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ESTIMATING A MONOTONE DENSITY

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Some local and global results on estimating a monotone density are discussed. In particular, it is shown that a centered version of the L_1-distance between a smooth strictly decreasing density and its MLE is asymptotically normal and has an asymptotic variance which is independent of the density. The results are derived from the structure of a jump process generated by Brownian motion.

1980 MATHEMATICS SUBJECT CLASSIFICATION: Primary: 62E20, 62G05, Secondary: 60J65, 60J75.
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1. INTRODUCTION

Let $\mathcal{F}$ be the class of nonincreasing left continuous densities on the interval $[0,\infty)$. It was shown by Grenander(1956) that the maximum likelihood estimator (MLE) of a density f under the (order) restriction that it belongs to $\mathcal{F}$ is given by the slope of the concave majorant $\hat{F}_n$ of the empirical distribution function F_n (see figure 1).

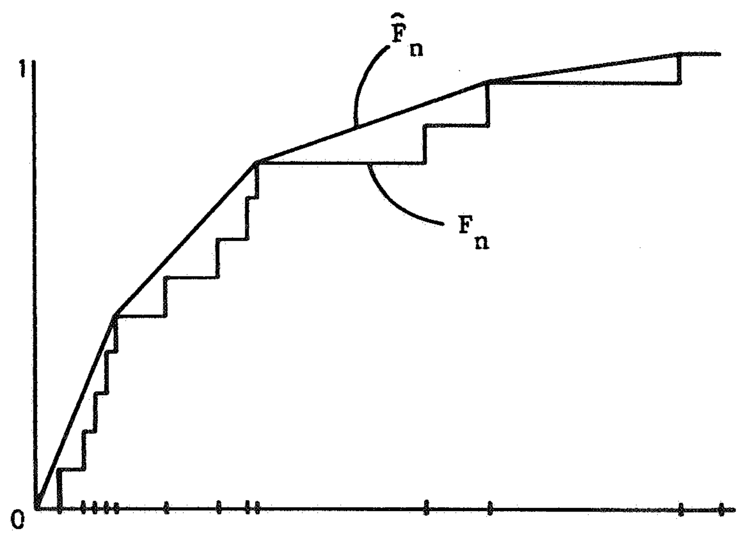

Figure 1: The empirical distribution function F_n and its concave majorant $\hat{F}_n$ for n = 12

For a discussion of this result and analogous results for monotone failure rates, see Barlow et al (1972). Not very much is known about the distribution theory of these estimators; in particular, distribution theory for their *global* behavior is missing. We will sketch an approach to the distribution theory using properties of certain jump processes generated by Brownian motion.

As examples of the use of this approach first a simple proof of a result of Prakasa Rao (1969) will be given in Section 2, and next the limiting distribution of the L_1-distance between a decreasing density and its Grenander maximum likelihood estimator will be derived in Section 3. The methods are similar to those in Groeneboom (1983).

Finally, in Section 4, the analytical properties of the underlying process are discussed. A Volterra integral equation is derived which seems to be much better tractable than the original heat equation.

In the following only statements of the results and sketches of (some of) the proofs will be given in the hope that this will give a flavor of the kind of results one can obtain along these lines. Full details and proofs will be given elsewhere.

2. A RESULT OF PRAKASA RAO.

In Prakasa Rao (1969) the following result is proved.

THEOREM 2.1. (Prakasa Rao). Let $X_1,...,X_n$ be independent observations generated by a decreasing density f on $[0,\infty)$ which has a non-zero derivative $f'(t)$ at a point $t \in (0,\infty)$. If f_n is the Grenander maximum likelihood estimator of f, based on $X_1,...,X_n$, then

$$n^{1/3}|\tfrac{1}{2}f(t)f'(t)|^{-1/3}(f_n(t)-f(t)) \xrightarrow{\mathcal{L}} 2Z, \tag{2.1}$$

where Z is distributed as the location of the maximum of the process $(W(t)-t^2,\ t \in \mathbb{R})$, and W is standard two-sided Brownian motion on $\mathbb{R}$, originating from zero (i.e. $W(0)=0$).

We will now show that this result can be derived from rather simple observations on the scaling properties of Brownian motion, together with the "Hungarian embedding" of Komlós et al (1975).

First of all, the problem of finding the distribution of f_n can be reduced right away to the problem of finding the distribution of locations of maxima of the processes $(F_n(t)-at, t \geq 0)$, $a>0$, where F_n is the empirical distribution function of $X_1,...,X_n$.

For suppose $U_n(a) = \sup\{t \geq 0: F_n(t) - at \text{ is maximal}\}$. Then we have

$$(2.2) \qquad f_n(t) \leq a \iff U_n(a) \leq t$$

(see figure 2).

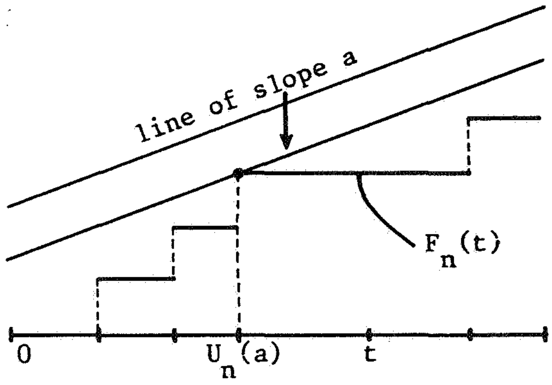

Figure 2

Relation (2.2) also was an essential tool in Groeneboom (1983). The usefulness of (2.2) comes from the fact that the process $(U_n(a),\ a \geq 0)$ is much better tractable than the (inverse) process $(f_n(t),\ t \geq 0)$.

By (2.2) we have

$$(2.3) \qquad P_f\{f_n(t) - f(t) \leq xn^{-1/3}|\tfrac{1}{2}f'(t)f(t)|^{1/3}\} = P_f\{U_n(f(t)+\delta_n) \leq t\}$$

where $\delta_n = xn^{-1/3}|\tfrac{1}{2}f'(t)f(t)|^{1/3}$. By definition,

$$U_n(a+\delta_n) = \sup\{t \geq 0: F_n(t) - (a+\delta_n)t \text{ is maximal}\}.$$

Hence we can write

$$U_n(a+\delta_n) = \sup\{t \geq 0: n^{\frac{1}{2}}(F_n(t) - F(t)) + n^{\frac{1}{2}}(F(t) - (a+\delta_n)t) \text{ is maximal}\}.$$

By Komlós et al (1975), $n^{\frac{1}{2}}(F_n(t) - F(t)) = B_n(F(t)) + O_p(n^{-\frac{1}{2}}\log n)$, where $(B_n,\ n \in \mathbb{N})$ is a sequence of Brownian bridges, constructed on the same space as the F_n's. Therefore, the limiting distribution of

$n^{1/3}(U_n(a+\delta_n)-t)$ (if it exists) will be the same as the limiting distribution of $n^{1/3}(U(a+\delta_n)-t)$, where $U(b)$ is the location of the maximum of the process $(B(F(u))+n^{\frac{1}{2}}(F(u)-bu), u \geq 0)$, and B is (standard) Brownian bridge on $[0,1]$ (with probability one, there is only one such maximum). Put $a = f(t)$ and $c = -\frac{1}{2}f'(t)$.

Now note that the location of the **maximum of the process** $(B(F(u))+n^{\frac{1}{2}}(F(u)-(a+\delta_n)u), u \geq 0)$ behaves, as $n\to\infty$, as the location of the maximum of the process

$$(B(F(t)+a(u-t))-n^{\frac{1}{2}}c(u-t)^2-n^{\frac{1}{2}}\delta_n(u-t),\ u \geq 0),$$

since $c = -\frac{1}{2}f'(t) > 0$. Define $z = (nc^2/a)^{1/3}(u-t)$. Since a Brownian bridge behaves locally as Brownian motion (at an interior point of $[0,1]$), the limiting distribution of $(nc^2/a)^{1/3}(U_n(a+\delta_n)-t)$ will be the same as that of the location of the maximum of the process

$$(W(a^{4/3}(nc^2)^{-1/3}z)-n^{-1/6}a^{2/3}c^{-1/3}(z^2+xz),\ z\in\mathbb{R})$$

(using $\delta_n = xn^{-1/3}(ac)^{1/3}$), where W is two-sided Brownian motion, originating from zero. By Brownian scaling this distribution is the same as that of the location of the maximum of $(W(z)-(z^2+xz),\ z\in\mathbb{R})$ which is equal to the location of the maximum of $(W(z)-(z+\frac{1}{2}x)^2,\ z\in\mathbb{R})$.

Let $V(a)$ denote the location of the maximum of the process $(W(z)-(z-a)^2,\ z\in\mathbb{R})$. Then $(V(a)-a,\ a\in\mathbb{R})$ is a *stationary* process and hence $P\{V(a)\leq t\} = P\{V(0)\leq t-a)\}$. This gives (2.1), since

$$P_f\{f_n(t)-a \leq xn^{-1/3}(ac)^{1/3}\} = P_f\{U_n(a+\delta_n)-t \leq 0\} \to$$

$$\to P\{V(-\tfrac{1}{2}x)\leq 0\} = P\{2V(0)\leq x\},\ \text{as } n\to\infty,$$

for each x, and thus $(ac)^{-1/3}n^{1/3}(f_n(t)-f(t)) \xrightarrow{\mathcal{L}} 2V(0)$.

Remark 2.1. The difference between the proof given above and the proof given by Prakasa Rao is that in the proof above (2.2) is used and that Brownian motion is introduced at an earlier stage. The remainder terms which arise from replacing the empirical processes by Brownian bridges are taken care of by the "Hungarian embedding".

Remark 2.2. The assumption $f'(t) \neq 0$ is essential in Theorem 2.1. For example, if f is the uniform density on [0,1], then, for $t \in (0,1)$,

$$(2.4)\qquad n^{\frac{1}{2}}(f_n(t) - f(t)) \xrightarrow{\mathcal{L}} S_t,$$

where S_t is the slope of the concave majorant of the Brownian bridge at t. The density of V_t is a function of the standard normal distribution function and the standard normal density, see Groeneboom (1983), formula (3.11). Note that the rate of convergence in (2.4) is $n^{-\frac{1}{2}}$ instead of the rate $n^{-1/3}$ in (2.1).

3. ASYMPTOTIC NORMALITY OF THE L_1-NORM $\| f_n - f \|_1$.

Let f be a decreasing density, concentrated on a bounded interval [0,B], with a bounded continuous second derivative, and such that $f'(t) \neq 0$, for $t \in (0,B)$. Let f_n be the MLE of f in the class $\mathcal{F}$, based on n observations from f (see Section 1). Furthermore, let $(W(t),\ t \in \mathbb{R})$ be two-sided Brownian motion on $\mathbb{R}$, originating from zero, and let the process $(V(a),\ a \in \mathbb{R})$ be defined by $V(a) = \sup\{t \in \mathbb{R}\colon W(t) - (t-a)^2 \text{ is maximal}\}$. We note that V is an increasing pure jump process, generated by Brownian motion sample paths. A picture of the situation is given below in figure 3.

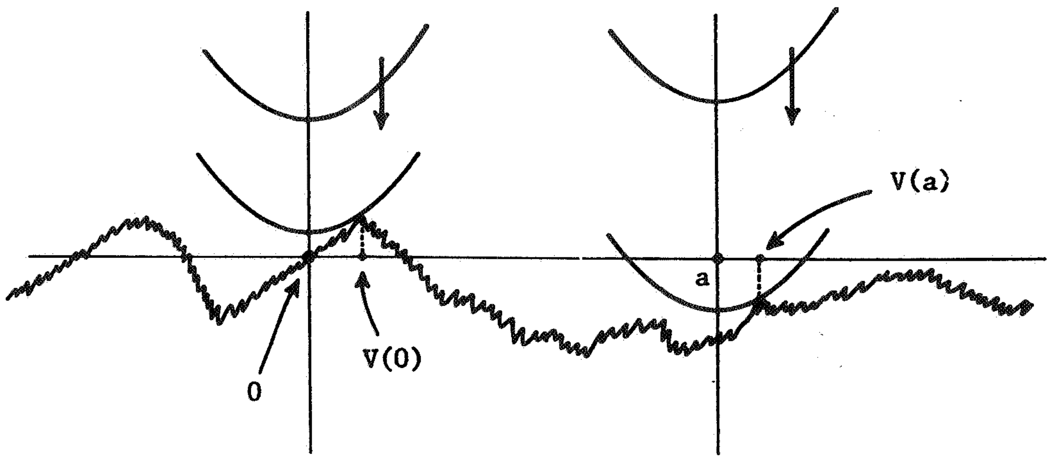

Figure 3: V(a) is the location of the point where the parabola $f(t) = (t-a)^2 + c$, sliding down along the line $t = a$, hits two-sided Brownian motion, originating from zero.

Let $\| f_n - f \|_1 = \int_0^B |f_n(t) - f(t)|\, dt$. The asymptotic behavior of $\| f_n - f \|_1$ is given in the following theorem.

Theorem 3.1. $n^{1/6}\{n^{1/3} \| f_n - f \|_1 - C\} \xrightarrow{\mathcal{L}} N(0,\sigma^2)$,
where

$$C = 2\, E|V(0)| \int_0^B |\tfrac{1}{2} f'(t) f(t)|^{1/3}\, dt,$$

$$\sigma^2 = 8 \int_0^\infty \mathrm{Covar}(|V(0)|, |V(a) - a|)\, da,$$

and $N(0,\sigma^2)$ is a normal distribution with mean zero and variance σ^2.

Remark 3.1. Note that the asymptotic variance of $\| f_n - f \|_1$ is *independent of* f, and that this variance tends to zero at a faster rate (i.e. the rate n^{-1}) than the variance of $f_n(t) - f(t)$ at a fixed point t (in the latter case the rate is $n^{-2/3}$).

Remark 3.2. The assumption $f' \neq 0$ is again essential (like in Theorem 2.1). For example, if f is the uniform density on [0,1], then

$$n^{\frac{1}{2}} \| f_n - f \|_1 \xrightarrow{\mathcal{L}} 2 \max_{t \in [0,1]} B(t),$$

where B is the Brownian bridge on [0,1]. We do not yet know what the limiting behavior of the L_1-norm will be, if the density f has some "flat" and some "non-flat" parts.

Remark 3.3. We have $n^{1/3} E \| f_n - f \|_1 \to 2E|V(0)| \int_0^B |\tfrac{1}{2} f'(t) f(t)|^{1/3} dt$, as $n \to \infty$, and this "asymptotic risk" is invariant under scale changes in f:

$$\int_0^B |f'(t) f(t)|^{1/3} dt = \int_0^{B/\lambda} |\lambda^2 f'(\lambda t) \lambda f(\lambda t)|^{1/3} dt.$$

We will now give a sketch of the proof of Theorem 3.1. For this result we need the structure of the *process* $(f_n(t),\ t \in [0,B])$. As in Section 2, the better tractable inverse process $(U_n(a),\ a \geq 0)$, where $U_n(a) = \sup\{t \geq 0:\ f_n(t) - at \text{ is maximal}\}$, will be considered.

First we note that the asymptotic behavior of $\int_0^B |f_n(t) - f(t)|dt$ is the same as that of $\int_0^M |U_n(a) - g(a)|da$, where $M = \sup_t f(t)$, and g is the inverse of f (this follows from integration by parts and the limiting behavior of $\int_0^B |U_n(a) - g(a)|da$, which will be derived below).

The process $(n^{1/3}(U_n(a) - g(a)), a \in [0,M])$ will behave locally in the limit (after rescaling) as the process $(V(a) - a, a \in \mathbb{R})$, where $V(a) = \sup\{t \in \mathbb{R}: W(t) - (t-a)^2 \text{ is maximal}\}$ (see figure 3).
An analytical characterization of the process $(V(a), a \in \mathbb{R})$ will be given in Section 4, and the following calculations can be justified by using this characterization.

Let $C(a,b) = \text{Covar}(|V(a) - a|, |V(b) - b|)$. Then we have, for $0 < t < u < M$, as $n \to \infty$,

$$\text{Var}(n^{1/3}\int_t^u |U_n(a) - g(a)|da) =$$

$$= 2\, n^{2/3} \int\int_{t<a<b<u} \text{Covar}(|U_n(a) - g(a)|, |U_n(b) - g(b)|)dadb \sim$$

$$\sim 2 \int_t^u \{ \int_a^\infty (4a/(f'(g(a)))^2)^{2/3} C(g(a), g(a) + n^{1/3} g'(a)(b-a) . .((f'(g(a)))^2/(4a))^{1/3}) db\} da$$

$$= 8kn^{-1/3} \int_t^u a/\{g'(a)(f'(g(a)))^2\} da = 8kn^{-1/3} \int_{g(t)}^{g(u)} f(x)dx =$$

$$= 8kn^{-1/3}(F(g(u)) - F(g(t))),$$

where $k = \int_0^\infty C(0,b)db$, and F is the distribution function corresponding to the density f.

In particular, $\text{Var}(n^{1/3}\int_0^M |U_n(a) - g(a)|da) \sim 8kn^{-1/3}$, $n \to \infty$, which shows that the asymptotic variance is independent of f.
Now, let $A_n(t) = E \int_0^t |U_n(a) - g(a)|da$, and let the process $(B_n(t), t \in [0,M])$ be defined by

$$B_n(t) = n^{\frac{1}{2}}\{\int_0^t |U_n(a) - g(a)|da - A_n(t)\}, \quad 0 \le t \le M.$$

The process B_n converges weakly in $C[0,M]$ to a Gaussian process. This follows by standard arguments, using

$$n^2 E\{\int_t^u |U_n(a) - g(a)|da - (A_n(u) - A_n(t))\}^4 \sim$$

$$\sim 3n^2\{\text{Var}(\int_t^u |U_n(a) - g(a)|da)\}^2 \sim 192\, k^2\{F(g(u)) - F(g(t))\}^2,$$

and the fact that B_n has asymptotically independent increments (as defined in Billingsley (1968), p.157). Since $E(B_n(t)) = 0$, for each t, the limiting process is a zero-mean Gaussian process. Thus $n^{\frac{1}{2}}B_n(M)$ tends in law to a normal distribution with mean zero and variance 8k.

Finally,

$$n^{\frac{1}{2}}A_n(M) - 2n^{1/6}E|V(0)|\int_0^B|\tfrac{1}{2}f'(t)f(t)|^{1/3}dt =$$

$$= n^{\frac{1}{2}}A_n(M) - 2n^{1/6}E|V(0)|\int_0^M\{\tfrac{1}{2}a(g'(a))^2\}^{1/3}da \to 0, \text{ as } n \to \infty.$$

Thus we have the conclusion of Theorem 3.1:

$$n^{1/6}\{n^{1/3}\| f_n - f \|_1 - 2E|V(0)|\int_0^B|\tfrac{1}{2}f'(t)f(t)|^{1/3}dt\} \xrightarrow{\mathcal{L}} N(0,8k). \quad \square$$

Remark 3.4. In the proof of Theorem 3.1, the remainder terms have to be treated with some care. Here the condition that f has a bounded and continuous second derivative is used, together with the "Hungarian embedding" technique (see Section 2).

It can be deduced from Theorem 4.1 in Section 4 that the conditional expectation $k(a,t) \equiv E(\,|V(0)|\;|V(a) = t)$ satisfies, for $a < 0$, the integral equation

$$(3.1) \qquad k(a,t) = |t| + \int_a^0\{\int_t^\infty g(t-b,w-b)\{k(b,w) - k(b,t)\}dw\}db,$$

where g(t,w) is the density appearing at the right-hand side of (4.9), and k(a,t) satisfies the boundary condition $k(0,t) = |t|$.

Clearly, the variance 8k of the limiting distribution of the standardized version of $\| f_n - f\|_1$ can be deduced from the values of k(a,t), using the stationarity of the process $(V(a) - a,\ a \in \mathbb{R})$. At present, a computer program is being developed for calculating the values of k(a,t), using (3.1). Preliminary results suggest a rather rapid convergence of k(a,t+a) to the limiting value $E|V(0)| \approx 0.41$, as $a \to -\infty$.

Nothing seems to be known about the asymptotic distribution of the L_2-distance between f_n and f under the conditions of Theorem 3.1. It is proved in Groeneboom & Pyke (1983) that a standardized version of this L_2-distance tends (very slowly) in law to a normal limiting distribution, if f is the uniform density on [0,1]. A proof of this last result via Brownian motion is given in Groeneboom (1983).

4. ANALYTICAL PROPERTIES OF THE UNDERLYING PROCESS.

It was shown in Section 3 that the limiting distribution of the L_1-distance $\| f_n - f \|_1$ between a monotone density and its MLE f_n can be derived from the structure of the process $(V(a),\ a \in \mathbb{R})$, where $V(a) = \sup\{t \in \mathbb{R} : W(t) - (t-a)^2 \text{ is maximal}\}$. Moreover, the limiting distribution of $n^{1/3}(f_n(t) - f(t))$ at a fixed point t is given by the distribution of V(0) (apart from a scale factor). In this section we will briefly discuss the analytical properties of the process $(V(a), a \in \mathbb{R})$, and also show how the density of V(0) can be found from the solution of an integral equation (instead of the usual approach, using the numerical solution of a heat equation). This will throw some new light on the nature of the density of V(0).

It was shown by Chernoff (1964) that the density of V(0) is given by

$$g(t) = \tfrac{1}{2} \lim_{x \uparrow t^2} \frac{\partial}{\partial x} u(t,x) . \frac{\partial}{\partial x} u(-t,x), \tag{4.1}$$

where

$$u(t,x) = P^{(t,x)}\{W(z) > z^2, \text{ for some } z \geq t\} \tag{4.2}$$

is the probability that space-time Brownian motion, starting at (t,x), will cross the parabola $f(z) = z^2$ at some time $z \geq t$. It is also shown in Chernoff (1964) that u(t,x) is a solution to the (backward) heat equation (in the domain $\{(t,x): x < t^2\}$)

$$\frac{\partial}{\partial t} u(t,x) = -\tfrac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x), \tag{4.3}$$

under the boundary conditions

$$u(t,t^2) = \lim_{x \uparrow t^2} u(t,x) = 1, \quad \lim_{x \to -\infty} u(t,x) = 0. \tag{4.4}$$

The process V can also be characterized by solutions to certain heat equations. Let v(t,x,w) be defined by

$$v(t,x,w) = P^{(t,x)}\{W(z) > z^2, \text{ for some } z \in (t,w)\}, \tag{4.5}$$

that is, v(t,x,w) is the probability that space-time Brownian motion, starting at (t,x), will cross the parabola $f(z) = z^2$ *before time* w. Then v(t,x,w) is a solution to the heat equation

(4.6) $$\frac{\partial}{\partial t} v(t,x,w) = -\tfrac{1}{2}\frac{\partial^2}{\partial x^2} v(t,x,w)$$

in the domain $\{(t,x)\colon t < w,\ x < t^2\}$, under the boundary conditions

(4.7) $$v(t,x,w) = \begin{cases} 1, & x = t^2, \\ 0, & t = w,\ x < t^2 \end{cases} \qquad \lim_{x\to-\infty} v(t,x,w) = 0.$$

For notational convenience, we define

(4.8) $$u_2(t) = \lim_{x \uparrow t^2} \frac{\partial}{\partial x} u(t,x),$$

where $u(t,x)$ is defined by (4.2). We now have the following "infinitesimal" characterization of the process V.

<u>Theorem 4.1</u>. Let $(V(a),\ a \in \mathbb{R})$ be the process defined by

$$V(a) = \sup\{t \in \mathbb{R}\colon W(t) - (t-a)^2 \text{ is maximal}\},$$

where W is two-sided Brownian motion originating from zero. Then, for $w > t$,

(4.9) $$\lim_{h \downarrow 0} P\{V(h) \in dw \mid V(0) = t\}/h =$$
$$= 2(w-t)\{u_2(w)/u_2(t)\}\{\lim_{x\uparrow t^2} \frac{-\partial^2}{\partial x \partial w} v(t,x,w)\}dw.$$

By the stationarity of the process $(V(a) - a, a \in \mathbb{R})$, the process $(V(a),\ a \in \mathbb{R})$ is completely characterized by relation (4.9), and we now discuss some analytical properties of the right-hand side of (4.9). For numerical purposes, the right-hand side of (4.9) looks rather unpromising, and in fact our original attempts in getting the value of this expression numerically (using the same kind of methods as can be used for solving the equation (4.3)) failed completely. Therefore another approach, using integral equations, was tried, and this not only gave a satisfactory numerical evaluation of (4.9), but also yielded considerably more insight into the solutions of the heat equations (4.3) and (4.6) (under their respective boundary conditions).

Let $v(t,x,w)$ be defined by (4.5) and let $h(t,x,w)$ be defined by

(4.10) $$h(t,x,w) = \frac{\partial}{\partial w} v(t,x,w)/\phi((w^2-x)/(w-t)^{\frac{1}{2}}),\quad x < t^2,\ t < w,$$

where ϕ is the standard normal density $\phi(u) = (2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}u^2)$.

Then $h(t,x,w)$ satisfies the integral equation

(4.11) $$h(t,x,w) = (w-t)^{-3/2}(2tw - x - w^2) + (2\pi)^{-\frac{1}{2}}\int_t^w (w-y)^{\frac{1}{2}}\exp\{-\tfrac{1}{2}(w-y).\,.\frac{((w-t)(y-t)-(t^2-x))^2}{(w-t)(y-t)}\}h(t,x,y)dy$$

Relation (4.11) follows from Ferebee (1982), formula (2.7), where an integral equation is given for Brownian exit densities with respect to quite general boundaries. This integral equation was discovered independently by Durbin (1981).

Since $h(t,x,w)$ only depends on $w-t$ and t^2-x, we can write $h(t,x,w) = h_1(u,a)$, where $u = w-t$, $a = t^2-x$, and where $h_1(u,a)$ satisfies the integral equation

(4.12) $$h_1(u,a) = u^{-3/2}(a-u^2) + (2\pi)^{-\frac{1}{2}}\int_0^u (u-y)^{\frac{1}{2}}.\,.\exp\{-\tfrac{1}{2}\frac{(u-y)(uy-a)^2}{uy}\}h_1(y,a)dy.$$

The factor $\lim_{x\uparrow t^2} -\frac{\partial^2}{\partial x\partial w}v(t,x,w)$ in the characterization (4.9) of the process V can be found from this by determining $\lim_{a\downarrow 0} a^{-1}h_1(u,a)$. However, since $\lim_{a\downarrow 0} a^{-1}h_1(u,a) \sim u^{-3/2}$, for $u\downarrow 0$, we consider instead the function $p:[0,\infty)\to\mathbb{R}$, defined by

(4.13) $$p(u) = \lim_{a\downarrow 0}\,(h_1(u,a) - au^{-3/2})/a,$$

removing the singularity at zero. This function satisfies the integral equation

(4.14) $$p(t) = (2\pi)^{-\frac{1}{2}}\int_0^t p(y)\exp\{-\tfrac{1}{2}ty(t-y)\}(t-y)^{\frac{1}{2}}dy + r(t),$$

where

(4.15) $$r(t) = -(\pi/2)^{\frac{1}{2}} + t^{3/2} - (2\pi)^{-\frac{1}{2}}\int_0^1(1-y)^{\frac{1}{2}}(1-\exp\{-\tfrac{1}{2}t^3y(1-y)\})y^{-3/2}dy,$$

and p satisfies the initial condition $p(0) = -(\pi/2)^{\frac{1}{2}}$. Equation (4.14) can be derived by first considering the function $p_1(t,a) = (h_1(t,a) - at^{-3/2})/a$, which, by (4.12), satisfies the integral equation

(4.16) $$p_1(t,a) = (2\pi)^{-\frac{1}{2}}\int_0^t(t-y)^{\frac{1}{2}}\exp\{-\tfrac{1}{2}\frac{(t-y)(ty-a)^2}{ty}\}p_1(y,a)dy - t^{\frac{1}{2}}/a + (2\pi)^{-\frac{1}{2}}\int_0^t y^{-3/2}(t-y)^{\frac{1}{2}}\exp\{-\tfrac{1}{2}\frac{(t-y)(ty-a)^2}{ty}\}dy,$$

and by analyzing the asymptotic behavior of the right-hand side of (4.16), as $a \downarrow 0$.

The remainder term $r(t)$, defined by (4.15), has the representation

(4.17) $$r(t) = t^{3/2} - (\pi/2)^{\frac{1}{2}} {}_2F_2(-\tfrac{1}{2}, 3/2; 1, \tfrac{1}{2}; -t^3/8)$$

in terms of the hypergeometric function ${}_2F_2(a_1,a_2;b_1,b_2;z)$, defined by the power series

(4.18) $${}_2F_2(a_1,a_2;b_1,b_2;z) = \Sigma_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n} \frac{z^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(n)$. This representation of $r(t)$ was brought to my attention by Nico Temme.

The solution of the integral equation (4.14) can be written as a power series in powers of $t^{3/2}$, where the coefficients are defined recursively. Specifically, let

$${}_2F_2(-\tfrac{1}{2}, 3/2; 1, \tfrac{1}{2}; -t^3/8) = \Sigma_{n=0}^{\infty} c_n t^{3n};$$

note that the coefficients c_n can be defined recursively by $c_0=1$, $c_n = -2^{-4}\{(2n-3)(2n+1)/(n^2(2n-1))\}c_{n-1}$. We have the following result.

<u>Theorem 4.2.</u>

(i) The factor $f(t,w) = -\lim_{x \uparrow t^2} \frac{\partial^2}{\partial x \partial w} v(t,x,w)$ in the characterization (4.9) of the process V can be written

(4.19) $$f(t,w) = (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}(w-t)(w+t)^2\}\{(w-t)^{-3/2} + p(w-t)\},$$

where

(4.20) $$p(t) = -(\pi/2)^{\frac{1}{2}} \Sigma_{n=0}^{\infty} a_n t^{3n} + \Sigma_{n=1}^{\infty} b_n t^{3(n-\frac{1}{2})},$$

and the coefficients a_n and b_n are defined recursively by $a_0=1$, $b_1=2/3$,

(4.21) $$a_n = c_n - \Sigma_{k=0}^{n-1} \frac{(-\frac{1}{2})^k}{k!} b_{n-k} B(3n-2k-\tfrac{1}{2}, k+3/2)/\pi, \quad n \geq 1,$$

(4.22) $$b_n = \Sigma_{k=0}^{n-1} \frac{(-\frac{1}{2})^{k+1}}{k!} a_{n-k-1} B(3n-2k-2, k+3/2), \quad n \geq 2,$$

and where $B(x,y)$ is the the value of the Beta function in (x,y).

(ii) $$t^{-3/2} + p(t) \sim 2(2\pi)^{\frac{1}{2}} \exp\{-t^3/6 - ct\}, \text{ as } t \to \infty,$$

where $c = 2.9458\ldots$ satisfies the equation

$$(4.23)\qquad (2\pi)^{-\frac{1}{2}}\int_0^\infty \sin(\tfrac{1}{2}.3^{\frac{1}{2}}.cu)\ \exp\{-u^3/6 + \tfrac{1}{2}cu\}\sqrt{u}\ du = -1.$$

The function $u_2 : \mathbb{R} \to \mathbb{R}$, defined by (4.8), which is needed in the characterization of the process V, and also needed in the definition of the density of V(0) (see (4.1)), can be expressed in terms of the function p. This is described in the following theorem.

Theorem 4.3. The function u_2, defined by (4.8), can be written as

$$(4.24)\qquad u_2(t) = 2t - (2\pi)^{-\frac{1}{2}}\int_0^\infty p(u)\exp\{-\tfrac{1}{2}u(2t+u)^2\}du \ +$$

$$+\ (2\pi)^{-\frac{1}{2}}\int_0^\infty (4t^2 + 8tu + 3u^2)\exp\{-\tfrac{1}{2}u(2t+u)^2\}u^{-\frac{1}{2}}du,$$

where the function p is defined as in Theorem 4.2. Furthermore,

$$(4.25)\qquad u_2(t) \sim 4t, \text{ as } t \to \infty,$$

and

$$(4.26)\qquad u_2(t) \sim c_1\exp\{-(2/3)|t|^3 - c|t|\}, \text{ as } t \to -\infty,$$

where the constant $c = 2.9458\ldots$ is the same as in Theorem 4.2, and where $c_1 \approx 2.2638$.

Theorem 4.3 gives a much easier way of determining the density $\frac{1}{2}u_2(t)u_2(-t)$ of V(0) (Chernoff's (1964) result) than first solving the heat equation (4.3) numerically and then computing numerical derivatives on the boundary. Proofs of Theorems 4.2 and 4.3 will be given in a joint publication with Nico Temme.

In conclusion, we believe that the analytical results, discussed in this section, make the limiting distributions of a class of isotonic estimators more tangible (and at least more tractable). These results and methods can also be used to specify the local and global behavior of estimators of failure rates.

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