$$

We can write

$$N_{r,nr}=|Q_{nr}\cap\mathcal{C}(0)|-|Q_r\cap\mathcal{C}(0)|. \tag{5.17}$$

By Theorem 1.6, when n is large enough,

$$\mathbb{E}_{p_c}[N_{r,nr}]=\mathbb{E}_{p_c}[|Q_{nr}\cap\mathcal{C}(0)|]-\mathbb{E}_{p_c}[|Q_r\cap\mathcal{C}(0)|]\geq c_3(nr)^{(2\wedge\alpha)}-C_4r^{(2\wedge\alpha)}\geq c_5r^{(2\wedge\alpha)}. \tag{5.18}$$

We can write $N_{r,nr}^2$ as

$$N_{r,nr}^2=\#\{\text{pairs } x,y: x,y\in Q_{nr}\setminus Q_r \text{ and } 0\leftrightarrow x, 0\leftrightarrow y\}. \tag{5.19}$$

Obviously,

$$N_{r,nr}^2\leq \#\{\text{triplets } x,y,z: x,y\in Q_{nr}\setminus Q_r, z\in\mathbb{Z}^d \text{ and } \{0\leftrightarrow z\}\circ\{z\leftrightarrow x\}\circ\{z\leftrightarrow y\}\}. \tag{5.20}$$

Using the BK-inequality and techniques similar to those used in the proof of Theorem 1.6, we can show

$$\mathbb{E}_{p_c}[N_{r,nr}^2]\leq \sum_{x,y\in Q_{nr}}\sum_{z\in\mathbb{Z}^d}\tau_{p_c}(z)\tau_{p_c}(x-z)\tau_{p_c}(y-z)\leq C_6r^{3(2\wedge\alpha)}. \tag{5.21}$$

(In particular, first bound the sum over y for fixed x,z and then bound the remaining sum in the same way as was done in Section 4.4.1.

Hence, it follows that

$$\mathbb{P}_{p_c}(0\leftrightarrow Q_r^c)\geq \frac{c_5r^{2(2\wedge\alpha)}}{C_6r^{3(2\wedge\alpha)}}\geq \frac{c''}{r^2}. \tag{5.22}$$

Finally, we combine the bounds (5.13) and (5.22). This yields

$$\mathbb{P}_{p_c}(0\leftrightarrow Q_r^c)\geq \max\left\{\frac{c'}{r^{\alpha/2}},\frac{c''}{r^2}\right\}\geq \frac{c}{r^{(4\wedge\alpha)/2}}, \tag{5.23}$$

completing the proof. □

6. Bounds on lace expansion coefficients: proof of Proposition 2.5 and Lemma 2.7

In this section and the next we prove Proposition 2.5 and Lemma 2.7. We start by showing that the functions $\xi^{(n)}$ and $\gamma^{(n)}$ can both be bounded in terms of one-arm probabilities, $\pi^{(n)}$, and another function, $\phi^{(n)}$. Then we bound the complex expressions $\pi^{(n)}$ and $\phi^{(n)}$ in terms of simpler two-point functions. These bounds are known as *diagrammatic estimates*.

Using the diagrammatic estimates we are able to get the bounds needed to prove Proposition 2.5 and Lemma 2.7, but it involves a lot of machinery to do so.

In the case of Lemma 2.7(i), this is mainly due to the fact that the function $\phi^{(n)}$ has not appeared in any other lace expansion (though a similar function is considered for oriented percolation in [25]), so there is little to fall back on.

In the case of Proposition 2.5, the reason for the difficulties is more fundamental. The bound that we require is quite strong while our knowledge of the two-point functions is limited and mainly consists of its properties in Fourier space. Significant effort is needed to evaluate these functions in Fourier space without sacrificing too much accuracy in the bounds.

In the course of the proof of Proposition 2.5 we introduce a method for getting lace expansion diagrams *in Fourier space*. This construction uses ideas from graph theory, and in principle applies to any lace expansion whose terms can be bounded by 'planar' diagrams (e.g. self-avoiding walk, lattice animals and lattice trees). Moreover, the Fourier space diagrams have a simple combinatorial structure and are fairly easy to bound.

6.1. Proof of Lemma 2.7(i)

In this section we prove Lemma 2.7(i) subject to Proposition 2.5, which we prove in the next section, and subject to Lemma 6.1, which is stated further along in the section and proved in the final subsection. The techniques that we use are similar to those used in [25], but much less refined, as we only need an upper bound.

Proof of Lemma 2.7(i) subject to Proposition 2.5 and Lemma 6.1. Recall definitions (2.28) – (2.32).

We start by showing that $\xi^{(n)}(r;F)$ and $\gamma^{(n)}(r;F)$ can be bounded as follows:

$$\xi^{(n)}(r;F) \quad\leq\quad \frac{1}{2}\sum_{x\in Q_r}\theta^{(n)}(x,r;F)\mathbb{P}_{p_c}(x\leftrightarrow Q_r^c)+\varrho^{(n)}(r;F); \tag{6.1}$$

$$\gamma^{(n)}(r;F) \quad\leq\quad \sum_{x\in Q_r}\theta^{(n)}(x,r;F)\mathbb{P}_{p_c}(x\leftrightarrow Q_r^c), \tag{6.2}$$

for the function $\theta^{(n)}$ defined below in (6.5) and (6.7) and $\varrho^{(n)}$ defined below in (6.12) and (6.14).

Then we show that $\varrho^{(n)}$ can be bounded by

$$\varrho^{(n)}(r;F)\leq\sum_{x\in Q_r^c}\pi^{(n)}(x,r;F) \tag{6.3}$$

and that $\theta^{(n)}$ can be bounded further by

$$\theta^{(n)}(x,r;F)\leq\sum_{z\in Q_r}\phi^{(n)}(z,x,r;F)\mathbb{P}_{p_c}(z\leftrightarrow Q_r^c) \tag{6.4}$$

for the function $\phi^{(n)}$ defined below in (6.21) and (6.22).

After showing that such bounds exist, we get diagrammatic bounds on $\theta^{(n)}$ and $\phi^{(n)}$ that suffice to prove Lemma 2.7(i) (subject to Proposition 2.5).

Define

$$\theta^{(0)}(x,r;F)=\sum_{y\in Q_r}p_{y,x}\mathbb{E}_{p_c}[\mathbb{1}_{F\cap\{\{0\leftrightarrow y,Q_m\Longleftrightarrow y,Q_m\leftrightarrow Q_r^c\}\text{ on }\tilde{\mathcal{C}}^{(y,x)}(0)\}}], \tag{6.5}$$

then (6.2) for $n=0$ follows immediately from (2.8) and the simple fact that

$$\mathbb{P}^A_{p_c}(x\leftrightarrow Q_r^c)\leq\mathbb{P}_{p_c}(x\leftrightarrow Q_r^c). \tag{6.6}$$

For $n \geq 1$, define,

$$\theta^{(n)}(x,r;F) = \sum_{y\in Q_r} p_{y,x} \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0,v_0} \cdots \sum_{(u_{n-1},v_{n-1})\in\mathcal{E}_r} p_{u_{n-1}v_{n-1}} \mathbb{E}_0\Big[\mathbb{1}_{F\cap\{0\leftrightarrow u_0, Q_m \Longleftrightarrow Q_r\}} \times \mathbb{E}_1\Big[\mathbb{1}_1\mathbb{E}_2\big[\mathbb{1}_2\cdots\mathbb{E}_{n-1}[\mathbb{1}_{\{E'(v_{n-1},y;\tilde{\mathcal{C}}_{n-1})\cap\{v_{n-1}\leftrightarrow Q_r^c\}\text{ on }\tilde{\mathcal{C}}^{(y,x)}(v_{n-1})\}}]\cdots\big]\Big]\Big]. \tag{6.7}$$

Combined with (2.30), (6.6) and (6.7) give (6.2) for $n \geq 1$.

Although the purpose of the functions $\theta^{(n)}$ is to bound the probability of events E' that are restricted to be connected to Q_r^c, it will come in handy later on to use that the bound

$$\theta^{(n)}(x,r;F) \leq \pi^{(n)}(x,r;F) \tag{6.8}$$

also holds, since

$$E'(v_{n-1},y;\tilde{\mathcal{C}}_{n-1}) \cap \{v_{n-1} \leftrightarrow Q_r^c\} \subset E'(v_{n-1},y;\tilde{\mathcal{C}}_{n-1}). \tag{6.9}$$

We need to do a bit more work to show (6.1). In a similar fashion as in [25], we define the set

$$\mathcal{P}_A = \big\{\text{edges } b|\text{ the event } E'(v,\underline{b};A)\cap\{b\text{ open}\}\cap\{\overline{b}\leftrightarrow Q_r^c\text{ off }\tilde{\mathcal{C}}^b(v)\}\text{ occurs}\big\}. \tag{6.10}$$

In words, $\mathcal{P}_A$ is the (unordered) set of *cutting edges*, i.e., edges in $\mathcal{P}_A$ have the property that they are open and they are the first edge after A that is pivotal for at least one connection from v to Q_r^c. (This means that these edges are not necessarily pivotal for all connections from v to Q_r^c.)

Using that $\sum_{b\in\mathcal{E}_r}\mathbb{1}_{\{b\in\mathcal{P}_A\}} = |\mathcal{P}_A|$, we can decompose the event $E''(v,r;A)$ according to the size of $\mathcal{P}_A$:

$$\begin{aligned}\mathbb{P}_{p_c}(F\cap E''(v,r;A)) =& \mathbb{P}_{p_c}(F\cap E''(v,r;A)\cap\{\mathcal{P}_A=\varnothing\}) \\ &+\sum_{l=1}^{\infty}\frac{1}{l}\sum_{b\in\mathcal{E}_r}\mathbb{P}_{p_c}(F\cap E''(v,r;A)\cap\{b\in\mathcal{P}_A\}\cap\{|\mathcal{P}_A|=l\}) \\ =&\frac{1}{2}\sum_{b\in\mathcal{E}_r}\mathbb{P}_{p_c}(F\cap E''(v,r;A)\cap\{b\in\mathcal{P}_A\}) + \tilde{\varrho}^{(0)}(v,r;A,F)\end{aligned} \tag{6.11}$$

where

$$\tilde{\varrho}^{(0)}(v,r;A,F) = \mathbb{P}_{p_c}(F\cap E''(v,r;A)\cap\{\mathcal{P}_A=\varnothing\}) + \sum_{l=1}^{\infty}\left(\frac{1}{l}-\frac{1}{2}\right)\sum_{b\in\mathcal{E}_r}\mathbb{P}_{p_c}(F\cap E''(v,r;A)\cap\{b\in\mathcal{P}_A\}\cap\{|\mathcal{P}_A|=l\}). \tag{6.12}$$

Define $\varrho^{(0)}(r;F) = \tilde{\varrho}^{(0)}(0,r;Q_m,F)$, then it follows that

$$\xi^{(0)}(r;F) = \frac{1}{2}\sum_{b\in\mathcal{E}_r}\mathbb{P}_{p_c}(F\cap E''(0,r;Q_m)\cap\{b\in\mathcal{P}_{Q_m}\}) + \varrho^{(0)}(r;F). \tag{6.13}$$

Similarly, by replacing the final expectation in (2.29) by (6.11), we can isolate a term $\varrho^{(n)}$ from $\xi^{(n)}(r;F)$:

$$\varrho^{(n)}(r;F) = \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0,v_0}\cdots\sum_{(u_n,v_n)\in\mathcal{E}_r} p_{u_nv_n}\mathbb{E}_0\Big[\mathbb{1}_{F\cap\{0\leftrightarrow u,Q_m\Longleftrightarrow u_0\}} \times \mathbb{E}_1\Big[\mathbb{1}_1\mathbb{E}_2\big[\mathbb{1}_2\cdots\mathbb{E}_{n-1}[\tilde{\varrho}^{(0)}(v_{n-1},r;\tilde{\mathcal{C}}^{(u_n,v_n)}(v_{n-1}),\Omega)]\cdots\big]\Big]\Big], \tag{6.14}$$

where Ω denotes the full state space.

From [25, Proposition 4.3] we have the following useful identity: for $A\subseteq\mathbb{Z}^d$, $v\in\mathbb{Z}^d$, $r\geq 1$ and $b\in\mathcal{E}_r$,

$$E''(v,r;A)\cap\{b\in\mathcal{P}_A\} = \{E'(v,\underline{b};A)\cap\{v\xleftrightarrow{A}Q_r^c\}\text{ on }\tilde{\mathcal{C}}^b(v)\}\cap\{b\text{ open}\}\cap\{\overline{b}\leftrightarrow Q_r^c\text{ off }\tilde{\mathcal{C}}^b(v)\}. \tag{6.15}$$

This equality is proved in [25] for oriented percolation, but the proof is easily adapted to the unoriented case.

Applying (6.15) with $A = Q_m$ and $v = 0$ to (6.13) and using

$$\{x \xleftrightarrow{A} Q_r^c\} \subseteq \{x \leftrightarrow Q_r^c\} \tag{6.16}$$

and the Factorization Lemma yields (6.1) for $n = 0$. Applying (6.15) with $A = \tilde{\mathcal{C}}^b(v_{n-1})$ and $v = v_{n-1}$ to (6.13) and again using (6.16) and the Factorization Lemma yields (6.1) for $n \geq 1$.

Now we show (6.3). Observe that the sum in (6.12) only has positive contributions when $l = 0, 1$, so we do not have to consider the terms $l \geq 2$ for an upper bound. Therefore,

$$\tilde{\varrho}^{(0)}(v, r; A, \Omega) \leq \mathbb{P}_{p_c}(E''(v, r; A) \cap \{|\mathcal{P}_A| \leq 1\}). \tag{6.17}$$

From [25, Proposition 4.6] we have

$$E''(v, r; A) \cap \{|\mathcal{P}_A| \leq 1\} \subseteq \bigcup_{x \in Q_r^c} E'(v, x; A). \tag{6.18}$$

Again, (6.18) is proved in [25] for oriented percolation, and again the proof is straightforwardly adapted to the unoriented case.

Applying (6.18) with $A = Q_m$ and $v = 0$ to (6.12) and applying (6.18) with $A = \tilde{\mathcal{C}}^b(v_{n-1})$ and $v = v_{n-1}$ to (6.14) yields (6.3) for $n \geq 0$.

Next is the bound (6.4). From the definition of E' it follows that

$$\begin{aligned} E'(v, x; A) \cap \{v \leftrightarrow Q_r^c\} \quad &\subseteq \quad \bigcup_{y \in Q_r} \bigcup_{t \in \mathbb{Z}^d} \bigcup_{z \in A} \big(\{v \leftrightarrow t\} \circ \{t \leftrightarrow y\} \circ \{y \leftrightarrow x\} \circ \{t \leftrightarrow z\} \circ \{z \leftrightarrow x\} \circ \{y \leftrightarrow Q_r^c\}\big) \\ &\qquad \cup \big(\{v \leftrightarrow y\} \circ \{y \leftrightarrow t\} \circ \{t \leftrightarrow x\} \circ \{t \leftrightarrow z\} \circ \{z \leftrightarrow x\} \circ \{y \leftrightarrow Q_r^c\}\big) \\ &\equiv \quad \bigcup_{y \in Q_r} H'(v, x, y; A). \end{aligned} \tag{6.19}$$

Hence, by the union bound and the BK-inequality,

$$\mathbb{P}_{p_c}(E'(v, x; A) \cap \{v \leftrightarrow Q_r^c\}) \leq \sum_{y \in Q_r} \mathbb{P}_{p_c}\big(H'(v, x, y; A)\big) \mathbb{P}_{p_c}(y \leftrightarrow Q_r^c). \tag{6.20}$$

Define

$$\phi^{(0)}(x, y, r; F) = \mathbb{P}_{p_c}\big(F \cap H'(0, x, y; Q_m)\big) \tag{6.21}$$

(as with $\pi^{(0)}(x, r; F)$, this function is independent of r, but we write r anyway for consistency). Also define, for $n \geq 1$,

$$\begin{aligned} \phi^{(n)}(x, y, r; F) = \sum_{(u_0, v_0) \in \mathcal{E}_r} p_{u_0, v_0} \cdots \sum_{(u_{n-1}, v_{n-1}) \in \mathcal{E}_r} p_{u_{n-1} v_{n-1}} \mathbb{E}_0 \Big[\mathbb{1}_{F \cap \{0 \leftrightarrow u_0, Q_m \Longleftrightarrow Q_r\}} \\ \times \mathbb{E}_1 \Big[\mathbb{1}_1 \mathbb{E}_2 \big[\mathbb{1}_2 \cdots \mathbb{E}_{n-1} [\mathbb{1}_{\{H'(v_{n-1}, x, y; \tilde{\mathcal{C}}_{n-1})\}}] \cdots \big] \Big] \Big]. \end{aligned} \tag{6.22}$$

Now it follows from (6.5), (6.7), (6.20) and (6.19) that (6.4) holds.

For future use we define

$$\begin{aligned} \widetilde{\Pi}(x, r; F) &= \sum_{n=0}^{\infty} \pi^{(n)}(x, r; F), \\ \widetilde{\Theta}(x, r; F) &= \sum_{n=0}^{\infty} \theta^{(n)}(x, r; F), \\ \widetilde{\Phi}(x, y, r; F) &= \sum_{n=0}^{\infty} \phi^{(n)}(x, y, r; F). \end{aligned} \tag{6.23}$$

Before we proceed we state the following lemma:

Lemma 6.1. *For a model that satisfies the assumptions of Proposition 2.5 and for the same choice of $\delta > 0$ as in Proposition 2.5, for all $r \in \mathbb{N}$, there exists constants $C_i = C_i(F, L, d, \alpha, \delta)$ for $i = 1, 2, 3$, such that*

$$\sum_{x,y\in\mathbb{Z}^d} \widetilde{\Phi}(x,y,r;F) \leq C_1; \tag{6.24}$$

$$\sum_{x,y\in\mathbb{Z}^d} |x-y|^{\delta}\widetilde{\Phi}(x,y,r;F) \leq C_2; \tag{6.25}$$

$$\sum_{x\in\mathbb{Z}^d} |x|^{(2\wedge\alpha)+\delta}\widetilde{\Pi}(x,r;F) \leq C_3. \tag{6.26}$$

We do not prove Lemma 6.1 since it can be proved in a similar way as Proposition 2.5. At the end of Section 7 we do briefly discuss this proof for Lemma 6.1.

From (6.1) – (6.4), (6.8) and (6.23) it follows that

$$\Xi(r;F) \leq \frac{1}{2}\sum_{x\in Q_r} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) + \sum_{x\in Q_r^c} \widetilde{\Pi}(x,r;F) \tag{6.27}$$

and

$$\Gamma(r;F) \leq \sum_{x\in Q_r} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c). \tag{6.28}$$

Hence, Lemma 2.7(i) is proved once we show that

$$\sum_{x\in Q_r} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) \leq \frac{C_4}{r^{1/\varrho+\delta}} \tag{6.29}$$

for a constant C_4 that may depend on F, L, d, α and δ, and that

$$\sum_{x\in Q_r^c} \widetilde{\Pi}(x,r;F) \leq \frac{C_3}{r^{(2\wedge\alpha)+\delta}}. \tag{6.30}$$

That (6.30) holds follows immediately from (6.26): $x \in Q_r^c$ implies $|x|/r > 1$, so

$$\sum_{x\in Q_r^c} \widetilde{\Pi}(x,r;F) \leq \sum_{x\in Q_r^c} \frac{|x|^{(2\wedge\alpha)+\delta}}{r^{(2\wedge\alpha)+\delta}}\widetilde{\Pi}(x,r;F) \leq \frac{C_3}{r^{(2\wedge\alpha)+\delta}}. \tag{6.31}$$

To bound (6.29) we introduce the following notation: for $a, b \in \mathbb{N}$ and $a > b$,

$$Q_{a,b} = Q_a \setminus Q_b. \tag{6.32}$$

The sum on the left-hand side of (6.29) can be split into the contributions of $x \in Q_{r/4}$ and those of $x \in Q_{r,r/4}$:

$$\sum_{x\in Q_r} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) = \sum_{x\in Q_{r/4}} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) + \sum_{x\in Q_{r,r/4}} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c). \tag{6.33}$$

The second term can be bounded using (6.8) and (6.26):

$$\sum_{x\in Q_{r,r/4}} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) \leq \sum_{x\in Q_{r/4}^c} \widetilde{\Pi}(x,r;F) \leq \frac{C_3}{r^{(2\wedge\alpha)+\delta}}. \tag{6.34}$$

To bound the first term we use (6.4):

$$\sum_{x\in Q_{r/4}} \widetilde{\Theta}(x,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c) \leq \sum_{x\in Q_{r/4}}\sum_{y\in Q_r} \widetilde{\Phi}(x,y,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c)\mathbb{P}_{p_c}(y \leftrightarrow Q_r^c). \tag{6.35}$$

For convenience, write

$$S^{\Phi}(x,y,r;F) = \widetilde{\Phi}(x,y,r;F)\mathbb{P}_{p_c}(x \leftrightarrow Q_r^c)\mathbb{P}_{p_c}(y \leftrightarrow Q_r^c). \tag{6.36}$$

The right-hand side of (6.35) can again be split into the contribution of $y \in Q_{r/2}$ and of $y \in Q_{r,r/2}$:

$$\sum_{x\in Q_{r/4}} \sum_{y\in Q_r} S^\Phi(x,y,r;F) = \sum_{x\in Q_{r/4}} \sum_{y\in Q_{r/2}} S^\Phi(x,y,r;F) + \sum_{x\in Q_{r/4}} \sum_{y\in Q_{r,r/2}} S^\Phi(x,y,r;F). \tag{6.37}$$

To bound the first term on the right-hand side, we note for $a,b \in \mathbb{N}$, $a > b$ and $x \in Q_b$, by (1.28) we have

$$\mathbb{P}_{p_c}(x \leftrightarrow Q_a^c) \le \mathbb{P}_{p_c}(0 \leftrightarrow Q_{a-b}^c) \le \frac{C}{(a-b)^{1/\varrho}} \tag{6.38}$$

so that, by Lemma 6.1,

$$\sum_{x\in Q_{r/4}} \sum_{y\in Q_{r/2}} S^\Phi(x,y,r;F) \le \frac{C}{r^{2/\varrho}} \sum_{x\in Q_{r/4}} \sum_{y\in Q_{r/2}} \widetilde{\Phi}(x,y,r;F) \le \frac{C_5}{r^{2/\varrho}} \tag{6.39}$$

for some constant C_5 that may depend on m, L, d, α and δ.

Finally, the second term in (6.37) can also be bounded using (6.38) and Lemma 6.1: since $x \in Q_{r/4}$ and $|x-y| > r/4$, we have

$$\begin{aligned} \sum_{x\in Q_{r/4}} \sum_{y\in Q_{r,r/2}} S^\Phi(x,y,r) &\le \frac{C}{r^{1/\varrho}} \sum_{x\in Q_{r/4}} \sum_{y\in Q_{r,r/2}} \widetilde{\Phi}(x,y,r;F) \\ &\le \frac{C}{r^{1/\varrho}} \sum_{x\in Q_{r/4}} \sum_{y\in Q_{r,r/2}} \frac{|x-y|^\delta}{r^\delta} \widetilde{\Phi}(x,y,r;F) \le \frac{C_6}{r^{1/\varrho+\delta}} \end{aligned} \tag{6.40}$$

for some constant C_6 that may depend on m, L, d, α and δ. Combining (6.34), (6.39) and (6.40) gives the desired bound (6.29) and completes the proof. □

6.2. The proof of Lemma 2.7(ii)

From the definition of $R^{(N)}(r;F)$ in (2.33) and of $\pi^{(n)}(x,r;F)$ in (2.31) it is easy to see that

$$R^{(N)}(r;F) \le p_c \sum_{x\in\mathbb{Z}^d} \pi^{(N-1)}(x,r;F). \tag{6.41}$$

It is a simple consequence of (6.23) and (6.26) that $\lim_{N\to\infty} \sum_x \pi^{(N-1)}(x,r;F) = 0$. Furthermore, for all $N \ge 1$, $R^{(N)}(r;F) \ge 0$ and $\pi^{(N-1)}(x,r;F) \ge 0$,

$$\lim_{N\to\infty} R^{(N)}(r;F) = \lim_{N\to\infty} p_c \sum_{x\in\mathbb{Z}^d} \pi^{(N-1)}(x,r;F) = 0. \tag{6.42}$$

□

6.3. Diagrammatic estimates

In this subsection we derive diagrammatic estimates on the functions $\pi_m^{(n)}$ and $\phi_m^{(n)}$. We need them to prove Proposition 2.5 and Lemma 6.1. Our derivation is based on the derivation given in [7]. The derivation below is in broad strokes identical to what is already in the literature. Therefore, a reader already familiar with this procedure could skip to the conclusion at the end of this section.

We start with $\pi^{(0)}$ and $\phi^{(0)}$. From the definition of E' in (2.16) it is easy to see that

$$E'(0,x;Q_m) \subseteq \bigcup_{w\in Q_m} (\{0 \leftrightarrow x\} \circ \{w \leftrightarrow x\}). \tag{6.43}$$

Hence, by the BK-inequality,

$$\begin{aligned} \pi^{(0)}(x,r;F) &\le \mathbb{P}_{p_c}(E'(0,x;Q_m)) \le \mathbb{P}_{p_c}\left(\bigcup_{w\in Q_m} (\{0 \leftrightarrow x\} \circ \{w \leftrightarrow x\})\right) \\ &\le \sum_{w\in Q_m} \tau_{p_c}(x)\tau_{p_c}(x-w). \end{aligned} \tag{6.44}$$

Similarly, from the definition of H' in (6.19) it follows that

$$H'(0,x,y;Q_m) \subseteq \bigcup_{w\in Q_m} \big(\{0\leftrightarrow x\}\circ\{w\leftrightarrow y\}\circ\{y\leftrightarrow x\}\big). \tag{6.45}$$

Therefore, by (6.21) and the BK-inequality,

$$\begin{aligned}\phi^{(0)}(x,y,r;F) \le& \mathbb{P}_{p_c}\left(\bigcup_{w\in Q_m}\big(\{0\leftrightarrow x\}\circ\{w\leftrightarrow y\}\circ\{y\leftrightarrow x\}\big)\right)\\ &\le \sum_{w\in Q_m}\tau_{p_c}(x)\tau_{p_c}(y-x)\tau_{p_c}(y-w).\end{aligned} \tag{6.46}$$

Furthermore, since on both right-hand sides of (6.44) and (6.46) we sum w over the finite ball Q_m, we can bound both by Q_m-independent functions:

$$\pi^{(0)}(x,r;F) \le C'_m\bar{\pi}^{(0)}(x) \equiv C'_m\tau_{p_c}(x)^2 \tag{6.47}$$

and

$$\phi^{(0)}(x,y,r;F) \le C'_m\bar{\phi}^{(0)}(x,y) \equiv C'_m\tau_{p_c}(x)\tau_{p_c}(x-y)\tau_{p_c}(y) \tag{6.48}$$

where C'_m is a constant given by

$$C'_m = \max_{x\in\mathbb{Z}^d}\frac{\sum_{w\in Q_m}\tau_{p_c}(x-w)}{\tau_{p_c}(x)} < \infty. \tag{6.49}$$

Let $\mathbb{P}^{(n)}_{p_c}$ denote the product measure of $n+1$ copies of critical percolation on $\mathbb{Z}^d$. We write A_i to signify that the event A occurs on the ith copy. By Fubini's theorem and (2.31) and (6.22), for $n\ge 1$,

$$\begin{aligned}\pi^{(n)}(x,r;F) = \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0}\cdots\sum_{(u_{n-1},v_{n-1})\in\mathcal{E}_r} p_{u_{n-1}v_{n-1}}\mathbb{P}^{(n)}_{p_c}\Bigg(&\{F\cap\{0\leftrightarrow u_0, Q_m\Longleftrightarrow u_0\}\}_0\\ &\cap\left(\bigcap_{i=1}^{n-1}E'(v_{i-1},u_i;\tilde{\mathcal{C}}_{i-1})_i\right)\cap E'(v_{n-1},x;\tilde{\mathcal{C}}_{n-1})_n\Bigg)\end{aligned} \tag{6.50}$$

and

$$\begin{aligned}\phi^{(n)}(x,y,r;F) = \sum_{(u_0,v_0)\in\mathcal{E}_r} p_{u_0v_0}\cdots\sum_{(u_{n-1},v_{n-1})\in\mathcal{E}_r} p_{u_{n-1}v_{n-1}}\mathbb{P}^{(n)}_{p_c}\Bigg(&\{F\cap\{0\leftrightarrow u_0, Q_m\Longleftrightarrow u_0\}\}_0\\ &\cap\left(\bigcap_{i=1}^{n-1}E'(v_{i-1},u_i;\tilde{\mathcal{C}}_{i-1})_i\right)\cap H'(v_{n-1},x,y;\tilde{\mathcal{C}}_{n-1})_n\Bigg).\end{aligned} \tag{6.51}$$

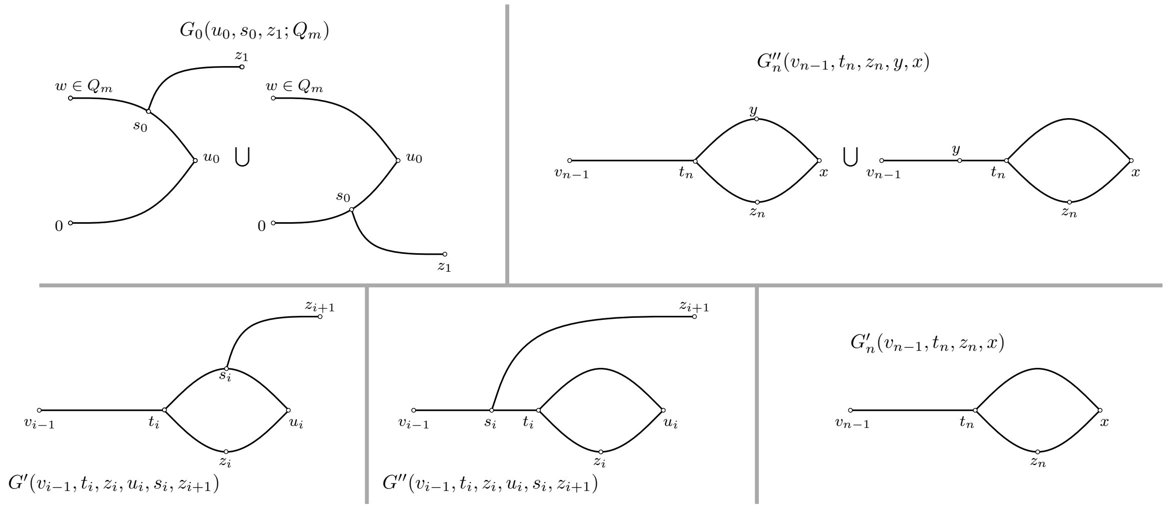

Figure 1. Depictions of the events G_0, G', G'', G'_n and G''_n.

To estimate these functions, we define the events

$$\begin{aligned} G_0(u_0,s_0,z_1;Q_m) &= \Big(\bigcup_{w\in Q_m}\{0\leftrightarrow u_0\}\circ\{w\leftrightarrow s_0\}\circ\{s_0\leftrightarrow u_0\}\circ\{s_0\leftrightarrow z_1\}\Big)\\ &\quad\cup\Big(\bigcup_{w\in Q_m}\{0\leftrightarrow s_0\}\circ\{s_0\leftrightarrow u_0\}\circ\{w\leftrightarrow u_0\}\circ\{s_0\leftrightarrow z_1\}\Big); \qquad (6.52)\\ G'(v_{i-1},t_i,z_i,u_i,s_i,z_{i+1}) &= \{v_{i-1}\leftrightarrow t_i\}\circ\{t_i\leftrightarrow z_i\}\circ\{t_i\leftrightarrow s_i\}\circ\{z_i\leftrightarrow u_i\}\\ &\quad\circ\{s_i\leftrightarrow u_i\}\circ\{s_i\leftrightarrow z_{i+1}\}; \qquad (6.53)\\ G''(v_{i-1},t_i,z_i,u_i,s_i,z_{i+1}) &= \{v_{i-1}\leftrightarrow s_i\}\circ\{s_i\leftrightarrow t_i\}\circ\{t_i\leftrightarrow z_i\}\circ\{t_i\leftrightarrow u_i\}\\ &\quad\circ\{z_i\leftrightarrow u_i\}\circ\{s_i\leftrightarrow z_{i+1}\}; \qquad (6.54)\\ G(v_{i-1},t_i,z_i,u_i,s_i,z_{i+1}) &= G'(v_{i-1},t_i,z_i,u_i,s_i,z_{i+1})\cup G''(v_{i-1},t_i,z_i,u_i,s_i,z_{i+1}); \qquad (6.55)\\ G'_n(v_{n-1},t_n,z_n,x) &= \{v_{n-1}\leftrightarrow t_n\}\circ\{t_n\leftrightarrow z_n\}\circ\{t_n\leftrightarrow x\}\circ\{z_n\leftrightarrow x\}; \qquad (6.56)\\ G''_n(v_{n-1},t_n,z_n,y,x) &= \big(\{v_{n-1}\leftrightarrow t_n\}\circ\{t_n\leftrightarrow z_n\}\circ\{z_n\leftrightarrow x\}\circ\{t_n\leftrightarrow y\}\circ\{y\leftrightarrow x\}\big) \qquad (6.57)\\ &\quad\cup\big(\{v_{n-1}\leftrightarrow y\}\circ\{y\leftrightarrow t_n\}\circ\{t_n\leftrightarrow z_n\}\circ\{z_n\leftrightarrow x\}\circ\{t_n\leftrightarrow x\}\big). \end{aligned}$$

See Figure 1 for depictions of these events. The events G' and G'' are new in the context of diagrammatic expansions, all other events have appeared before, e.g. in [39]. All the events above are constructed of disjointly occurring, increasing events, and hence the BK-inequality can be used to factorize their probabilities.

The events inside $\pi^{(n)}$ and $\phi^{(n)}$ can be contained in constructions of the events (6.52) – (6.57): by definitions (2.16) and (6.19),

$$E'(v_{n-1},x;\tilde{\mathcal{C}}_{n-1})_n\subset\bigcup_{z_n\in\tilde{\mathcal{C}}_{n-1}}\bigcup_{t_n\in\mathbb{Z}^d}G'_n(v_{n-1},t_n,z_n,x)_n \qquad (6.58)$$

and

$$H'(v_{n-1},x,y;\tilde{\mathcal{C}}_{n-1})_n\subset\bigcup_{z_n\in\tilde{\mathcal{C}}_{n-1}}\bigcup_{t_n\in\mathbb{Z}^d}G''_n(v_{n-1},t_n,z_n,y,x)_n. \qquad (6.59)$$

Observe that

$$F\cap\{0\leftrightarrow u_0,Q_m\Longleftrightarrow u_0\}\cap\{z_1\in\tilde{\mathcal{C}}_0\}\subset\bigcup_{s_0\in\mathbb{Z}^d}G_0(u_0,s_0,z_1;Q_m), \qquad (6.60)$$

and note that the right-hand side is independent of F (but still depends on Q_m).

Similarly, for $n \geq 2$ and $i \in \{1, \ldots, n-1\}$,

$$E'(v_{i-1}, u_i; \tilde{\mathcal{C}}_{i-1}) \cap \{z_{i+1} \in \tilde{\mathcal{C}}_i\} \subset \bigcup_{z_i \in \tilde{\mathcal{C}}_{i-1}} \bigcup_{t_i, s_i \in \mathbb{Z}^d} G(v_{i-1}, t_i, z_i, u_i, s_i, z_{i+1})_i. \tag{6.61}$$

The relations (6.58) and (6.61) lead to

$$\begin{aligned}\{F \cap \{0 \leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}\}_0 &\cap \left(\bigcap_{i=1}^{n-1} E'(v_{i-1}, u_i; \tilde{\mathcal{C}}_{i-1})_i\right) \cap E'(v_{n-1}, x; \tilde{\mathcal{C}}_{n-1})_n \\ &\subset \bigcup_{\vec{t},\vec{s},\vec{z}} \left(G_0(u_0, s_0, z_1; Q_m)_0 \cap \left(\bigcap_{i=1}^{n-1} G(v_{i-1}, t_i, z_i, u_i, s_i, z_{i+1})_i\right) \cap G'_n(v_{n-1}, t_n, z_n, x)_n\right),\end{aligned} \tag{6.62}$$

where $\vec{t} = (t_1, \ldots, t_n)$, $\vec{s} = (s_0, \ldots, s_{n-1})$ and $\vec{z} = (z_1, \ldots, z_n)$, and all elements are allowed to take values in $\mathbb{Z}^d$. The relations (6.59) and (6.61) lead to

$$\begin{aligned}\{F \cap \{0 \leftrightarrow u_0, Q_m \Longleftrightarrow u_0\}\}_0 &\cap \left(\bigcap_{i=1}^{n-1} E'(v_{i-1}, u_i; \tilde{\mathcal{C}}_{i-1})_i\right) \cap H'(v_{n-1}, x, y; \tilde{\mathcal{C}}_{n-1})_n \\ &\subset \bigcup_{\vec{t},\vec{s},\vec{z}} \left(G_0(u_0, s_0, z_1; Q_m)_0 \cap \left(\bigcap_{i=1}^{n-1} G(v_{i-1}, t_i, z_i, u_i, s_i, z_{i+1})_i\right) \cap G''_n(v_{n-1}, t_n, z_n, y, x)_n\right).\end{aligned} \tag{6.63}$$

Therefore, we can get an upper bound on $\pi_m^{(n)}$ and $\xi_m^{(n)}$:

$$\begin{aligned}\pi^{(n)}(x, r; F) \leq \sum_{\vec{z},\vec{t},\vec{s},\vec{u},\vec{v}} &\left[\prod_{i=0}^{n-1} p_{u_i v_i}\right] \mathbb{P}_{p_c}(G_0(u_0, s_0, z_1; Q_m)) \\ &\times \prod_{i=1}^{n-1} \mathbb{P}_{p_c}(G(v_{i-1}, t_i, u_i, s_i, z_{i+1})) \mathbb{P}_{p_c}(G'_n(v_{n-1}, t_n, z_n, x)),\end{aligned} \tag{6.64}$$

where $\vec{u} = (u_0, \ldots, u_{n-1})$ and $\vec{v} = (v_0, \ldots, v_{n-1})$ with all elements are restricted to $\mathbb{Z}^d$, and

$$\begin{aligned}\phi^{(n)}(x, y, r; F) \leq \sum_{\vec{z},\vec{t},\vec{s},\vec{u},\vec{v}} &\left[\prod_{i=0}^{n-1} p_{u_i v_i}\right] \mathbb{P}_{p_c}(G_0(u_0, s_0, z_1; Q_m)) \\ &\times \prod_{i=1}^{n-1} \mathbb{P}_{p_c}(G(v_{i-1}, t_i, u_i, s_i, z_{i+1})) \mathbb{P}_{p_c}(G''_n(v_{n-1}, t_n, z_n, y, x)).\end{aligned} \tag{6.65}$$

The probabilities in (6.64) and (6.65) factorize because $G_0, \ldots, G'_n$ and $G_0, \ldots, G''_n$ are events on different percolation models. The separate probabilities can all be estimated using the BK-inequality. To organize the resulting sum, define

$$\widetilde{\tau}_{p_c}(x) = p_c(D * \tau_{p_c})(x) \tag{6.66}$$

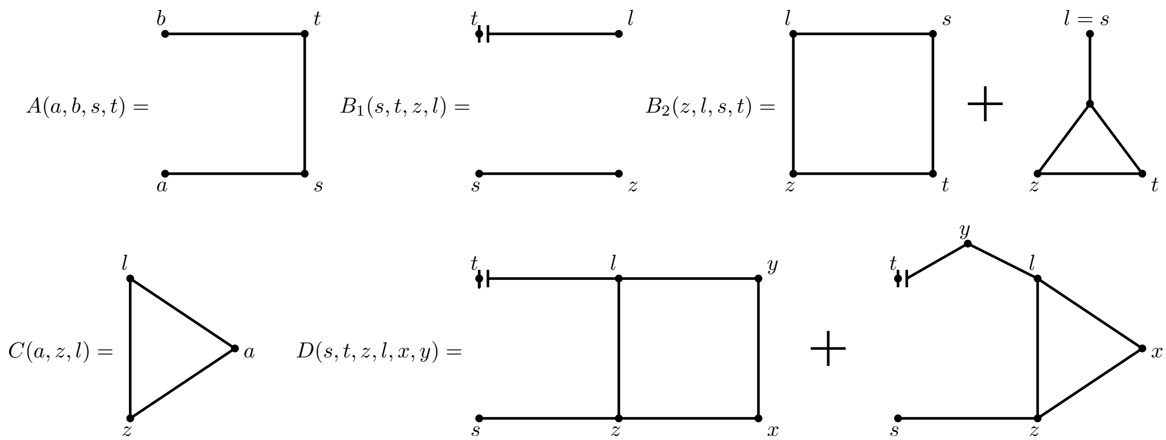

Figure 2. Diagrammatic representations of A, B_1, B_2, C and D. Unbroken lines represent τ's, lines that start with a gap represent $\widetilde{\tau}$'s.

and

$$\begin{aligned}
A(a,b,s,t) &= \tau_{p_c}(a-s)\tau_{p_c}(s-t)\tau_{p_c}(t-b); && (6.67)\\
B_1(s,t,z,l) &= \widetilde{\tau}_{p_c}(l-t)\tau_{p_c}(z-s); && (6.68)\\
B_2^{(0)}(z,l,s,t) &= \tau_{p_c}(l-z)\tau_{p_c}(t-z)\tau_{p_c}(s-l)\tau_{p_c}(t-s) && (6.69)\\
B_2^{(1)}(z,l,s,t) &= \sum_{a\in\mathbb{Z}^d}\delta_{l,s}\tau_{p_c}(a-s)\tau_{p_c}(z-a)\tau_{p_c}(t-a)\tau_{p_c}(t-z); && (6.70)\\
B_2(z,l,s,t) &= B_2^{(0)}(z,l,s,t)+B_2^{(1)}(z,l,s,t); && (6.71)\\
C(a,z,l) &= A(a,a,z,l)=\tau_{p_c}(a-z)\tau_{p_c}(l-a)\tau_{p_c}(z-l); && (6.72)\\
D^{(0)}(s,t,z,l,x,y) &= B_1(s,t,z,l)\tau_{p_c}(z-l)A(z,l,x,y); && (6.73)\\
D^{(1)}(s,t,z,l,x,y) &= \widetilde{\tau}_{p_c}(y-t)\tau_{p_c}(l-y)\tau_{p_c}(z-s)C(x,z,l); && (6.74)\\
D(s,t,z,l,x,y) &= D^{(0)}(s,t,z,l,x,y)+D^{(1)}(s,t,z,l,x,y). && (6.75)
\end{aligned}$$

See Figure 2 for diagrammatic representations of these functions.

Application of the BK-inequality yields

$$\mathbb{P}_{p_c}(G_0(s_0,t_0,z_1;Q_m)) \le \sum_{w\in Q_m}(A(0,w,s_0,t_0)+A(0,w,t_0,s_0))\tau_{p_c}(s_0,z_1), \tag{6.76}$$

$$\sum_{v_{n-1}\in\mathbb{Z}^d} p_{t_{n-1}v_{n-1}}\mathbb{P}_{p_c}(G_n'(v_{n-1},t_n,z_n,x)) \le \frac{B_1(s_{n-1},t_{n-1},z_n,t_n)}{\tau_{p_c}(z_n-s_{n-1})}C(x,z_n,t_n), \tag{6.77}$$

(note that we have switched from writing u_i-1 to writing t_{i-1}: this is for consistency in what follows) and

$$\sum_{v_{n-1}\in\mathbb{Z}^d} p_{t_{n-1}v_{n-1}}\mathbb{P}_{p_c}(G_n''(v_{n-1},t_n,z_n,y,x)) \le \frac{D(s_{n-1},t_{n-1},z_n,t_n,x,y)}{\tau_{p_c}(z_n-s_{n-1})}. \tag{6.78}$$

For G' and G'' we get

$$\sum_{v_{i-1}\in\mathbb{Z}^d} p_{t_{i-1}v_{i-1}}\mathbb{P}_{p_c}(G'(v_{i-1},l_i,z_i,s_i,t_i,z_{i+1}))$$
$$\le \frac{B_1(s_{i-1},t_{i-1},z_i,l_i)}{\tau_{p_c}(z_i-s_{i-1})}B_2^{(0)}(z_i,l_i,s_i,t_i)\tau_{p_c}(z_{i+1}-s_i); \tag{6.79}$$

and

$$\sum_{v_{i-1}\in\mathbb{Z}^d} p_{t_{i-1}v_{i-1}}\mathbb{P}_{p_c}(G''(v_{i-1},l_i,z_i,s_i,t_i,z_{i+1})) \le \frac{B_1(s_{i-1},t_{i-1},z_i,l_i)}{\tau_{p_c}(z_i-s_{i-1})}B_2^{(1)}(z_i,l_i,l_i,t_i)\tau_{p_c}(z_{i+1}-s_i). \quad (6.80)$$

The Kronecker delta in $B_2^{(1)}$ guarantees that it can only be nonzero when its second and third argument are equal, so we can replace the third argument of $B_2^{(1)}$ by s_i and combine (6.79) and (6.80) to get

$$\sum_{v_{i-1}\in\mathbb{Z}^d} p_{t_{i-1}v_{i-1}}\mathbb{P}_{p_c}(G(v_{i-1},l_i,z_i,s_i,t_i,z_{i+1})) \le \frac{B_1(s_{i-1},t_{i-1},z_i,l_i)}{\tau_{p_c}(z_i-s_{i-1})}B_2(z_i,l_i,s_i,t_i)\tau_{p_c}(z_{i+1}-s_i). \quad (6.81)$$

Substituting (6.76), (6.77) and (6.81) into (6.64), and (6.76), (6.78) and (6.81) into (6.65), respectively, we get, for $n\ge 1$

$$\pi^{(n)}(x,r;F) \le \sum_{\vec{s},\vec{t},\vec{z},\vec{l}}\sum_{w\in Q_m} A(0,w,s_0,t_0)\prod_{i=1}^{n-1}\left[B_1(s_{i-1},t_{i-1},z_i,l_i)B_2(z_i,l_i,s_i,t_i)\right] \times B_1(s_{n-1},t_{n-1},z_n,l_n)C(x,z_n,l_n) \quad (6.82)$$

and

$$\phi^{(n)}(x,y,r;F) \le \sum_{\vec{s},\vec{t},\vec{z},\vec{l}}\sum_{w\in Q_m} A(0,w,s_0,t_0)\prod_{i=1}^{n-1}\left[B_1(s_{i-1},t_{i-1},z_i,l_i)B_2(z_i,l_i,s_i,t_i)\right] \times D(s_{n-1},t_{n-1},z_n,l_n,x,y). \quad (6.83)$$

The summation over the vectors $\vec{s}=(s_0,\dots,s_{n-1})$, $\vec{t}=(t_0,\dots t_{n-1})$, $\vec{z}=(z_1,\dots,z_n)$ and $\vec{l}=(l_1,\dots,l_n)$ on the right-hand sides of (6.82) and (6.83) is over all of $\mathbb{Z}^d$ for each element, so in both cases the dependence of r has been removed. Also observe that the sum over w is again restricted to Q_m, so that once again we may replace $A(0,w,s_0,t_0)$ by $C(0,s_0,t_0)$ in both instances, to bound, for $n\ge 1$,

$$\pi^{(n)}(x,r;F) \le C'_m\bar{\pi}^{(n)}(x) \equiv C'_m\sum_{\vec{s},\vec{t},\vec{z},\vec{l}} C(0,s_0,t_0)\prod_{i=1}^{n-1}\left[B_1(s_{i-1},t_{i-1},z_i,l_i)B_2(z_i,l_i,s_i,t_i)\right] \times B_1(s_{n-1},t_{n-1},z_n,l_n)C(x,z_n,l_n) \quad (6.84)$$

and

$$\phi^{(n)}(x,y,r;F) \le C'_m\bar{\phi}^{(n)}(x,y) \equiv C'_m\sum_{\vec{s},\vec{t},\vec{z},\vec{l}} C(0,s_0,t_0)\prod_{i=1}^{n-1}\left[B_1(s_{i-1},t_{i-1},z_i,l_i)B_2(z_i,l_i,s_i,t_i)\right] \times D(s_{n-1},t_{n-1},z_n,l_n,x,y). \quad (6.85)$$

The two bounds above are commonly referred to as diagrammatic estimates. In Figure 3 we show two examples of diagrams.

Finally, for ease of notation in the coming sections, we define

$$\bar{\Pi}(x) = \sum_{n=0}^{\infty}\bar{\pi}^{(n)}(x) \qquad \text{and} \qquad \bar{\Phi}(x,y) = \sum_{n=0}^{\infty}\bar{\phi}^{(n)}(x,y). \quad (6.86)$$

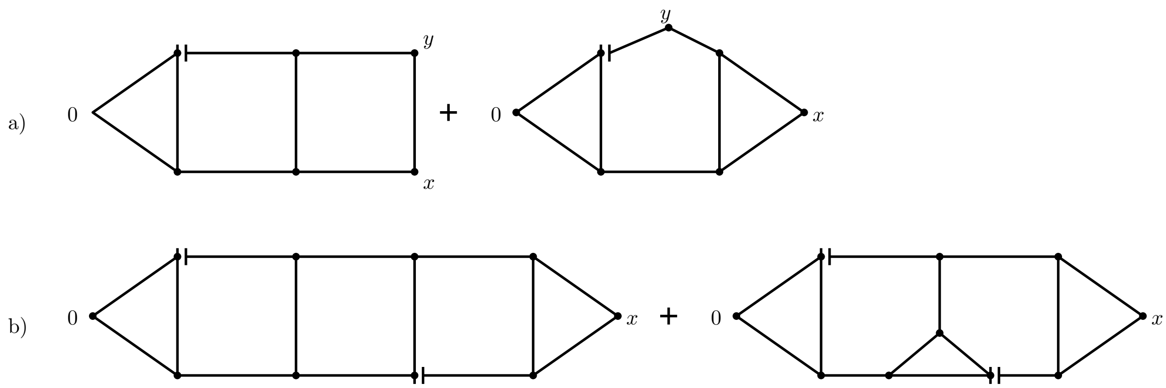

Figure 3. Diagrams bounding a) $\bar\phi^{(1)}(x,y)$ and b) $\bar\pi^{(2)}(x)$

7. Finite moments of $\bar\Pi(x)$: proof of Proposition 2.5

In this section we prove Proposition 2.5, which states that the $((2\wedge\alpha)+\delta)$'th moment of $|\Pi(x,r;F)|$ and of $|\Psi(x,r;F)|$ are finite for some $\delta>0$. Both claims follow once we show

$$\sum_{x\in\mathbb{Z}^d} |x|^{(2\wedge\alpha)+\delta}\bar\Pi(x) \le K \tag{7.1}$$

for $\bar\Pi(x)$ as defined in (6.86). In the course of the proof we derive certain quantities that are similar to quantities bounded by Chen and Sakai [10], and the proof of this bound is based in part on their proofs.

We assume $p=p_c$ throughout and suppress all subscripts p_c. We also omit the area of integration $[-\pi,\pi)^d$ below the integral signs, whenever it occurs.

Proof of Proposition 2.5. The proof is split up into three sections. In the first section we describe a way of distributing the weight $|x|^{(2\wedge\alpha)+\delta}$ over the path elements of the diagrams. The second section deals with taking the Fourier transform of lace expansion diagrams. In the third section we bound the elements of these Fourier space diagrams.

7.1. Distributing the weight

For $\alpha>0$ and $d>3(2\wedge\alpha)$ we choose ε and δ such that

$$0<\varepsilon<d/2(2\wedge\alpha)-3/2 \qquad\text{and}\qquad 0<\delta<\min\{\alpha,1,d/2(2\wedge\alpha)-3/2-\varepsilon\}. \tag{7.2}$$

Observe for any D that satisfies Assumption D this choice of δ implies that

$$\sum_x |x|^{(2\wedge\alpha)+\delta} D(x)\le C<\infty, \tag{7.3}$$

so that by (2.32),

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}|x|^{(2\wedge\alpha)+\delta}\psi^{(n)}(x,r;F) &\le \sum_{x\in\mathbb{Z}^d}\sum_{y\in Q_r}\sum_{n=0}^{\infty}|x-y|^{(2\wedge\alpha)+\delta}p_cD(x-y)|y|^{(2\wedge\alpha)+\delta}\pi^{(n)}(x,r;F)\\ &\le C\sum_{y\in Q_r}\sum_{n=0}^{\infty}|y|^{(2\wedge\alpha)+\delta}\bar\pi^{(n)}(y),\end{aligned} \tag{7.4}$$

so the finiteness of the $((2\wedge\alpha)+\delta)$-th moment of $\sum_n\psi^{(n)}(x,r;F)$ follows once we show that it holds for the $((2\wedge\alpha)+\delta)$-th moment of $\sum_n\bar\pi^{(n)}(x)$.

By the definition of $\bar{\Pi}(x)$,

$$\sum_{x\in\mathbb{Z}^d} |x|^{(2\wedge\alpha)+\delta}\bar{\Pi}(x) = \sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty} |x|^{(2\wedge\alpha)+\delta}\bar{\pi}^{(n)}(x). \tag{7.5}$$

For $x \in \mathbb{Z}^d$ we write $x = (x_1, x_2, \dots, x_d)$. Because the functions $\bar{\pi}^{(n)}(x)$ are invariant under the symmetries of $\mathbb{Z}^d$, we can bound (7.5) as follows:

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty} |x|^{(2\wedge\alpha)+\delta}\bar{\pi}^{(n)}(x) \le d^{((2\wedge\alpha)+\delta)/2+1}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty} |x_1|^{(2\wedge\alpha)+\delta}\bar{\pi}^{(n)}(x). \tag{7.6}$$

We can deal with this sum by distributing the weight $|x_1|^{(2\wedge\alpha)+\delta}$ along the top and bottom paths of the diagram. The first step is to rewrite (7.6) using the following identity: for $t > 0$ and $\zeta \in (0,2)$, let

$$K'_\zeta \equiv \int_0^\infty \frac{1-\cos(x)}{x^{1+\zeta}}\,\mathrm{d}x \in (0,\infty). \tag{7.7}$$

This gives the identity

$$t^\zeta = \frac{1}{K'_\zeta}\int_0^\infty \frac{1-\cos(st)}{s^{1+\zeta}}\,\mathrm{d}s. \tag{7.8}$$

For $u, v \in (0,\infty)$, define the d-dimensional vectors $\vec{u} = (u,0,\dots,0)$ and $\vec{v} = (v,0,\dots,0)$. Applying (7.8) twice to (7.6) with $\zeta = (2\wedge\alpha)-\varepsilon, \delta+\varepsilon$, we get

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty} &|x|^{(2\wedge\alpha)+\delta}\bar{\pi}^{(n)}(x) \\ &\le C\int_0^\infty \frac{\mathrm{d}u}{u^{1+(2\wedge\alpha)-\varepsilon}}\int_0^\infty \frac{\mathrm{d}v}{v^{1+\delta+\varepsilon}}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}[1-\cos(\vec{u}\cdot x)][1-\cos(\vec{v}\cdot x)]\bar{\pi}^{(n)}(x).\end{aligned} \tag{7.9}$$

The double integral can be split into four parts: $I_1 + I_2 + I_3 + I_4$, where

$$I_1 = O(1)\int_0^1 \frac{\mathrm{d}u}{u^{1+(2\wedge\alpha)-\varepsilon}}\int_0^1 \frac{\mathrm{d}v}{v^{1+\delta+\varepsilon}}\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}[1-\cos(\vec{u}\cdot x)][1-\cos(\vec{v}\cdot x)]\bar{\pi}^{(n)}(x) \tag{7.10}$$

and I_2, I_3 and I_4 are similarly defined but with different areas of integration A_i, $i = 2,3,4$, where

$$A_2 = [0,1]\times(1,\infty], \quad A_3 = (1,\infty]\times[0,1], \quad \text{and} \quad A_4 = (1,\infty]\times(1,\infty]. \tag{7.11}$$

It remains to show that $I_1,\dots,I_4$ are finite. To prove that this is so, we need an upper bound on

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}[1-\cos(\vec{u}\cdot x)][1-\cos(\vec{v}\cdot x)]\bar{\pi}^{(n)}(x). \tag{7.12}$$

Indeed, Proposition 2.5 follows once we show that there exists $\theta > \delta+\varepsilon$ such that

$$\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}[1-\cos(\vec{u}\cdot x)][1-\cos(\vec{v}\cdot x)]\bar{\pi}^{(n)}(x) = O\Big((u\wedge 1)^{(2\wedge\alpha)}(v\wedge 1)^{\theta}\Big) \tag{7.13}$$

The bounds are easy for u or v in $(1,\infty]$. In particular, $I_4 < \infty$ follows from the fact that $\sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}\bar{\pi}^{(n)}(x) \le C < \infty$ and $1-\cos(t) \le 2$.

The remainder of this section is devoted to proving (7.13) when both $u, v \in [0,1]$, that is, the bound needed for the finiteness of I_1. The bounds on I_2 and I_3 can be obtained in a similar, but much easier, way.

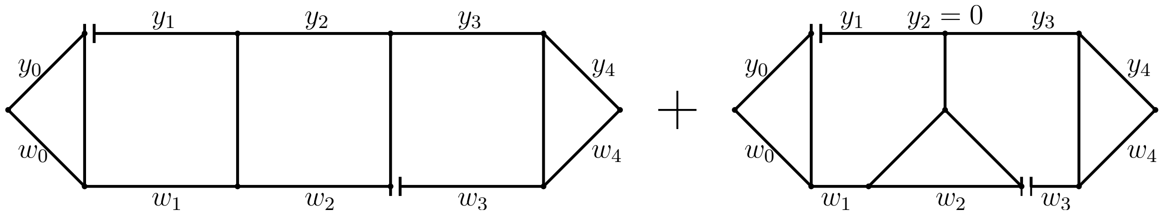

Figure 4. The path-elements of $\bar{\pi}^{(2)}(x)$ labeled according to the proposed scheme.

We start by only considering $n \geq 1$. The case $n = 0$ is much simpler, and we will comment on the bound for $n = 0$ when it is appropriate (around equation (7.50)). Using (6.82) we can rewrite the right-hand side of (7.9) (with the term for $n = 0$ omitted) as

$$C\int_0^1 \frac{\mathrm{d}u}{u^{1+(2\wedge\alpha)-\varepsilon}} \int_0^1 \frac{\mathrm{d}v}{v^{1+\delta+\varepsilon}} \sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty}[1-\cos(\vec{u}\cdot x)][1-\cos(\vec{v}\cdot x)] \sum_{\vec{s},\vec{t},\vec{z},\vec{l}} C(0,s_0,t_0)$$
$$\times \prod_{m=1}^{n-1}[B_1(s_{m-1},t_{m-1},z_m,l_m)B_2(z_m,l_m,s_m,t_m)]\, B_1(s_{n-1},t_{n-1},z_n,l_n)C(x,z_n,l_n). \quad (7.14)$$

Define for $i = 0,1,\dots,n$:

$$y_{2i} = \begin{cases} t_0 & \text{if } i=0;\\ t_i - z_i & \text{if } i \text{ is odd};\\ s_i - l_i & \text{if } i \text{ is even}; \end{cases} \qquad y_{2i+1} = \begin{cases} l_i - t_{i-1} & \text{if } i<n \text{ is odd};\\ z_i - s_{i-1} & \text{if } i<n \text{ is even}; \end{cases} \qquad y_{2n} = \begin{cases} x - z_n & \text{if } n \text{ is odd};\\ x - l_n & \text{if } n \text{ is even}; \end{cases} \quad (7.15)$$

$$w_{2i} = \begin{cases} s_0 & \text{if } i=0;\\ s_i - l_i & \text{if } i \text{ is odd};\\ t_i - z_i & \text{if } i \text{ is even}; \end{cases} \qquad w_{2i+1} = \begin{cases} t_i - z_{i-1} & \text{if } i<n \text{ is odd};\\ s_i - l_{i-1} & \text{if } i<n \text{ is even}; \end{cases} \qquad w_{2n} = \begin{cases} x - l_n & \text{if } n \text{ is odd};\\ x - z_n & \text{if } n \text{ is even}. \end{cases} \quad (7.16)$$

The y's and w's can be viewed as the path elements along the top and bottom of the diagram $\bar{\pi}^{(n)}$, respectively. An example is given in Figure 4.

The result is that we get two telescoping sums:

$$\sum_{i=0}^{2n} y_i = \sum_{i=0}^{2n} w_i = x. \quad (7.17)$$

By [7, (4.51)], for $a = \sum_{j=1}^{J} a_j$,

$$1 - \cos a \leq (2J+1)\sum_{j=1}^{J}[1-\cos a_j]. \quad (7.18)$$

Applying this with $a_i = \vec{u}\cdot y_i$ and $\vec{v}\cdot w_i$ gives that (7.14) is bounded from above by

$$C\int_0^1 \frac{\mathrm{d}u}{u^{1+(2\wedge\alpha)-\varepsilon}}\int_0^1 \frac{\mathrm{d}v}{v^{1+\delta+\varepsilon}}\sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty}(4n+3)^2\sum_{i,j=0}^{2n}\sum_{\vec{s},\vec{t},\vec{z},\vec{l}}[1-\cos(\vec{u}\cdot y_i)][1-\cos(\vec{v}\cdot w_j)]$$
$$\times C(0,s_0,t_0)\prod_{m=1}^{n-1}[B_1(s_{m-1},t_{m-1},z_m,l_m)B_2(z_m,l_m,s_m,t_m)]$$
$$\times B_1(s_{n-1},t_{n-1},z_n,l_n)C(x,z_n,l_n)$$
$$\equiv C\int_0^1 \frac{\mathrm{d}u}{u^{1+(2\wedge\alpha)-\varepsilon}}\int_0^1 \frac{\mathrm{d}v}{v^{1+\delta+\varepsilon}}\sum_{n=1}^{\infty}(4n+3)^2\sum_{i,j=0}^{2n}\mathcal{R}^{(n)}_{(i,j)}(\vec{u},\vec{v}). \quad (7.19)$$

Each of the $\mathcal{R}^{(n)}_{(i,j)}(\vec{u},\vec{v})$ is the sum of 2^{n-1} terms: one for each sequence of $B_2^{(0)}$ and $B_2^{(1)}$ diagrams possible. The possible sequences of $B_2^{(0)}$ and $B_2^{(1)}$ diagrams from left to right (say), corresponds one-to-one to the binary expansion of an integer between 0 and $2^{n-1}-1$, so we can write

$$\mathcal{R}^{(n)}_{(i,j)}(\vec{u},\vec{v}) \equiv \sum_{m=0}^{2^{n-1}-1}\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v}) \quad (7.20)$$

where each of the $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ corresponds to exactly one realization of a diagram. Furthermore, the diagrams are products of functions of two variables, the (possibly weighted) two-point functions. Hence, we can associate a graph to each of the $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ in such a way that the edges of the graph correspond to the two-variable functions of $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ and the vertices of the graph correspond to the variables in $\mathbb{Z}^d$ that are being summed over. This graph structure implies certain properties of the Fourier transform of the diagrams that are useful in getting upper bounds.

We use these properties to bound the diagrams in Fourier space. Our strategy is as follows: The first step is to use graph properties to write $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ as the integral over a function of $2n+1$ Fourier variables, rather than the $6n+2$ variables that we would get from taking the Fourier transform for each of the $6n+2$ two-point functions that are contained in $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ separately. Then, using a duality argument on the graph structure, we determine the order in which to integrate over these $2n+1$ variables (similar approaches exist for bounding Feynman diagrams in the quantum field theory literature, cf. [13], [14]). We show that when we choose the correct order of integration, we can integrate over the product of at most three functions of the same variable. Roughly speaking, this corresponds to integration over the triangle diagram in Fourier space. We assumed that this integral is bounded by a small constant in the statement of Proposition 2.5. This way we are able to show that $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ has an upper bound of the order of $\beta^{n-3}u^{(2\wedge\alpha)}v^{\theta}$ for some $\theta > \delta+\varepsilon$, and this is enough to show (7.13) and hence Proposition 2.5 holds.

7.2. Fourier space diagrams

We start by carrying out the above program with some general considerations. Let $\mathcal{V}$ be a finite set of vertices with $|\mathcal{V}| = V$ and let $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ with $|\mathcal{E}| = E$ be a set of unoriented edges (i.e., $\{i,j\} = \{j,i\}$). Below, we will assume that there is a fixed (but arbitrary) order to the elements of $\mathcal{V}$ and $\mathcal{E}$. The graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ plays the role of an index set for a diagram. We call a function $F : (\mathbb{Z}^d)^V \mapsto \mathbb{R}^+$ an *edge diagram* if it can be written as a product of functions on 'edges' as indexed by $\mathcal{E}$, i.e.,

$$F(x_1,\dots,x_V) = \prod_{\{i,j\}\in\mathcal{E}} f_{i,j}(x_i-x_j), \quad (7.21)$$

where $f_{i,j} : \mathbb{Z}^d \mapsto \mathbb{R}^+$. We call the edge diagram simple and connected, respectively, if the associated graph $\mathcal{G}$ is simple and connected. Define the *anchored sum of F* as

$$I_0 = \sum_{x_2 \in \mathbb{Z}^d} \cdots \sum_{x_V \in \mathbb{Z}^d} F(0, x_2, \ldots, x_V). \tag{7.22}$$

The upcoming lemma and its proof use certain elementary graph theoretic notions that we will briefly review here. It is a basic fact from graph theory that associated to every graph $\mathcal{G}$ there is a vector space $\mathcal{C}(\mathcal{G})$ whose elements represent formal combinations of cycles in $\mathcal{G}$. This vector space is known as the *cycle space of* $\mathcal{G}$. Given a spanning tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}')$ of $\mathcal{G}$, a *fundamental cycle of* $\mathcal{T}$ is defined as the single cycle in the graph $\mathcal{S} = (\mathcal{V}, \mathcal{E}' \cup e)$ for the edge $e \in \mathcal{E} \setminus \mathcal{E}'$. It is a well known result that a graph with V vertices and E edges has $E - V + 1$ cycles. For further definitions and a proof of the above statement we refer the reader to the literature of this field (e.g. [12]).

Lemma 7.1 [An integral representation for edge diagrams]. *Let $F(x_1, \ldots, x_V)$ be a translation invariant simple and connected edge diagram indexed by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with V vertices and E edges and let $F \in L^1(\mathbb{Z}^{d(V-1)})$. Then there exists an $E \times (E - V + 1)$ matrix M with rows indexed by the edges $\{i, j\} \in \mathcal{E}$ such that,*

$$I_0 = \int_{[-\pi,\pi)^d} \frac{\mathrm{d}^d p_1}{(2\pi)^d} \cdots \int_{[-\pi,\pi)^d} \frac{\mathrm{d}^d p_{E-V+1}}{(2\pi)^d} \prod_{\{i,j\} \in \mathcal{E}} \hat{f}_{i,j}\big((M \cdot \vec{p})_{\{i,j\}}\big), \tag{7.23}$$

where $\vec{p} = (p_1, \ldots, p_{E-V+1})^T$ and $p_i \in [-\pi, \pi)^d$ for all $i = 1, \ldots, E - V + 1$. Furthermore, the matrix M can be chosen in such a way that its columns correspond to a basis of the cycle space $\mathcal{C}(\mathcal{G})$ of $\mathcal{G}$.

Proof. Define $g_{i,j}(x_i, x_j) \equiv f_{i,j}(x_i - x_j)$. We start by examining the part of the Fourier transform of F that corresponds to the factor $g_{i,j}(x_i, x_j)$. When we express $g_{i,j}(x_i, x_j)$ in terms of its Fourier transform we get

$$g_{i,j}(x_i, x_j) = \int_{[\pi,\pi)^d} \frac{\mathrm{d}^d k_i}{(2\pi)^d} \int_{[\pi,\pi)^d} \frac{\mathrm{d}^d k_j}{(2\pi)^d} e^{i k_i \cdot x_i} e^{i k_j \cdot x_j} \hat{g}_{i,j}(k_i, k_j). \tag{7.24}$$

When we shift x_i and x_j by a vector $a \in \mathbb{Z}^d$ we get

$$\begin{aligned} g_{i,j}(x_i + a, x_j + a) &= \int_{[\pi,\pi)^d} \frac{\mathrm{d}^d k_i}{(2\pi)^d} \int_{[\pi,\pi)^d} \frac{\mathrm{d}^d k_j}{(2\pi)^d} e^{i k_i \cdot (x_i + a)} e^{i k_j \cdot (x_j + a)} \hat{g}_{i,j}(k_i, k_j) \\ &= \int_{[\pi,\pi)^d} \frac{\mathrm{d}^d k_i}{(2\pi)^d} \int_{[\pi,\pi)^d} \frac{\mathrm{d}^d k_j}{(2\pi)^d} e^{i k_i \cdot x_i} e^{i k_j \cdot x_j} e^{i (k_i + k_j) \cdot a} \hat{g}_{i,j}(k_i, k_j). \end{aligned} \tag{7.25}$$

By the definition of $g_{i,j}$ the left-hand sides of (7.24) and (7.25) are equal, and so the right-hand sides must also be equal. This is only the case for every $a \in \mathbb{Z}^d$ when

$$e^{i(k_i + k_j) \cdot a} = 1 \quad \text{or, equivalently,} \quad k_i + k_j = 0 \quad \text{for} \quad k_i, k_j \in [-\pi, \pi)^d. \tag{7.26}$$

It follows that when we take the Fourier transform of F we get such a constraint for every pair k_i, k_j for which $\{i, j\} \in \mathcal{E}$.

Let A denote the *incidence matrix* of $\mathcal{G}$, i.e., the $V \times E$ matrix with entries $(A)_{v,\{i,j\}} = 1$ if $v \in \{i, j\}$ and 0 otherwise. Then we can write the constraints on the variables of the Fourier transform of F as a system of linear equations in terms of A, so that in matrix notation we have $A^T \cdot \vec{k} = \vec{0}$, where $\vec{k}$ is a vector of length V with entries $k_i \in [-\pi, \pi)^d$, $i \in \mathcal{V}$. It is an elemental result from graph theory ([12, Proposition 1.9.7]), that the rank of A is equal to the dimension of $\mathcal{C}(\mathcal{G})$, the cycle space of $\mathcal{G}$. Another elemental result is that the dimension of $\mathcal{C}(\mathcal{G})$ is $E - V + 1$ ([12, Theorem 1.9.6]), so the

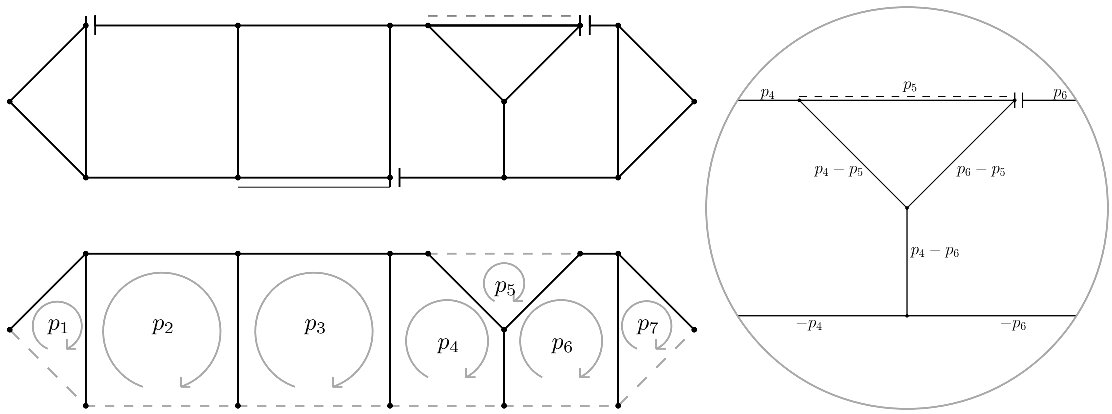

Figure 5. On the left the diagram $\mathcal{R}^{(3,2)}_{(5,3)}$ and its spanning tree, with associated loop momenta. The solid and the dashed line in the upper diagram represent the weighted paths. On the right a portion of the diagram with loop momenta associated to the lines.

rank of A is $E-V+1$. This implies that there are $E-V+1$ linearly independent Fourier variables associated to the Fourier transform of F.

Furthermore, the kernel of A is $\mathcal{C}(\mathcal{G})$ ([12, Proposition 1.9.7]), so we can express these linearly independent Fourier variables in terms of a basis of $\mathcal{C}(\mathcal{G})$. Therefore, there exists a matrix M such that (7.23) holds.

There are of course many realisations of the matrix M that satisfy (7.23). Since we can associate a basis of the cycle space with a spanning tree, the following procedure can be used to give a realisation for M:

Given a spanning tree $\mathcal{T}$ we can get a *pointed spanning tree* $\mathcal{T}_\uparrow$ by choosing an orientation for one of the edges of the tree. It is then a simple exercise to show that this can be used to give a unique orientation to all edges of $\mathcal{T}_\uparrow$ (e.g. edges are pointing away from the pointed edge), and moreover, we can do the same thing for all edges in $\mathcal{E}$. We write $\mathcal{E}_\uparrow$ for this set of oriented edges.

Furthermore, we can independently of this give the set of fundamental cycles its own orientation by giving an orientation to each element of $\mathcal{F}(\mathcal{T})$ independently. We call this oriented fundamental cycle set $\mathcal{F}_\downarrow(\mathcal{T})$. We assume below that $\mathcal{F}_\downarrow(\mathcal{T})$ has some fixed (but arbitrary) ordering. We can use these definitions to give a construction for the matrix M in terms of $\mathcal{E}_\uparrow$ and $\mathcal{F}_\downarrow(\mathcal{T})$. Given a pointed spanning tree $\mathcal{T}_\uparrow$, define the *cycle-adjacency matrix* $M_{\uparrow,\downarrow}$ as the $E\times E-V+1$ matrix whose rows are indexed by elements in $\mathcal{E}_\uparrow$ and whose columns are indexed by the elements of $\mathcal{F}_\downarrow(\mathcal{T})$, such that, for any edge $(i,j)\in\mathcal{E}_\uparrow$ and any oriented cycle in $\mathsf{c}\in\mathcal{F}_\downarrow(\mathcal{T})$,

$$\left(M_{\uparrow,\downarrow}\right)_{(i,j),\mathsf{c}} = \begin{cases} 1 & \text{if } (i,j)\in\mathsf{c}; \\ -1 & \text{if}(j,i)\in\mathsf{c}; \\ 0 & \text{otherwise.} \end{cases} \tag{7.27}$$

The columns of $M_{\uparrow,\downarrow}$ correspond to a (signed) basis of $\mathcal{C}(\mathcal{G})$ and the row (i,j) corresponds to a solution of $k_{\{i,j\}}$ in terms of $p_1,\dots,p_{E-V+1}\in[-\pi,\pi)^d$. Thus, $M_{\uparrow,\downarrow}$ is a matrix that satisfies (7.23) and this completes the proof. □

Recall definition (7.20). Every $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ is an edge diagram with one variable fixed at 0 and all other variables summed over, so we can write it as the anchored integral of an edge diagram,

$$\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v}) = \sum_x \sum_{\vec{z},\vec{t},\vec{s},\vec{l}} F^\iota(0,\vec{z},\vec{t},\vec{s},\vec{l},x) \tag{7.28}$$

where ι is a shorthand for the four indices n,m,i,j and the dependence of F^ι on $\vec{u}$ and $\vec{v}$ is implicit. Since $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})<\infty$, it follows that $F^\iota\in L^1(\mathbb{Z}^{d(4n+1)})$ so F^ι satisfies the assumptions of

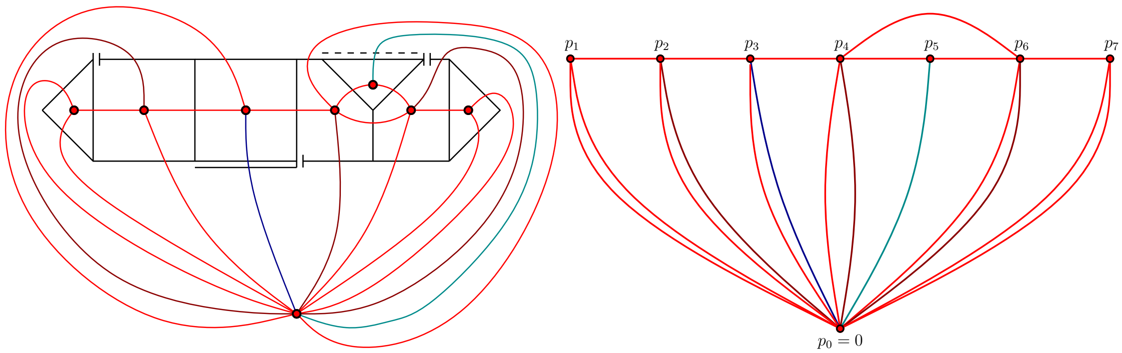

Figure 6. On the left the diagram $\mathcal{R}^{(3,2)}_{(5,3)}$ and its dual. On the right an isomorphism of the dual.

Lemma 7.1. Let $\mathcal{G}^\iota$ be the graph associated to $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$. From the construction of the diagrams it follows that all $\mathcal{G}^\iota$ are planar graphs. Furthermore, $\mathcal{G}^\iota$ has $6n+2$ edges and $4n+2$ vertices, so by Lemma 7.1, the Fourier transform of F^ι has $2n+1$ independent variables. We choose a spanning tree that separates all the internal faces of $\mathcal{G}^\iota$, so that the Fourier variables can be associated to loops along the $2n+1$ internal faces of the graph $\mathcal{G}^\iota$ (cf. Figure 5). Since there are $2n+2$ faces (also counting the external face) and $2n+1$ linearly independent variables, the Fourier variable associated to the external face can be set to zero. Furthermore, since we are free to choose the direction of the variables, we will always take the variables to run clockwise along a face. In the physics literature, such variables are commonly known as *loop momenta* and hence we use the same term.

A property of planar graphs is that each edge lies between exactly two faces (where the area on the 'outside' of the graph is also considered a face). Furthermore, it is a well-known fact from graph theory that each planar graph $\mathcal{G}$ has a unique dual (multi-)graph $\mathcal{G}^\star$ (up to isomorphisms) such that each vertex of $\mathcal{G}^\star$ can be associated to a face of $\mathcal{G}$, and each edge of $\mathcal{G}$ is crossed by exactly one dual edge of $\mathcal{G}^\star$ and vice versa. It follows that the degree of vertices in $\mathcal{G}^\star$ corresponds to the number of sides of the associated face in $\mathcal{G}$, and therefore, it corresponds to the number of separate occurrences of the associated loop momentum in the Fourier transform of the edge diagram that $\mathcal{G}$ indexes.

Hence, the dual graph $(\mathcal{G}^\iota)^\star$ indexes $\widehat{F^\iota}$, the Fourier transform of F^ι. The dual graph $(\mathcal{G}^\iota)^\star$ again has a very simple structure that allows us to write $\widehat{F^\iota}$ as the product of $2n+2$ simple elements. In Figure 6 we show an example of a diagram and its dual diagram.

The construction for Fourier-space diagrams that follows does not work for one particular subset of $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$, namely those where a weight is associated to a path element that is forced to be zero by the Kronecker delta in the definition of $B^{(1)}_2$, (6.70). These weights are an artifact of our notation: they are trivially zero. From here on, we assume that the weights lie on path elements that have a non-zero displacement.

Define

$$\mathcal{B}(p_a,p_b) = \hat{\tau}(p_a)\hat{\tau}(p_a-p_b)\hat{\tau}(p_b), \tag{7.29}$$

$$\widetilde{\mathcal{B}}(p_a,p_b) = \hat{D}(p_a)\hat{\tau}(p_a)\hat{\tau}(p_a-p_b)\hat{\tau}(p_b), \tag{7.30}$$

$$\mathcal{C}(p_a,p_b,p_c) = \hat{\tau}(p_a-p_b)\hat{\tau}(p_b)\hat{\tau}(p_b-p_c). \tag{7.31}$$

Also define the functions

$$\tau_q(x) = [1-\cos(q\cdot x)]\tau(x), \tag{7.32}$$

$$\widetilde{\tau}_q(x) = [1-\cos(q\cdot x)](D*\tau)(x), \tag{7.33}$$

$$\overline{\mathcal{D}}_q(p_i) = \begin{cases} \hat{\hat{\tau}}_q(p_i)/\hat{\hat{\tau}}(p_i) & \text{if } i/2 \text{ is an odd integer;} \\ \hat{\tau}_q(p_i)/\hat{\tau}(p_i) & \text{otherwise} \end{cases} \tag{7.34}$$

and

$$\underline{\mathcal{D}}_q(p_i) = \begin{cases} \hat{\hat{\tau}}_q(p_i)/\hat{\hat{\tau}}(p_i) & \text{if } i/2 \text{ is an even integer;} \\ \hat{\tau}_q(p_i)/\hat{\tau}(p_i) & \text{otherwise.} \end{cases} \tag{7.35}$$

Write m as a binary expansion, i.e., $m = m_{n-1}\cdots m_2 m_1$. Taking the Fourier transform of $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$, using the definitions (6.68) – (6.72), and rewriting the Fourier variables in terms of the loop momenta as described above, we get

$$\begin{aligned} \mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v}) &= \int \frac{\mathrm{d}^d p_1}{(2\pi)^d} \cdots \int \frac{\mathrm{d}^d p_{2n+1}}{(2\pi)^d} \hat{\tau}(p_1)\mathcal{B}(p_1,p_2) \\ &\times \Big[\prod_{\ell=1}^{n-1} \delta_{0,m_\ell} \widetilde{\mathcal{B}}(p_{2\ell},p_{2\ell+1})\mathcal{B}(p_{2\ell+1},p_{2\ell+2}) + \delta_{1,m_\ell}\widetilde{\mathcal{B}}(p_{2\ell},p_{2\ell+2})\mathcal{C}(p_{2\ell},p_{2\ell+1},p_{2\ell+2}) \Big] \\ &\times \widetilde{\mathcal{B}}(p_{2n},p_{2n+1})\hat{\tau}(p_{2n+1})\overline{\mathcal{D}}_{\vec{u}}(p_i)\underline{\mathcal{D}}_{\vec{v}}(p_j) \end{aligned} \tag{7.36}$$

where δ_{0,m_ℓ} and δ_{1,m_ℓ} are Kronecker deltas.

7.3. A recursive scheme for bounding $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$

The simple structure that $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ has in Fourier space allows us to recursively integrate over all the variables in such a way that all integrals converge. This is not necessarily obvious if we perform the integrals in some arbitrary order. Indeed, there may be as many as six functions of the same loop momentum, while we know that the integrals do not converge when there are more than three two-point functions present. Furthermore, the weights on p_i and p_j will make the integrals even more divergent (although the weight on p_i has a greater effect than the one on p_j). We can show that $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ is small despite these concerns by performing the integrals in the correct order.

One of the main tools we need for bounding $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u},\vec{v})$ is the following iterative version of Hölders inequality:

Lemma 7.2 [An application of Hölder's inequality]. *For any $n \geq 2$, let $\alpha_1,\dots,\alpha_n \in \mathbb{R}^+$. Let $S_n = \sum_{i=1}^n \alpha_i$. Let $f_1,\dots,f_n$ be L^{S_n}-integrable functions. Then*

$$\int \prod_{i=1}^n f_i(x)^{\alpha_i}\,\mathrm{d}x \leq \prod_{i=1}^n \left(\int f_i(x)^{S_n}\,\mathrm{d}x \right)^{\alpha_i/S_n}. \tag{7.37}$$

Proof. The proof is by induction over n. The case $n = 2$ follows directly from Hölder's inequality with conjugates S_2/α_1 and S_2/α_2. The inductive step is performed by applying Hölder's inequality with conjugates S_n/α_n and S_n/S_{n-1} to establish that the hypothesis holds for n if it holds for $n-1$. □

Note that for any function $f\colon \mathbb{Z}^d \mapsto \mathbb{R}$, its Fourier transform $\hat{f}(k)$ will be periodic with period 2π in all dimensions, and therefore, we have for any vector $\vec{q}$ and any $s \in \mathbb{R}$,

$$\int_{[-\pi,\pi)^d} \mathrm{d}^d k \hat{f}(k+\vec{q})^s = \int_{[-\pi,\pi)^d+\vec{q}} \mathrm{d}^d k' \hat{f}(k')^s = \int_{[-\pi,\pi)^d} \mathrm{d}^d k' \hat{f}(k')^s. \tag{7.38}$$

One of the bounds that the recursion is based on is

$$\begin{aligned}\int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\hat{\tau}(p_a)\mathcal{B}(p_a,p_b) &= \hat{\tau}(p_b)\int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\hat{\tau}(p_a)^2\hat{\tau}(p_a-p_b)\\ &\le \hat{\tau}(p_b)\left(\int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\hat{\tau}(p_a)^3\right)^{2/3}\left(\int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\hat{\tau}(p_a-p_b)^3\right)^{1/3} \\ &= \hat{\tau}(p_b)\int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\hat{\tau}(p_a)^3 \le \bar{\triangle}\hat{\tau}(p_b).\end{aligned} \tag{7.39}$$

where $\bar{\triangle}$ is given in (4.2). Note that the factor $\hat{\tau}(p_b)$ is unaffected in this bound and will carry through to the next bound. The first inequality follows from Lemma 7.2, the second equality follows from (7.38), and the second inequality is a consequence of the triangle condition. In a similar vein, but with a slightly longer calculation, it can be shown that

$$\int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\hat{\tau}(p_a)\widetilde{\mathcal{B}}(p_a,p_b) \le T\,\hat{\tau}(p_b) \tag{7.40}$$

where T is given in (4.3). Furthermore, it also follows from Lemma 7.2 and (7.38) that

$$\int \frac{\mathrm{d}^d p_b}{(2\pi)^d}\mathcal{C}(p_a,p_b,p_c) \le \bar{\triangle}. \tag{7.41}$$

From the bounds (7.39), (7.40) and (7.41) it is easy to see that we can perform the integrals over the Fourier variables that are not associated with a term $\overline{\mathcal{D}}_{\vec{u}}$ or $\underline{\mathcal{D}}_{\vec{v}}$ in (7.36) in such a way that we can bound every integral by either a factor T or a factor $\bar{\triangle}$.

We associate the terms $\overline{\mathcal{D}}_{\vec{u}}(p_i)$ and $\underline{\mathcal{D}}_{\vec{v}}(p_j)$ with the first term $\hat{\tau},\mathcal{B},\widetilde{\mathcal{B}}$ or $\mathcal{C}$ of the same variable, as seen when viewed from left to right in the Fourier diagram's construction in (7.36).

We assume for the moment that $i \neq j$. We sequentially integrate over all other Fourier variables from the left until we come to the ith or jth variable using the bounds (7.39), (7.40) and (7.41). Then we integrate from the right until we come to the other weighted variable, using the same bounds. Once all these variables are integrated over, the resulting expression either contains an integral of the form

$$\mathcal{X}(\vec{v}) \equiv \int \frac{\mathrm{d}^d p_j}{(2\pi)^d}\hat{\tau}(p_j)\underline{\mathcal{D}}_{\vec{v}}(p_j)\mathcal{B}(p_j,p_a), \tag{7.42}$$

$$\underline{\mathcal{X}}(\vec{v}) \equiv \int \frac{\mathrm{d}^d p_j}{(2\pi)^d}\hat{\hat{\tau}}(p_j)\underline{\mathcal{D}}_{\vec{v}}(p_j)\mathcal{B}(p_j,p_a) \tag{7.43}$$

(where the value of the second index depends on the structure of the diagram) or an integral of the form

$$\mathcal{X}'(\vec{v}) \equiv \int \frac{\mathrm{d}^d p_i}{(2\pi)^d}\mathcal{C}(p_{i-1},p_i,p_{i+1})\underline{\mathcal{D}}_{\vec{v}}(p_i). \tag{7.44}$$

It can be shown that there exists $\theta > \delta+\varepsilon$ such that

$$\mathcal{X}(\vec{v}) = O(v^\theta)\hat{\tau}(p_a), \qquad \underline{\mathcal{X}}(\vec{v}) = O(v^\theta)\hat{\tau}(p_a) \qquad \text{and} \qquad \mathcal{X}'(\vec{v}) = O(v^\theta). \tag{7.45}$$

Assume that these bounds hold (we will discuss this assumption below). We continue integrating over the Fourier variables that lie between p_i and p_j until, for $i \neq j$, we end up with either of the following integrals:

$$\mathcal{Y}(\vec{u}) \equiv \int \frac{\mathrm{d}^d p_i}{(2\pi)^d}\int \frac{\mathrm{d}^d p_a}{(2\pi)^d}\hat{\tau}(p_i)\overline{\mathcal{D}}_{\vec{u}}(p_i)\mathcal{B}(p_i,p_a)\hat{\tau}(p_a), \tag{7.46}$$

the integral $\overline{\mathcal{Y}}(\vec{u})$, which we define to be the same integral but with $\hat{\tau}(p_i)$ replaced by $\hat{\widetilde{\tau}}(p_i)$, or

$$\mathcal{Y}'(\vec{u}) \equiv \int \frac{\mathrm{d}^d p_{i-1}}{(2\pi)^d} \int \frac{\mathrm{d}^d p_i}{(2\pi)^d} \int \frac{\mathrm{d}^d p_{i+1}}{(2\pi)^d} \hat{\tau}(p_{i-1}) \mathcal{B}(p_{i-1}, p_{i+1}) \mathcal{C}(p_{i-1}, p_i, p_{i+1}) \hat{\tau}(p_{i+1}) \overline{\mathcal{D}}_{\vec{u}}(p_i). \quad (7.47)$$

Indeed,

$$\mathcal{Y}(\vec{u}) = O(u^{(2\wedge\alpha)}), \qquad \overline{\mathcal{Y}}(\vec{u}) = O(u^{(2\wedge\alpha)}) \qquad \text{and} \qquad \mathcal{Y}'(\vec{u}) = T\bar{\triangle} O(u^{(2\wedge\alpha)}), \quad (7.48)$$

as we discuss below.

When $i = j$ we integrate over variables from the left and the right until we get

$$\mathcal{Z}(\vec{u}, \vec{v}) \equiv \int \frac{\mathrm{d}^d p_i}{(2\pi)^d} \hat{\tau}(p_i)^2 \overline{\mathcal{D}}_{\vec{u}}(p_i) \underline{\mathcal{D}}_{\vec{v}}(p_i) = \int \frac{\mathrm{d}^d p_i}{(2\pi)^d} \hat{\tau}_{\vec{u}}(p_i) \hat{\tau}_{\vec{v}}(p_i), \quad (7.49)$$

the integral $\underline{\mathcal{Z}}(\vec{u}, \vec{v})$, or the integral $\overline{\mathcal{Z}}(\vec{u}, \vec{v})$, depending on the value of i, where $\underline{\mathcal{Z}}$ and $\overline{\mathcal{Z}}$ follow the same definition as $\mathcal{Z}$, but with $\hat{\tau}_{\vec{v}}$ and $\hat{\tau}_{\vec{u}}$ replaced by $\hat{\widetilde{\tau}}_{\vec{v}}$ and $\hat{\widetilde{\tau}}_{\vec{u}}$, respectively. (Note that we do not treat the case $i = j = 2\ell + 1$ when the ℓ-th factor contains $\widetilde{\mathcal{B}}(p_{2\ell}, p_{2\ell+2}) \times \mathcal{C}(p_{2\ell}, p_{2\ell+1}, p_{2\ell+2})$, since these cases do not occur in nonzero diagrams $\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u}, \vec{v})$. This is the artifact of our notation that we mentioned earlier.)

This is the right time to mention the case $n = 0$, because then, by (6.47) and the Fourier techniques described above we can write

$$\sum_{x \in \mathbb{Z}^d} [1 - \cos(\vec{u} \cdot x)][1 - \cos(\vec{v} \cdot x)] \bar{\pi}^{(0)}(x) = \mathcal{Z}(\vec{u}, \vec{v}). \quad (7.50)$$

We will show below that there exists a $\delta + \varepsilon < \theta < d/(2 \wedge \alpha) - 3$ such that

$$\mathcal{Z}(\vec{u}, \vec{v}) = O(u^{(2\wedge\alpha)} v^{\theta}). \quad (7.51)$$

Very similar proofs can be given for the following bounds:

$$\underline{\mathcal{Z}}(\vec{u}, \vec{v}) = O(u^{(2\wedge\alpha)} v^{\theta}) \qquad \text{and} \qquad \overline{\mathcal{Z}}(\vec{u}, \vec{v}) = O(u^{(2\wedge\alpha)} v^{\theta}). \quad (7.52)$$

When the bounds (7.45), (7.48), (7.51) and (7.52) hold, it follows that

$$\mathcal{R}^{(n,m)}_{(i,j)}(\vec{u}, \vec{v}) \le T^{n-3} \bar{\triangle}^{n+1} O(u^{(2\wedge\alpha)} v^{\theta}), \quad (7.53)$$

and therefore

$$\mathcal{R}^{(n)}_{(i,j)}(\vec{u}, \vec{v}) = \sum_{m=0}^{2^{n-1}} \mathcal{R}^{(n,m)}_{(i,j)}(\vec{u}, \vec{v}) \le (2^{n-1} + 1) T^{n-3} \bar{\triangle}^{n+1} O(u^{(2\wedge\alpha)} v^{\theta}) \quad (7.54)$$

and finally, by (7.19) and Lemma 4.1,

$$\sum_{x \in \mathbb{Z}^d} \sum_{n=0}^{\infty} [1 - \cos(\vec{u} \cdot x)][1 - \cos(\vec{v} \cdot x)] \bar{\pi}^{(n)}(x) \le \sum_{n=0}^{\infty} (4n+3)^2 \mathcal{R}^{(n)}_{(i,j)}(\vec{u}, \vec{v}) = O(u^{(2\wedge\alpha)} v^{\theta}) \quad (7.55)$$

when β is sufficiently small, as we set out to prove.

We complete the proof by establishing (7.51) and the third bound in (7.48). The two other bounds in (7.48) and those in (7.45) can be obtained similarly.

Before we start with the proof of (7.51), we briefly explain how to deal with factors $\hat{\widetilde{\tau}}_q(k)$ when they appear. Define

$$D_q(x) = [1 - \cos(q \cdot x)] D(x). \quad (7.56)$$

Recall the definition of $\widetilde{\tau}$, (7.33). We begin by distributing the weight once more, now over D and τ:

$$\begin{aligned}
[1 - \cos(q \cdot x)](D * \tau)(x) &= \sum_{y \in \mathbb{Z}^d} [1 - \cos(q \cdot x)] D(y) \tau(x - y) \\
&\le 5 \sum_{y \in \mathbb{Z}^d} ([1 - \cos(q \cdot y)] + [1 - \cos(q \cdot (x - y))]) D(y) \tau(x - y) \qquad (7.57) \\
&= 5(D_q * \tau)(x) + 5(D * \tau_q)(x)
\end{aligned}$$

where we used (7.18) for the inequality. The Fourier transform of $(D_q * \tau)(x)$ can be bounded as follows:

$$\begin{aligned}\widehat{(D_q * \tau)}(k) = \hat{D}_q(k)\hat{\tau}(k) &= \left(\sum_{x\in\mathbb{Z}^d} \cos(k\cdot x)[1-\cos(q\cdot x)]D(x)\right)\hat{\tau}(k) \\ &\le \left(\sum_{x\in\mathbb{Z}^d}[1-\cos(q\cdot x)]D(x)\right)\hat{\tau}(k) = [1-\hat{D}(q)]\hat{\tau}(k) = O(q^{(2\wedge\alpha)})\hat{\tau}(k).\end{aligned} \tag{7.58}$$

For the second term of (7.57), we observe that $\hat{D}(k) \le 1$ and $\hat{\tau}_q(k) \ge 0$, both uniformly in k, so

$$\widehat{(D * \tau_q)}(k) = \hat{D}(k)\hat{\tau}_q(k) \le \hat{\tau}_q(k). \tag{7.59}$$

Hence, we can bound

$$\hat{\hat{\tau}}_q(k) \le O(q^{(2\wedge\alpha)})\hat{\tau}(k) + 5\hat{\tau}_q(k). \tag{7.60}$$

Applying this bound whenever a weighted factor $\hat{\hat{\tau}}$ occurs, we can use the bounds on weighted and unweighted factors $\hat{\tau}$ for an upper bound.

Now we give the full proof of (7.51). Recall definition (7.32). From symmetry of the cosine it follows that that

$$\hat{\tau}_q(k) = \hat{\tau}(k) - \tfrac{1}{2}\hat{\tau}(k-q) - \tfrac{1}{2}\hat{\tau}(k+q) = -\tfrac{1}{2}\Delta_q\hat{\tau}(k), \tag{7.61}$$

where Δ_q is the discrete Laplacian operator with shift q. Therefore, using (7.61) we get

$$\mathcal{Z}(\vec{u},\vec{v}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\hat{\tau}_{\vec{u}}(k)\hat{\tau}_{\vec{v}}(k) \le \int \left|\tfrac{1}{2}\Delta_{\vec{u}}\hat{\tau}(k)\right|\left|\tfrac{1}{2}\Delta_{\vec{v}}\hat{\tau}(k)\right| \frac{\mathrm{d}^d k}{(2\pi)^d}. \tag{7.62}$$

Define

$$\hat{C}(k) = \frac{1}{1-\hat{D}(k)}. \tag{7.63}$$

By (4.6) it follows that

$$\hat{\tau}(k) = O(1)\hat{C}(k). \tag{7.64}$$

Hence,

$$\left|\tfrac{1}{2}\Delta_q\hat{\tau}(k)\right| \le O(1)(\hat{C}(k-q)+\hat{C}(k)+\hat{C}(k+q)). \tag{7.65}$$

Define

$$\mathcal{U}(q,k) = \frac{1}{\hat{C}(q)}\left\{\hat{C}(k-q)\hat{C}(k)+\hat{C}(k)\hat{C}(k+q)+\hat{C}(k-q)\hat{C}(k+q)\right\}. \tag{7.66}$$

From [23, (2.19) and Proposition 2.6] we also have the following bound:

$$\left|\tfrac{1}{2}\Delta_q\hat{\tau}(k)\right| \le O(1)\mathcal{U}(q,k). \tag{7.67}$$

We can interpolate the bounds (7.65) and (7.67) for $\theta \in (0,1)$:

$$\left|\tfrac{1}{2}\Delta_q\hat{\tau}(k)\right| \le O(1)\mathcal{U}(q,k)^\theta[\hat{C}(k-q)+\hat{C}(k)+\hat{C}(k+q)]^{1-\theta}. \tag{7.68}$$

Now we apply (7.67) to $|\tfrac{1}{2}\Delta_{\vec{u}}\hat{\tau}(k)|$ and (7.68) with $\delta+\varepsilon<\theta<d/(2\wedge\alpha)-3$ to $|\tfrac{1}{2}\Delta_{\vec{v}}\hat{\tau}(k)|$ in (7.62). This gives

$$\begin{aligned}\mathcal{Z}(\vec{u},\vec{v}) \le\ & O(1)[1-\hat{D}(\vec{u})][1-\hat{D}(\vec{v})]^\theta \int \mathrm{d}^d k[\hat{C}(k-\vec{u})\hat{C}(k)+\hat{C}(k)\hat{C}(k+\vec{u})+\hat{C}(k-\vec{u})\hat{C}(k+\vec{u})] \\ &\times[\hat{C}(k-\vec{v})+\hat{C}(k)+\hat{C}(k+\vec{v})]^{1-\theta} \\ &\times[\hat{C}(k-\vec{v})\hat{C}(k)+\hat{C}(k)\hat{C}(k+\vec{v})+\hat{C}(k-\vec{v})\hat{C}(k+\vec{v})]^\theta \\ \le\ & O(1)[1-\hat{D}(\vec{u})][1-\hat{D}(\vec{v})]^\theta \int \mathrm{d}^d k[\hat{C}(k-\vec{u})\hat{C}(k)+\hat{C}(k)\hat{C}(k+\vec{u})+\hat{C}(k-\vec{u})\hat{C}(k+\vec{u})] \\ &\times[\hat{C}(k-\vec{v})^{1-\theta}+\hat{C}(k)^{1-\theta}+\hat{C}(k+\vec{v})^{1-\theta}] \\ &\times[\hat{C}(k-\vec{v})^\theta\hat{C}(k)^\theta+\hat{C}(k)^\theta\hat{C}(k+\vec{v})^\theta+\hat{C}(k-\vec{v})^\theta\hat{C}(k+\vec{v})^\theta]\end{aligned} \tag{7.69}$$

where we used for the second inequality that $(x+y)^\theta \le x^\theta + y^\theta$ for $x, y \ge 0$ and $\theta \in (0,1)$. The integral contains 27 distinct product-terms of the function $\hat{C}$ with different shifts and different powers. One term, for instance, is $\hat{C}(k-\vec{u})\hat{C}(k)^{2-\theta}\hat{C}(k-\vec{v})^\theta\hat{C}(k+\vec{v})^\theta$. To generalize the structure of these terms, we write

$$\int \hat{C}(k-\vec{u})^{a_1}\hat{C}(k+\vec{u})^{a_2}\hat{C}(k-\vec{v})^{b_1}\hat{C}(k+\vec{v})^{b_2}\hat{C}(k)^{c_1+c_2}\,\mathrm{d}^d k. \tag{7.70}$$

Here, c_1 is the exponent due to the bound on $\tau_{\vec{u}}$, whereas c_2 is due to the bound on $\tau_{\vec{v}}$. Note that for every term the following relation holds for the exponents:

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 3 + \theta. \tag{7.71}$$

Applying Lemma 7.2 to (7.70) we get the upper bound

$$\begin{aligned}&\left(\int \hat{C}(k-\vec{u})^{3+\theta}\,\mathrm{d}^d k\right)^{\frac{a_1}{3+\theta}}\left(\int \hat{C}(k+\vec{u})^{3+\theta}\,\mathrm{d}^d k\right)^{\frac{a_2}{3+\theta}}\\ &\qquad\times\left(\int \hat{C}(k-\vec{v})^{3+\theta}\,\mathrm{d}^d k\right)^{\frac{b_1}{3+\theta}}\left(\int \hat{C}(k+\vec{v})^{3+\theta}\,\mathrm{d}^d k\right)^{\frac{b_2}{3+\theta}}\left(\int \hat{C}(k)^{3+\theta}\,\mathrm{d}^d k\right)^{\frac{c_1+c_2}{3+\theta}}.\end{aligned} \tag{7.72}$$

Using (7.38) and (7.71) we get the following bound on (7.69)

$$\int \hat{C}(k)^{3+\theta}\,\mathrm{d}^d k \le C < \infty, \tag{7.73}$$

where the boundedness of the integral follows from Assumption D and the choice $\theta < d/(2\wedge\alpha) - 3$. Plugging this bound into (7.69) and using Assumption D again, we get

$$\mathcal{Z}(\vec{u},\vec{v}) = O(1)[1-\hat{D}(\vec{u})][1-\hat{D}(\vec{v})]^\theta = O\left(u^{(2\wedge\alpha)}v^\theta\right), \tag{7.74}$$

establishing (7.51).

The Fourier space diagram corresponding to the integrated function in $\mathcal{Y}'(\vec{u})$ has two vertices of degree four and only one vertex of degree three, which, unfortunately, is the weighted vertex. As we saw while bounding $\mathcal{Z}$, the integral associated to the weighted vertex is only just convergent for d near the critical dimension when it is of degree two. The other two vertices correspond to integrals that are divergent near the critical dimension.

But the diagram has three integrated variables and eight functions, so we should be able to bound it by two triangles and a weighted bubble. To see this, we need to bound the integral by something simpler before we evaluate it. We use the Cauchy-Schwarz inequality for this. Roughly speaking, using the Cauchy-Schwarz inequality and the symmetry of the integral under relabeling allows us to bound the diagram by the same diagram with one factor $\hat{\tau}(p_{i-1})$ replaced by a factor $\hat{\tau}(p_{i+1})$. See Figure 7 for an illustration of this.

Applying the bound described above, and by positivity of the $\hat{\tau}$-functions, we get

$$\begin{aligned}\mathcal{Y}'(\vec{u}) &\le \int\frac{\mathrm{d}^d p_{i-1}}{(2\pi)^d}\int\frac{\mathrm{d}^d p_i}{(2\pi)^d}\int\frac{\mathrm{d}^d p_{i+1}}{(2\pi)^d}\widetilde{\mathcal{B}}(p_{i-1},p_{i+1})\mathcal{C}(p_{i-1},p_i,p_{i+1})\hat{\tau}(p_{i+1})^2\overline{\mathcal{D}}_{\vec{u}}(p_i)\\ &= \int\frac{\mathrm{d}^d p_i}{(2\pi)^d}\int\frac{\mathrm{d}^d p_{i+1}}{(2\pi)^d}\hat{\tau}_{\vec{u}}(p_i)\hat{\tau}(p_i-p_{i+1})\hat{\tau}(p_{i+1})^3\\ &\qquad\times\int\frac{\mathrm{d}^d p_{i-1}}{(2\pi)^d}\hat{D}(p_{i-1})\hat{\tau}(p_{i-1})\hat{\tau}(p_{i-1}-p_i)\hat{\tau}(p_{i-1}-p_{i+1})\\ &\le T\int\frac{\mathrm{d}^d p_{i+1}}{(2\pi)^d}\hat{\tau}(p_{i+1})^3\int\frac{\mathrm{d}^d p_i}{(2\pi)^d}\hat{\tau}_{\vec{u}}(p_i)\hat{\tau}(p_i-p_{i+1})\\ &\le TO(u^{(2\wedge\alpha)})\int\frac{\mathrm{d}^d p_{i+1}}{(2\pi)^d}\hat{\tau}(p_{i+1})^3 \le T\bar{\triangle}O(u^{(2\wedge\alpha)}) \le T\bar{\triangle}O(u^{(2\wedge\alpha)}).\end{aligned} \tag{7.75}$$

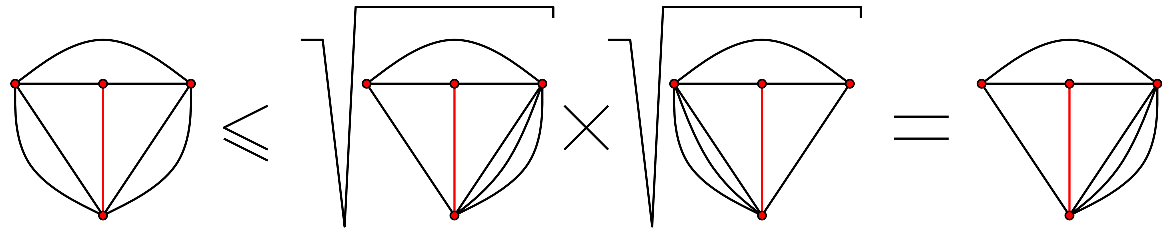

Figure 7. A graphic representation of the bound on $\mathcal{Y}'(\vec{u})$. The red (vertical) line corresponds to the weighted edge.

The second, third, and fourth inequality follow from a calculation similar to (7.39) and $\mathcal{Z}(\vec{u},\vec{v})$. The final bound is just there to fit the statement of (7.13). This completes the proof of Proposition 2.5. □

7.4. About the proof of Lemma 6.1

The equality (6.26) in Lemma 6.1 follows immediately from (6.44), (6.84), (6.86) and the proof of Proposition 2.5.

The equality (6.25) in Lemma 6.1 can be proved in the same way as Proposition 2.5, but with much less bookkeeping, so we do not give it. Heuristically, we can understand that the claim is true by noting that the diagrams $\bar\phi^{(n)}(x,y)$ are like $\bar\pi^{(n)}(x)$ diagrams with an extra point y placed on either the last or second-to-last upper path element (cf. Figure 3). From (7.67) it can be seen that in Fourier space, adding a point to a path element has roughly the same effect as having a 'heavy' weight on that path element (i.e. the factor $1-\cos(\vec{u}\cdot x)$ in the above analysis). Indeed, using (7.65) twice, we can bound

$$\widehat{(\tau*\tau)}(k) \le O(1)\hat{C}(k)^2, \tag{7.76}$$

which is similar to the right-hand side of (7.67). Therefore, the diagrams $\bar\phi^{(n)}(x,y)$ with the small weight $|x-y|^\delta$ can be bounded in a similar way as the diagrams $\bar\pi^{(n)}(x)$ with the weight $|x|^{(2\wedge\alpha)+\delta}$, and hence the bounds should also be similar. Following the proof of Proposition 2.5 confirms that this is the case. In the course of this proof, (6.24) in Lemma 6.1 also follows naturally.

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