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Recent investigations about the radical of a ring

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Recent investigations about the radical of a ring [1]).

The concept of radical is used in the structure theory of linear associative algebras by Wedderburn in 1908 [1] [2]). The join of all nilpotent ideals of an algebra (lin. ass.) H is a nilpotent ideal R and is called the radical of H. We write R = Rad(H). As Rad(R) = R and Rad(H/R)= = 0 hold, the structure problem is in a certain sense reduced to the same problem for those algebras in which the radical coincides with the whole algebra and those with zero radical.

In 1928 Artin [2] proved the Wedderburn structure theory to be valid for rings satisfying both chain conditions for left ideals. He actually proved it for rings with operator domain and chain conditions for admissable left ideals, thus including the Wedderburn theory as a special case.

In trying to weaken the conditions of Artin in 1938 [9] Hopkins showed the maximum condition to be superfluous, thus enlarging the scope of the theory by now grown classical. Later on (1942-1944) Brauer [13] and Levitzki [14,15] perfected Hopkins' original proof.

On the other hand the investigations aimed to develop a structure theory for general rings if necessary different from that of Artin. So new kinds of radicals appeared, the classical one not being very useful in the general case, e.g. it needs not to be a nilpotent ideal.

A radical has to satisfy the following requirements:

i It has to be an ideal.

ii The residue class ring modulo the radical must have zero radical according to the same radical definition.

For sake of beauty a third condition is often required.

iii In case the descending chain condition holds (for left ideals) the radical has to coincide with the classical one.

Koethe [4] was the first (1930) who defined a new radical (K-radical) as the set union of all nil ideals, provided it includes all nil left ideals (and then also all nil right ideals). It is an interesting open

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1) The contents of this report cover a lecture delivered at Amsterdam 28-III-'53. It is a sequel to rapport ZW 1950-013: W. Peremans, Het radicaal van een ring.

2) The numbers in the square brackets refer to the bibliography at end of the paper.

problem whether the K-radical always exists.

Fitting [6] in 1935 defined the F-radical as the set of all properly nilpotent elements. Here an element a is properly nilpotent if the ideal generated by a is a nil ideal.

According to Baer [17] an ideal I in a ring B is called a radical ideal if I is a nil ideal such that B/I contains no nonzero nilpotent ideal. The join of all radical ideals is a radical ideal and is called the upper radical of Baer. The intersection of all radical ideals is also a radical ideal and is called Baer's lower radical. It can also be defined as the intersection of all ideals having a residue class ring without nonzero nilpotent ideals. In case the upper and the lower radical coincide there exists only one radical ideal called the Baer radical (B-radical). It does not need to exist.

In 1943 Levitzki [14,15] introduced a radical by means of the concept of semi-nilpotency. He calls a ring (ideal, left ideal) semi-nilpotent if every subring generated by a finite number of elements is nilpotent. The set union of all semi-nilpotent ideals appears to be a semi-nilpotent ideal, containing all semi-nilpotent one-sided ideals. It is a radical ideal known as the L-radical.

All the radicals listed above were related to the concept of nilpotency. We shall go on with some radicals and "ideals with radical-like properties" all based on some kind of regularity. An element a of a ring B is called regular if there exists an element $x \in B$ such that $axa = a$, or what amounts to the same if $a \in \{axa\}$, if x ranges over B. It was von Neumann [7] who introduced the concept of regular ring, i.e. a ring with unit element in which every element is regular. In the following we shall call a subset regular (or strongly-regular, F-regular, quasi-regular and so on) if each of its elements is so. Brown and Mc Coy [29] proved the existence of a unique maximal regular ideal, being the join of all regular ideals of a ring B. It will be denoted by M(B). It has the following properties:

i $M(B/M(B)) = 0$.

ii If C is an ideal in B, then $M(C) = C \cap M(B)$, and

iii If B_n is the complete matrix ring over B, $M(B_n) = (M(B))_n$.

Jacobson [20] (see also [24]) uses right quasi-regularity. An element z of a ring B is said to be right quasi-regular with a right quasi-inverse $z' \in B$ if $z + z' - zz' = 0$. In case the ring has a unit element, $z \in B$ has a right quasi-inverse z' if and only if $1-z'$ is a right inverse of $1-z$.

A necessary and sufficient condition for the right quasi-regularity of an element of $z \in B$ is that the right ideal $J = \{zx - x\}$ where x ranges over B, equals B. If $J = B$ clearly $z = zz' - z'$ for some $z' \in B$. If z is right quasi-regular $z \in J$ and since J is a right ideal $zx \in J$, hence $x \in J$ for all $x \in B$. The Jacobson radical is the join of all right quasi-regular right ideals; it is a right quasi-regular two-sided ideal. Since a nilpotent element z with $z^n = 0$ has a right quasi-inverse $z' = -\sum_{i=1}^{n-1} z^i$, every nil ideal is contained in the J-radical. The J-radical in a regular ring vanishes. Suppose r is in the radical. By the regularity $rsr = r$ for some $s \in B$. Because the radical is a quasi-regular ideal sr has a quasi-inverse t. Now we have $0 = (r-rsr) + (rsr-r)t = r - r(sr+t-srt) = r$.

Brown and Mc Coy [23,25] generalize the concept of quasi-regularity as follows. If G(a) denotes the ideal generated by $J = \{ax-x\}$, an element $a \in B$ is called G-regular if $a \in G(a)$. The Brown-Mc Coy radical (B-M-radical) is the join of all G-regular ideals. Or more generally let F be a mapping of $a \in B$ on an ideal F(a) of B defined for each ring and such that any homomorphism $a \to \bar{a}$ satisfies $F(\bar{a}) = \overline{F(a)}$. The mapping $a \to G(a)$ is a special case. Call a F-regular if $a \in F(a)$ and the join of all F-regular ideals again is a radical.

In 1950 Brown and Mc Coy [30] gave a unified treatment of several kinds of regularity and of the J- and B-M-radicals. We shall consider this approach more closely. Let G be a group (in additive notation however not necessarily commutative) and A the set of its inner automorphisms and let Ω be a fixed set of endomorphisms of G such that $A \subseteq \Omega$. Moreover let F be a mapping of elements of G on subgroups F(a) of G, satisfying the following postulates P_1: $F(a+b) \subseteq F(a) + (b)$ (where (b) is the Ω-subgroup generated by b) and P_2: If $b \in F(a)$ then $F(a+b) \subseteq F(a)$. Under these conditions G is called an (F,Ω)-group. In the case F(a) is an Ω-subgroup for all $a \in G$ P_1 implies P_2.

An element a of an (F,Ω)-group is called F-regular if $a \in F(a)$. If we call $a \in G$ properly F-regular if (a) is F-regular, we can state the following theorem:

Theorem A. The set N of all properly F-regular elements of an (F,Ω)-group G is an F-regular Ω-subgroup of G containing every F-regular Ω-subgroup of G. The proof runs as follows: Let a and b be elements of N. If $x \in (a-b)$ then $x = u-v$ with $u \in (a)$ and $v \in (b)$ (since $(a+b) \subseteq (a)+(b)$). Because a is properly F-regular $u = (x+v) \in F(x+v) \subseteq F(x)+(v)$ so $u = -y+v'$ for some $y \in F(x)$ and $v' \in (v)$. Now $x = -y+v'-v = v''-y$ with $v'' \in (v)$ ((v) is a normal subgroup since $A \subseteq \Omega$, hence $-y+(v) = (v)-y$). Because of the regularity of (v) $v'' = x+y \in F(x+y) \subseteq F(x)$ (by P_2 since $y \in F(x)$). Hence $x \in F(x)$ and thus $a-b \in N$. Now let $\alpha \in \Omega$ be an endomorphism of G and $x \in N$. Then $(\alpha x) \subseteq (x)$, hence $\alpha x \in N$. So N is an Ω-subgroup of G and of course an F-regular one. If M is any F-regular Ω-subgroup of G and $b \in M$, also $(b) \in M$ and hence $b \in N$. So we have $M \subseteq N$.

In order to make the difference group of G modulo an Ω-subgroup ... an (F,Ω)-group the following theorem is used:

Theorem B. If $a \to a^*$ is an Ω-homomorphism of an (F,Ω)-group G on an Ω-group. Then it is possible to define $F(a^*) = [F(a)]^*$ such that G^* becomes an (F,Ω)-group.

Proof. In the first place $a^* = b^*$ implies $F(a^*) = F(b^*)$. Since, if K is the kernel of the Ω-homomorphism, $a = b+k$ for some $k \in K$, $F(a) = F(b+k) \subseteq F(b)+(k)$ and $[F(a)]^* = [F(b)]^*$. The postulate P_1 is satisfied because $F(a^*+b^*) = F([a+b]^*) = [F(a+b)]^* \subseteq [F(a)+(b)]^* = [F(a)]^* + (b)^* = F(a^*)+(b^*)$.

As for P_2, if $b^* \in F(a^*) = [F(a)]^*$ then $b = c+1$ for some $1 \in K$ and $c \in F(a)$. Hence $F(a^*+b^*) = F(a^*+c^*) = F(a+c)^* \subseteq F(a)^* = F(a^*)$. Let the set N of theorem A be denoted by N(G). Then the difference group G–N(G) is according to theorem B an (F,Ω)-group and N(G–N(G)) has a meaning.

Theorem C. N(G–N(G)) = 0.

Proof. If $b^* \in N(G-N(G))$ (b^* denotes the coset b+N(G)) and $a \in (b^*)$ then $a^* \in (b^*)$. Because (b^*) is F-regular $a^* \in F(a^*) = [F(a)]^*$ and $a+c \in N(G)$ for some $c \in F(a)$ whence $a+c \in F(a+c) \subseteq F(a)$ (by P_2). Since F(a) is a subgroup $a \in F(a)$ and (b) is F-regular. Therefore $b^* = 0$.

Now let B be an arbitrary ring and G its additive group and let Ω consist of all the right and left multiplications and the identity endomorphism. As for the mapping F we shall make several choices. In most cases one easily verifies that P_1 and P_2 are valid.

i $a \to \{ax-x\}$ thus establishing the existence (and some properties) of the J-radical.

ii $a \to \{ax-x+\sum^i[x_i a y_i - x_i y_i]\}$ leading to the B-M-radical.

iii $a \to F(a)$ where F(a) is the ideal F(a) of page 3. Here F(a) is an Ω-subgroup (ideal) hence we have only to verify P_1. Consider the Ω-homomorphism $a \to \bar{a} = a+(b)$ then $F(\overline{a+b}) = \overline{F(a)}$ since $\overline{a+b} = \bar{a}$ or $F(a+b) \subseteq F(a)+(b)$.

iv $a \to \{axa\}$ gives the existence of a unique maximal regular ideal.

v $a \to \{a^2x\}$ establishes the existence of a largest strongly regular ideal. Arens and Kaplansky [27] introduced the concept of strong regularity and in 1952 Kandô [35] proved the result of v essentially with the methods of theorem A without referring to [30].

vi $a \to a(a)$ gives the corresponding results for weak regularity.

vii $a \to (a^2-a)$ gives a radical later on defined by Nagata [34].

In 1949 Mc Coy published a paper entitled "Prime ideals in general rings" [28]. He first characterizes prime ideals in several ways.

An ideal $\mathfrak{p}$ in a ring B is prime if for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ of B $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. It makes no difference whether $\mathfrak{a}$ and $\mathfrak{b}$ are one- or two-sided. An ideal $\mathfrak{p}$ in B is also prime if $aBb \subseteq \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ for all $a,b \in B$. The intersection of an ideal $\mathfrak{a}$ and a prime ideal $\mathfrak{p}$ appears to be a prime ideal in the ring $\mathfrak{a}$.

As a generalization of the concept of a multiplicative system he defines an m-system as a subset M of a ring B such that if $a,b \in M$ there exists an $x \in B$ with $axb \in M$. Its importance lies in the fact that the complementary set of a prime ideal is an m-system (the converse does not hold). Here the radical of an ideal $\mathfrak{a}$ is the set of elements $r \in B$ such that every m-system containing r contains some element of $\mathfrak{a}$, the radical (M-radical) of a ring being the radical of (0). Let it be denoted b

Mc Coy proves that the radical of an ideal is the intersection of all prime over ideals. He uses Zorn's lemma.

The M-radical is a nil ideal, since the set of all powers of an element p is an m-system, which must contain 0 if $p \in M$, hence $p^n = 0$ for some integer n. Moreover every nilpotent ideal I is contained in M. For if

$I^n = 0$ we have $I^n \subseteq \mathfrak{P}$ for all prime ideals $\mathfrak{P}$, whence $I \subseteq \mathfrak{P}$, so $I \subseteq M$. In presence of the descending chain condition every nil ideal is nilpotent and thus M coincides with the classical radical in this case. Mc Coy also shows that M satisfies conditions i and ii of page 1. From ii it follows immediately that R/M contains no nonzero nilpotent ideal and hence M is a radical ideal in the sense of Baer. Mc Coy proposes as an unsolved problem the relation of the M-radical to those of Koethe and Levitzki. In 1951 Levitzki [32,33] and Nagata [34] proved that M equals the lower radical of Baer.

We shall give here a somewhat modified proof. If B is a ring we denote by B_1 the ring whose elements are pairs (m,a) with $a \in B$ and m an integer. The operations are defined as $(m,a)+(n,b) = (m+n, a+b)$ and $(m,a)(n,b) = (mn, mb+na+ab)$. The subset of B_1 consisting of pairs (0,a) is isomorphic to B and we shall identify a with (0,a) for all $a \in B$. An ideal $\mathfrak{a}$ in B is also an ideal in B_1 for if $a \in \mathfrak{a}$ and $x = (n,s) \in B_1$ we have $ax = (0,a)(n,s) = (0, na+as) \in \mathfrak{a}$; similarly $xa \in \mathfrak{a}$. In particular B is an ideal and because evidently B_1/B is isomorphic to the ring of integers B is even a prime ideal.

Now let L be the lower radical and $a \notin L$ $(a \in B)$. Since L is a radical ideal and aB_1 is a right ideal in B not contained in L, we have $aB_1aB_1 \nsubseteq L$ and thus $aB_1a \nsubseteq L$ (since L is an ideal in B_1). We can choose $b_0 \in B_1$ such that $a_1 = ab_0a \notin L$ $(a_1 \in B)$. By mathematical induction we thus construct a sequence $\{a_i\} \in B$ with the properties $a_i \notin L$ and $a_{i+1} = a_ib_ia_i$ for all i. But $\{a_i\}$ is an m-system in B (since $a_j = a_kxa_l$ with $x \in B$ provided $j > k+1, l+1$) not containing 0, hence $a \notin M$ and $M \subseteq L$. As M is a radical ideal it follows $M = L$.

Nagata [34] also tries to give a unified treatment of several radicals. Let C be a condition for rings; a ring satisfying C is called a C-ring and a ring isomorphic to a subdirect sum of C-rings is called a semi C-ring. An ideal $\mathfrak{P}$ in a ring B is called a (semi) C-ideal if $B/\mathfrak{P}$ is a (semi) C-ring. Hence a semi C-ideal is the intersection of C-ideal Finally the intersection of all C-ideals is called the C-radical; it is the smallest semi C-ideal. By appropriate choices of C he gets the radicals of Jacobson and Mc Coy, but there is by no means a gain in the proofs of the different properties of these and other radicals. He does not give a general theorem about C-radicals.

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