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EDGEWORTH EXPANSIONS FOR LINEAR COMBINATIONS OF ORDER STATISTICS WITH SMOOTH WEIGHT FUNCTIONS

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Edgeworth expansions for linear combinations of order statistics with smooth weight functions*)

by

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ABSTRACT

Edgeworth expansions for linear combinations of order statistics with smooth weight functions are established.

KEY WORDS & PHRASES: *Edgeworth expansions, linear combinations of order statistics.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Statistics of the form $T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{in}$, $n \geq 1$, where X_{in}, $i = 1,2,\ldots,n$ denotes the ith order statistic of a random sample $X_1,\ldots,X_n$ of size n from a distribution with distribution function (d.f.) F and the c_{in}, $i = 1,2,\ldots,n$ are known real numbers (weights), are said to be linear combinations of order statistics. In the last decade there has been considerable interest in these statistics with regard to the problem of their asymptotic normality, which has been investigated under different sets of conditions by many authors in this area. We refer to the important papers of SHORACK (1972) and STIGLER (1974) and the references given in these papers. More recently attention has been paid to the rate of convergence problem. Berry-Esseen type bounds for linear combinations of order statistics were established by BJERVE (1974) and HELMERS (1975). An account of these results is given by VAN ZWET (1977).

The purpose of this paper is to establish Edgeworth expansions for linear combinations of order statistics with remainder $o(n^{-1})$ for the case of smooth weights. BJERVE (1974) has shown that trimmed means admit asymptotic expansions. However his method employs special properties of the trimmed means and does not carry over to the more general linear combinations of order statistics we consider. Our method of proof was outlined by VAN ZWET (1977). In his paper he obtained a bound on the characteristic function of a linear combination of order statistics, which solves a crucial part of our problem. The paper is organized as follows: In section 2 we state our results in the form of two theorems. Section 3 contains a number of preliminaries. Theorem 2.1 is proved in section 4, theorem 2.2 in section 5. Extensions and applications are discussed in section 6.

2. THE RESULTS

Let J_1 and J_2 be bounded measurable functions on (0,1), where J_1 is twice differentiable with first and second derivative J_1' and J_1'' on (0,1). Let J_1'' be bounded on (0,1) and let F be a d.f. with finite fourth moment. The inverse of a d.f. will always be the left-continuous one. χ_E denotes the indicator of a set E. Let $\|h\| = \sup_{0<s<1} |h(s)|$ for any function h on

(0,1). Introduce functions h_1, h_2 and h_3 by

$$(2.1)\qquad h_1(u) = -\int_0^1 J_1(s)(\chi_{(0,s]}(u)-s)dF^{-1}(s)$$

$$(2.2)\qquad h_2(u,v) = -\int_0^1 J_1'(s)(\chi_{(0,s]}(u)-s)(\chi_{(0,s]}(v)-s)\,dF^{-1}(s)$$

$$(2.3)\qquad h_3(u,v,w) = -\int_0^1 J_1''(s)(\chi_{(0,s]}(u)-s)(\chi_{(0,s]}(v)-s)(\chi_{(0,s]}(w)-s)dF^{-1}(s)$$

for $0 < u,v,w < 1$. Furthermore define, for each $n \geq 1$ and real x, the function K_n by

$$(2.4)\qquad K_n(x) = \Phi(x)-\phi(x)\left[\frac{\kappa_3}{6n^{\frac{1}{2}}}(x^2-1)+\frac{\kappa_4}{24n}(x^3-3x)+\frac{\kappa_3^2}{72n}(x^5-10x^3+15x)\right]$$

where Φ and ϕ denote the d.f. and the density of the standard normal distribution. The quantities $\kappa_3 = \kappa_3(J_1,F)$ and $\kappa_4 = \kappa_4(J_1,F)$ are given by

$$(2.5)\qquad \kappa_3 = \kappa_3(J_1,F) = \frac{1}{\sigma^3(J_1,F)}\left[\int_0^1 h_1^3(u)du+3\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)dudv\right]$$

and

$$(2.6)\qquad \kappa_4 = \kappa_4(J_1,F) = \frac{1}{\sigma^4(J_1,F)}\left[\int_0^1 h_1^4(u)du-3\sigma^4(J_1,F)\; + \right.$$

$$+\;12\int_0^1\int_0^1 h_1^2(u)h_1(v)h_2(u,v)dudv + \int_0^1\int_0^1\int_0^1 (4h_1(u)h_1(v)h_1(w)h_3(u,v,w)\; +$$

$$\left. +\;12\,h_1(u)h_1(v)h_2(u,w)h_2(v,w))dudvdw\right]$$

where

$$(2.7)\qquad \sigma^2 = \sigma^2(J_1,F) = \int_0^1 h_1^2(u)du.$$

In our first theorem we shall establish an asymptotic expansion with remainder $o(n^{-1})$ for the d.f. $F_n^*(x) = P(T_n^* \leq x)$ for $-\infty < x < \infty$ where

(2.8) $$T_n^* = (T_n - E(T_n))/\sigma(T_n)$$

for the case of smooth weights.

THEOREM 2.1. *Suppose that positive numbers* $B, C_1, C_1', C_1'', C_2, D_4, K_1, K_2, M_1, M_2, \alpha_1, \alpha_2, c, m$ *and numbers* $\gamma > 1$ *and* $0 \le t_1 < t_2 \le 1$ *exist such that*

(2.9) $$\max_{1\le i\le n} \left| c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)ds \right| \le Bn^{-\gamma} \text{ for } n = 1,2,\ldots;$$

J_1 *is twice differentiable on* $(0,1)$ *with first and second derivative* J_1' *and* J_1'' *on* $(0,1)$;

(2.10) $$\|J_1\| \le C_1,\ \|J_1'\| \le C_1', \|J_1''\| \le C_1'', \|J_2\| \le C_2$$

and

(2.11) $$|J_1''(s_1)-J_1''(s_2)| \le K_1|s_1-s_2|^{\alpha_1} \text{ and } |J_2(s_1)-J_2(s_2)| \le K_2|s_1-s_2|^{\alpha_2}$$

for $0 < s_1, s_2 < 1$ *and*

(2.12) $$J_1(s) \ge c \quad \text{for } t_1 < s < t_2;$$

F *possesses a finite fourth moment* β_4 *with*

(2.13) $$\beta_4 \le D_4$$

and on $(F^{-1}(t_1), F^{-1}(t_2))$ F *is twice differentiable with density* f *and second derivative* f' *such that on* $(F^{-1}(t_1), F^{-1}(t_2))$

(2.14) $$m \le f \le M_1 \quad \text{and} \quad |f'| \le M_2.$$

Then there exists $A > 0$ *depending on* n, *the* c_{in} *and* F *only through* B, $C_1, C_1', C_1'', C_2, D_4, K_1, K_2, M_1, M_2, \alpha_1, \alpha_2, c, m, t_1$ *and* t_2 *and a sequence of positive numbers* $\delta_1, \delta_2, \ldots$ *with* δ_n *depending only on* n, α_1, α_2 *and* γ *and with*

$\lim_{n\to\infty} \delta_n = 0$, *such that for all* $n \geq 1$

$$\sup_x \mid F_n^*(x) - K_n(x) \mid \leq A\delta_n n^{-1}.$$

Note that the expansion K_n does not depend on the function J_2. This is due to the exact standardization we have employed in theorem 2.1.

Our second theorem is a modification of theorem 2.1 which lends itself better to applications. Since a different standardization is used in this case our expansion will not only depend on J_1 and F but also on J_2. We shall establish an asymptotic expansion with remainder $o(n^{-1})$ for the d.f. $G_n(x) = P(n^{\frac{1}{2}}(T_n-\mu)/\sigma \leq x)$ for $-\infty < x < \infty$ where

$$(2.15) \qquad \mu = \mu(J_1,F) = \int_0^1 F^{-1}(s)\ J_1(s)ds$$

and $\sigma^2 = \sigma^2(J_1,F)$ as in (2.7). Introduce a function h_4 by

$$(2.16) \qquad h_4(u) = -\int_0^1 J_2(s)\ (\chi_{(0,s]}(u)-s)\ d\ F^{-1}(s)$$

for $0 < u < 1$. Furthermore quantities $a = a(J_1,J_2,F)$ and $b = b(J_1,J_2,F)$ are given by

$$(2.17) \qquad a = a(J_1,J_2,F) = \frac{1}{\sigma(J_1,F)} \left[2^{-1} \int_0^1 s(1-s)\ J_1'(s)d\ F^{-1}(s) - \right.$$
$$\left. - \int_0^1 F^{-1}(s)\ J_2(s)\ ds\right]$$

and

$$(2.18) \qquad b = b(J_1,J_2,F) = \frac{1}{2\sigma^2(J_1,F)} \left[\int_0^1 (h_1(u)\ h_2(u,u) + 2\ h_1(u)h_4(u))\ du\ + \right.$$
$$\left. + \int_0^1 \int_0^1 (2^{-1}h_2^2(u,v) + h_1(u)h_3(u,v,v))\ dudv\right].$$

Finally define, for each $n \geq 1$ and real x, the function L_n by

$$(2.19) \qquad L_n(x) = K_n(x) - \phi(x)\left[-\frac{a}{n^{\frac{1}{2}}} + \left(\frac{a\kappa_3 + a^2 + 2b}{2n}\right)x - \frac{a\kappa_3}{6n}x^3\right].$$

THEOREM 2.2. *Suppose that the assumptions of theorem* 2.1 *are satisfied. Then there exists* $A > 0$ *depending on* n, *the* c_{in} *and* F *only through the same constants as in theorem* 2.1 *and a sequence* $\delta_1, \delta_2, \ldots$ *as in theorem* 2.1 *such that for all* $n \geq 1$

$$\sup_x \mid G_n(x) - L_n(x) \mid \leq A\, \delta_n\, n^{-1}.$$

It may be useful to comment briefly on assumption (2.9). In spite of its unusual appearance this assumption is satisfied in a number of interesting situations. Three examples of the validity of this assumption are provided by

$$(2.20) \qquad c_{in} = J_1\left(\frac{i}{n+1}\right)$$

$$(2.21) \qquad c_{in} = J_1\left(\frac{i}{n}\right)$$

and

$$(2.22) \qquad c_{in} = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds$$

where J_1 is a function on (0,1) satisfying the assumptions of theorem 2.1. In each of these three cases it is easy to verify that assumption (2.9) holds with $J_2(s) = (\frac{1}{2}-s)J_1'(s)$, $J_2(s) = \frac{1}{2}J_1'(s)$ and $J_2(s) = 0$ respectively. The weights (2.20) were considered by CHERNOFF et.al. (1967) and STIGLER (1974). MOORE (1968) studied weights of type (2.21) and BICKEL (1967) investigated weights of the form (2.22).

We conclude this section with a remark concerning the way we have presented our results. Although our theorems are formally stated as results for a fixed but arbitrary value of n they are in fact purely asymptotic results, because we do not make precise the way in which the constant A appearing in our estimate of the remainder depends on certain constants occurring in our assumptions. Since our estimate of the remainder depends on n, the c_{in} and F only through these constants, we have, in effect, indicated classes of linear combinations of order statistics and d.f.'s for which our expansions are valid uniformly.

3. PRELIMINARIES

In this section we present a number of preliminary results which will be needed in our proofs.

Let, for each $n \geq 1$, $U_1,\ldots,U_n$ be independent uniform (0,1) r.v.'s and let U_{in} $(1 \leq i \leq n)$ denote the i^{th} order statistic of $U_1,\ldots,U_n$. It is well-known that the joint distribution of $X_1,\ldots,X_n$ is the same as that of $(F^{-1}(U_1),\ldots, F^{-1}(U_n))$ for any d.f.F. Therefore we shall identify X_i with $F^{-1}(U_i)$ and also X_{in} with $F^{-1}(U_{in})$. The empirical d.f. based on $U_1,\ldots,U_n$ will be denoted by Γ_n. Throughout this paper we shall assume that all r.v.'s are defined on the same probability space (Ω,A,P). For any r.v.X with $0 < \sigma(X) < \infty$ we write $\hat{X} = X - E(X)$ and $X^* = \hat{X}/\sigma(X)$. For any positive number ℓ the ℓth absolute moment of F will be denoted by β_ℓ.
We start by stating a very simple but useful lemma.

LEMMA 3.1. *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be two sequences of r.v.'s and let there exist positive numbers* A *and* b *and a number* $\eta > 1$ *such that for all* $n \geq 1$

(i) $\sigma^2(X_n-Y_n) \leq A\, n^{-\eta}$, *and*

(ii) $\sigma^2(X_n) \geq bn^{-1}$ *holds.*

Then there exists $C > 0$ *depending on* A, b *and* η, *but not on* n, *such that* $\sigma^2(X_n^*-Y_n^*) \leq Cn^{-\eta+1}$ *for all* $n \geq 1$.

PROOF. Note first that

$$(3.1) \qquad \sigma^2(X_n-Y_n) = (\sigma(X_n)-\sigma(Y_n))^2 + 2(1-\rho_n)\,\sigma(X_n)\sigma(Y_n)$$

where ρ_n denotes the correlation coefficient of X_n and Y_n. Because of assumption (i) and the fact that each of the terms on the right of (3.1) is non-negative we find that

$$(3.2)\qquad \sigma(X_n) - \sigma(Y_n) \le A^{\frac{1}{2}} n^{-\frac{\eta}{2}}$$

and also that

$$(3.3)\qquad 2(1-\rho_n)\ \sigma(X_n)\ \sigma(Y_n) \le An^{-\eta}.$$

Using now assumption (ii) and (3.2) and noting that $\eta > 1$ we see that $\sigma^2(Y_n) \ge en^{-1}$ for some $e > 0$ depending only on A, b and η. Combining this and assumption (ii) with (3.3) we find that $2(1-\rho_n) \le Cn^{-\eta+1}$ for some $C > 0$ depending only on A, b and η. Because $\sigma^2(X_n^*-Y_n^*) = 2(1-\rho_n)$ we have proved the lemma. □

Secondly we present an obvious result concerning the finiteness of certain integrals.

LEMMA 3.2. a. *Let* ℓ *be a number* > 1 *and let, for some* $\delta > 0$, $\beta_{\ell+\delta} < \infty$. *Then there exists* $A > 0$, *depending only on* ℓ *and* δ, *such that*

$$(3.4)\qquad \int_0^1 (s(1-s))^{\frac{1}{\ell}}\, dF^{-1}(s) \le A\ \beta_{\ell+\delta}^{\frac{1}{\ell+\delta}} < \infty.$$

b. *If* $\ell = 1$ *and* $\delta = 0$ *then* (3.4) *holds with* $A = 1$.

PROOF. Applying integration by parts we obtain

$$(3.5)\qquad \int_0^1 (s(1-s))^{\frac{1}{\ell}}\, dF^{-1}(s) = (s(1-s))^{\frac{1}{\ell}}\, F^{-1}(s)\Big|_0^1 - \ell^{-1}\int_0^1 F^{-1}(s)(s(1-s))^{\frac{1}{\ell}-1}(1-2s)\, ds.$$

Both under the assumptions a. and b. the first term on the right of (3.5) is easily seen to be zero. To conclude the proof of part a. we apply Hölder's inequality to the second term on the right of (3.5):

$$\left| \ell^{-1} \int_0^1 F^{-1}(s)(s(1-s))^{\frac{1}{\ell}-1} ds \right| \le \int_0^1 |F^{-1}(s)| (s(1-s))^{\frac{1}{\ell}-1} ds \le$$

$$\le \beta_{\ell+\delta}^{\frac{1}{\ell+\delta}} \cdot \left(\int_0^1 (s(1-s))^{-1+\frac{\delta}{\ell(\ell+\delta-1)}} ds \right)^{\frac{\ell+\delta-1}{\ell+\delta}} < \infty .$$

The proof of part b. is immediate from (3.5) and the remark made after it. This completes the proof the lemma. □

The third lemma of this section will enable us to estimate certain moments.

<u>LEMMA 3.3</u>. *Let* ℓ *be a positive integer and let, for some* $\delta > 0$, $\beta_{\ell+\delta} < \infty$. *Then, for any number* p *for which* $p\,\ell \ge 2$, *there exists* $A > 0$ *depending only on* p, ℓ *and* δ, *such that*

$$E\left(\int_0^1 | \Gamma_n(s)-s |^p dF^{-1}(s)\right)^{\ell} \le A\, \beta_{\ell+\delta}^{\frac{\ell}{\ell+\delta}}\, n^{-\frac{p\ell}{2}} . \tag{3.6}$$

<u>PROOF</u>. By Fubini's theorem we have

$$E\left(\int_0^1 | \Gamma_n(s)-s |^p dF^{-1}(s)\right)^{\ell} =$$

$$= \int_0^1 \cdots \int_0^1 E \prod_{i=1}^{\ell} | \Gamma_n(s_i) - s_i|^p \, dF^{-1}(s_1) \ldots dF^{-1}(s_\ell).$$

An application of Hölder's inequality shows that

$$E \prod_{i=1}^{\ell} | \Gamma_n(s_i) - s_i |^p \le \prod_{i=1}^{\ell} (E\, | \Gamma_n(s_i) - s_i |^{p\ell})^{\frac{1}{\ell}}$$

for all $0 < s_1,\dots,s_\ell < 1$. Hence we know that

$$E\left(\int_0^1 | \Gamma_n(s)-s |^p d F^{-1}(s)\right)^\ell \le \left(\int_0^1 (E | \Gamma_n(s)-s |^{p\ell})^{\frac{1}{\ell}} d F^{-1}(s)\right)^\ell .$$

Since $\Gamma_n(s) = n^{-1} \sum_{i=1}^n (\chi_{(0,s]}(U_i)-s)$ for all $0 < s < 1$ and $n \ge 1$ the MARCINKIEVITZ, ZYGMUND, CHUNG inequality (see CHUNG(1951)) yields for $p\ell \ge 2$, $n \ge 1$ and $0 < s < 1$

$$E | \Gamma_n(s)-s |^{p\ell} \le B\, n^{-\frac{p\ell}{2}} s(1-s)$$

where $B > 0$ depends only on p and ℓ. It follows that

$$E\left(\int_0^1 | \Gamma_n(s)-s |^p d F^{-1}(s)\right)^\ell \le B\, n^{-\frac{p\ell}{2}} \left(\int_0^1 (s(1-s))^{\frac{1}{\ell}} d F^{-1}(s)\right)^\ell .$$

An application of lemma 3.2 completes our proof. □

To formulate the following lemma we introduce functions H_1, H_2, H_3 and H_4 by

(3.7) $$H_1(u) = \int_0^1 | J_1(s) | \cdot | \chi_{(0,s]}(u)-s | \, d F^{-1}(s)$$

(3.8) $$H_2(u) = \int_0^1 | J_1'(s) | \cdot | \chi_{(0,s]}(u)-s | \, d F^{-1}(s)$$

(3.9) $$H_3(u) = \int_0^1 | J_1''(s) | \cdot | \chi_{(0,s]}(u)-s | \, d F^{-1}(s)$$

(3.10) $$H_4(u) = \int_0^1 | J_2(s) | \cdot | \chi_{(0,s]}(u)-s | \, d F^{-1}(s)$$

for $0 < u < 1$. Note that the integrand of H_i majorize the integrand of h_i and hence that H_i majorize $h_i (1 \le i \le 4)$. Remark also that h_2 and h_3 are symmetric in their respectively arguments.

LEMMA 3.4. a. *Take* $\ell \ge 1$ *and suppose that there exist positive numbers* C_1, C_1', C_1'', C_2 *and* D_ℓ *such that* $\beta_\ell \le D_\ell$ *and assumption* (2.10) *is satisfied. Then*

$$E\, H_1^\ell(U_1) \le (4C_1)^\ell\, D_\ell \tag{3.11}$$

$$E\, H_2^\ell(U_1) \le (4C_1')^\ell\, D_\ell \tag{3.12}$$

$$E\, H_3^\ell(U_1) \le (4C_1'')^\ell\, D_\ell \tag{3.13}$$

$$E\, H_4^\ell(U_1) \le (4C_2)^\ell\, D_\ell . \tag{3.14}$$

b. *Let* J_1 *be twice differentiable with bounded second derivative* J_1'' *on* $(0,1)$, *let* J_2 *be bounded on* $(0,1)$ *and let* $\beta_1 < \infty$. *Then* $E\, h_1(U_i) = E\, h_4(U_i) = 0$ *for any* i, *and with probability one* $E(h_2(U_i,U_j) \mid U_j) = 0$ *for* $i \ne j$ *and* $E(h_3(U_i,U_j,U_k) \mid U_j,U_k) = 0$ *if* $i \ne j$ *and* $i \ne k$.

PROOF. a. We first prove (3.11). It is immediate from (3.7) that

$$H_1(U_1) \le \|J_1\| \cdot \Big(\int_{(0,U_1)} s\, d\, F^{-1}(s) + \int_{[U_1,1)} (1-s) d\, F^{-1}(s)\Big).$$

Applying the c_r-inequality we find

$$E\, H_1^\ell(U_1) \le 2^{\ell-1} \cdot \|J_1\|^\ell \cdot \Big[E\Big(\int_{(0,U_1)} s d\, F^{-1}(s)\Big)^\ell + E\Big(\int_{[U_1,1)} (1-s)\, d\, F^{-1}(s)\Big)^\ell\Big].$$

Using integration by parts, the finiteness of β_ℓ, and applying the c_r-inequality once more we see that

$$E\Big(\int_{(0,U_1)} s\,d\,F^{-1}(s)\Big)^{\ell} = E\,\Big|\,U_1\,F^{-1}(U_1) - \int_0^{U_1} F^{-1}(s)\,ds\,\Big|^{\ell} \leq$$

$$\leq 2^{\ell-1}\cdot\Big(E(|F^{-1}(U_1)|^{\ell} + \Big(\int_0^1 |F^{-1}(s)|ds\Big)^{\ell}\Big) \leq 2^{\ell-1}(E|X_1|^{\ell} + (E|X_1|)^{\ell}) \leq$$

$$\leq 2^{\ell}\,E|X_1|^{\ell}.$$

Similarly we can show that

$$E\Big(\int_{[U_1,1)} (1-s)\,dF^{-1}(s)\Big)^{\ell} \leq 2^{\ell}\,E|X_1|^{\ell}$$

so that $E\,H_1^{\ell}(U_1) \leq 4^{\ell}\,\|J_1\|^{\ell}\,E|X_1|^{\ell} \leq (4C_1)^{\ell}\,D_{\ell}$ which proves (3.11). The other statements of part a. can be proved in essentially the same way.
b. We shall prove that with probability one $E(h_3(U_i,U_j,U_k)|U_j,U_k) = 0$ for $i \neq j$ and $i \neq k$. Note first that using Fubini's theorem for non-negative integrands and applying (3.13) with $\ell = 1$ we see that with probability one

$$E\Big(\int_0^1 |J_1''(s)|\;|X_{(0,s]}(U_i)-s|\;|X_{(0,s]}(U_j)-s|\;|X_{(0,s]}(U_k)-s|$$

$$d\,F^{-1}(s)|U_j,U_k\Big) \leq E\,H_3(U_1) < \infty,$$

for all values of U_j and U_k. Therefore the conditional expectation $E(h_3(U_i,U_j,U_k)|U_j,U_k)$ is well-defined and Fubini's theorem can be applied to see that $E(h_3(U_i,U_j,U_k)|U_j,U_k) = 0$ with probability one. The other two statements of part b are easier and can be proved in essentially the same way. □

<u>REMARK</u>. Lemma 3.4 will be applied frequently in the following sections. In particular the proof of lemma 4.6 depends heavily on it. In this remark we give two typical examples of the way we shall use lemma 3.4.

(i) *Suppose that* $\beta_2 < \infty$ *and* J_1' *is bounded on* (0,1). *Then* $E\ h_1(U_1)h_2(U_1,U_2) = E\ h_1^2(U_1)\ h_2(U_1,U_2) = 0$.

(ii) *Suppose there exist numbers* C_1, C_1', C_1'' *and* D_4 *such that* $\|J_1\| \leq C_1$, $\|J_1'\| \leq C_1'$, $\|J_1''\| \leq C_1''$ *and* $\beta_4 \leq D_4$. *Then there exists* $A > 0$ *depending only on* C_1, C_1', C_1'' *and* D_4 *such that* $E(h_1(U_1)+h_1(U_2))^4|h_2(U_1,U_2)| \leq A$.

PROOF. (i) We first prove that $E\ h_1^2(U_1)\ h_2(U_1,U_2) = 0$. It follows directly from (2.1), (2.2), (3.11) and (3.12) and the independence of U_1 and U_2 that $|E\ h_1^2(U_1)\ h_2(U_1,U_2)| \leq E\ H_1^2(U_1)\ E\ H_2(U_2) < \infty$. Hence we can write

$$E\ h_1^2(U_1)\ h_2(U_1,U_2) = E\left[E(h_1^2(U_1)\ h_2(U_1,U_2)|U_1)\right] =$$

$$= E\left[h_1^2(U_1)E\left\{h_1(U_1,U_2)|U_1\right\}\right] = 0,$$

because of lemma 3.4.b. This proves the assertion. The other statement can be proved in essentially the same way.

(ii) We remark that $E(h_1(U_1)+h_1(U_2))^4\ |h_2(U_1,U_2)| \leq 8\ E\ h_1^4(U_1)|h_2(U_1,U_2)| + 8\ E\ h_1^4(U_2)|h_2(U_1,U_2)| = 16\ E\ h_1^4(U_1)|h_2(U_1,U_2)| \leq 16\ E\ H_1^4(U_1)\ H_2(U_2) = 16\ E\ H_1^4(U_1)\ E\ H_2(U_2) \leq 4^7 C_1^4 C_1' D_1 D_4 < \infty$,
using lemma 3.4 and the independence of U_1 and U_2. This completes the proof. □

The fifth and last lemma of this section gives conditions which guarantee that $\sigma^2 = \sigma^2(J_1,F)$ (c.f.(2.7)) is bounded away from zero.

LEMMA 3.5. *Let* J_1 *be bounded on* (0,1) *and let* $\beta_1 < \infty$. *Suppose that positive numbers* M_1 *and* c *and numbers* $0 \leq t_1 < t_2 \leq 1$ *exist such that on* $(F^{-1}(t_1), F^{-1}(t_2))$ F *possesses a density* f, *such that on* $(F^{-1}(t_1),F^{-1}(t_2))$ $f \leq M_1$ *and on* (t_1,t_2) $J_1 \geq c$. *Then there exists* $\sigma_0^2 > 0$ *depending only on* M_1, c, t_1 *and* t_2 *such that*

(3.15) $$\sigma^2(J_1,F) \geq \sigma_0^2.$$

PROOF. Note first that because J_1 is bounded on $(0,1)$ and $\beta_1 < \infty$ the function h_1 is well-defined and finite for every $0 < u < 1$. Secondly we remark that $\sigma^2(J_1,F) = \int_0^1 h_1^2(u)\,du \geq \int_{t_1}^{t_2} h_1^2(u)\,du$. It follows directly from (2.1) and the assumptions of the lemma that $h_1(u_2) - h_1(u_1) \geq c\, M_1^{-1}(u_2-u_1)$ for $t_1 < u_1 < u_2 < t_2$. The geometry of the situation ensures now that $\int_{t_1}^{t_2} h_1^2(u)\,du$ is minimized for $h_1(u) = (u-\frac{t_1}{2}-\frac{t_2}{2})\frac{c}{M_1}$. Hence

$$\sigma^2(J_1,F) \geq \frac{c^2(t_2-t_1)^3}{12\, M_1^2}.$$

This completes the proof of the lemma. □

4. PROOF OF THEOREM 2.1.

The purpose of this section is to provide a proof of theorem 2.1. Since our proofs will depend on characteristic function (c.f.) arguments let us denote by $\rho_n^*(t)$ the c.f. of T_n^* and by $\tilde{\rho}_n(t)$ the Fourier-Stieltjes transform $\tilde{\rho}_n(t) = \int_{-\infty}^{\infty} \exp(itx)\, d\, K_n(x)$ of K_n(see (2.4.)).

We shall show that for some sufficiently small $\varepsilon > 0$

$$(4.1) \qquad \int_{|t|\leq n^{\varepsilon}} |\rho_n^*(t) - \tilde{\rho}_n(t)|\ |t|^{-1}\,dt = o(n^{-1})$$

and that

$$(4.2) \qquad \int_{n^{\varepsilon}<|t|\leq n^{3/2}} |\rho_n^*(t)|\ |t|^{-1}\,dt = o\,(n^{-1})$$

and

(4.3) $$\int_{|t|>\log(n+1)} |\tilde{\rho}_n(t)|\ |t|^{-1}\, dt = o(n^{-1}),$$

hold as $n \to \infty$. An application of Esseen's smoothing lemma (ESSEEN (1945)) will then complete our proof.

We first prove (4.1). We shall essentially have to expand $\rho_n^*(t)$ for these "small" values of $|t|$. To start with we define for $0 < u < 1$

(4.4) $$\psi_i(u) = \int_u^1 J_i(s)\, ds - (1-u)\, \bar{J}_i$$

where $\bar{J}_i = \int_0^1 J_i(s)\, ds$ for $i = 1,2$. Then it is easy to check (see SHORACK (1972) for a similar approach) that with probability one

(4.5) $$T_n = \int_0^1 (\psi_1(\Gamma_n(s) + n^{-1}\psi_2(\Gamma_n(s))\, d\, F^{-1}(s) + $$
$$+ (\bar{J}_1 + n^{-1}\bar{J}_2)\, n^{-1} \sum_{i=1}^{n} F^{-1}(U_1) +$$
$$+ n^{-1} \sum_{i=1}^{n} (c_{in} - n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)\, ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)\, ds)\, F^{-1}(U_{in}).$$

Let J_1 be twice differentiable with first and second derivative J_1' and J_1'' on $(0,1)$. Let J_1'' and J_2 be bounded on $(0,1)$ and let $\beta_1 = E|X_1| < \infty$. Introduce for each $n \geq 1$ the r.v. S_n by (a prime denoting differentiation)

(4.6) $$S_n = \int_0^1 \Big\{\psi_1(s) + n^{-1}\psi_2(s) + (\Gamma_n(s)-s)\, (\psi_1'(s)+n^{-1}\psi_2'(s)) +$$

$$+ \frac{(\Gamma_n(s)-s)^2}{2} \psi_1''(s) + \frac{(\Gamma_n(s)-s)^3}{6} \psi_1'''(s) \Big\} d F^{-1}(s) +$$

$$+ (\bar{J}_1 + n^{-1}\bar{J}_2)\, n^{-1} \sum_{i=1}^{n} F^{-1}(U_i).$$

Note that $|\psi_i(u)| \le 4 \|J_i\| u(1-u)$ for $0 < u < 1$, $i = 1, 2$ and that $\psi_1' = - J_1 + \bar{J}_1$, $\psi_2' = - J_2 + \bar{J}_2$, $\psi_1'' = - J_1'$ and $\psi_1''' = -J_1''$ on $(0,1)$ so that it is easily verified that S_n is a well-defined r.v. Later on in this section it will become clear that $T_n^* - S_n^*$ is, under appropriate conditions, of negligible order for our purposes.

It is convenient to introduce some more notation. Define r.v.'s I_{mn} for $m = 1,2,3,4$ and $n \ge 1$ by

$$I_{1n} = - \int_0^1 J_1(s)\, (\Gamma_n(s)-s)\, d F^{-1}(s) = n^{-1} \sum_{i=1}^{n} h_1(U_i) \tag{4.7}$$

$$I_{2n} = - \int_0^1 J_1'(s) \frac{(\Gamma_n(s)-s)^2}{2} d F^{-1}(s) = 2^{-1} n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} h_2(U_i,U_j) \tag{4.8}$$

$$I_{3n} = - \int_0^1 J_1''(s) \frac{(\Gamma_n(s)-s)^3}{6} d F^{-1}(s) = 6^{-1} n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} h_3(U_i,U_j,U_k) \tag{4.9}$$

$$I_{4n} = - n^{-1} \int_0^1 J_2(s)\, (\Gamma_n(s)-s)\, d F^{-1}(s) = n^{-2} \sum_{i=1}^{n} h_4(U_i) \tag{4.10}$$

where the functions h_1, h_2, h_3 and h_4 are given by (2.1) - (2.3) and (2.16). It is easily checked that

$$\hat{S}_n = S_n - E\, S_n = \sum_{m=1}^{4} \hat{I}_{mn} = \sum_{m=1}^{4} (I_{mn} - E\, I_{mn}). \tag{4.11}$$

Furthermore define r.v.'s J_{mn} for m = 1,2,3,4 and n ≥ 1 by

(4.12) $$J_{mn} = \hat{I}_{mn} / \sigma(S_n) = (I_{mn} - E\, I_{mn}) / \sigma(S_n),$$

so that

(4.13) $$S_n^* = \sum_{m=1}^{4} J_{mn}.$$

The proof of (4.1) will now be split up in a number of lemma's. *Throughout the remaining part of this paper we shall frequently use order symbols in our proofs to indicate the order of certain remainder terms. We remark that these order symbols will always be uniform for fixed values of the constants appearing in the statement we are proving.*

LEMMA 4.1. *Suppose that positive numbers* C_1, C_1', C_1'', *and* D_2 *exist such that* $\beta_2 \le D_2$ *and assumption* (2.10) *is satisfied. Then there exists* A > 0 *depending on* n, J_1, J_2 *and* F *only through* C_1, C_1', C_1'', C_2 *and* D_2 *such that*

(4.14) $$|\sigma^2(S_n) - n^{-1}\sigma^2 - 2n^{-2}\sigma^2\, b| \le A\, n^{-\frac{5}{2}}$$

where $\sigma^2 = \sigma^2(J_1,F)$ *is as in* (2.7) *and* $b = b(J_1,J_2,F)$ *as in* (2.18). *In addition* $\sigma^2 \le A_1$ *and* $\sigma^2|b| \le A_2$ *for some positive constants* A_1 *and* A_2 *depending only on* C_1, D_2 *and* C_1, C_1', C_1'', C_2 *and* D_2 *respectively.*

PROOF. In view of (4.11) $\sigma^2(S_n) = \sigma^2(\sum_{m=1}^{4} I_{mn})$. It follows directly from (2.1) and (4.7) that $\sigma^2(I_{1n}) = n^{-1}\sigma^2$. Also note that it is immediate from (4.7), (4.8) and an application of lemma 3.4. that

$$2\,\mathrm{cov}(I_{1n},I_{2n}) = 2\, E\, I_{1n}\, I_{2n} = n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} E\, h_1(U_i)\, h_2(U_j,U_k) =$$

$$= n^{-2} \int_0^1 h_1(u)\, h_2(u,u)\, du.$$

Next we consider $\sigma^2(I_{2n})$. Using lemma 3.2 and lemma 3.4 once more we directly find that

$$\mathcal{E}\, I_{2n}^2 = 4^{-1}\, n^{-2}\, (\mathcal{E}\, h_2(U_1,U_1))^2 + 2^{-1}\, n^{-2}\, \mathcal{E}\, h_2^2(U_1,U_2) + + \mathcal{O}(n^{-3}), \text{ as } n \to \infty.$$

Because we know also that $(\mathcal{E}\, I_{2n})^2 = 4^{-1}\, n^{-2}(\mathcal{E}\, h_2(U_1,U_1))^2$ we have shown that

$$\sigma^2(I_{2n}) = 2^{-1}\, n^{-2} \int_0^1\int_0^1 h_2^2(u,v)\, dudv + \mathcal{O}(n^{-3}), \text{ as } n \to \infty.$$

Similarly we can prove that

$$2\, \mathrm{cov}(I_{1n},I_{3n}) = n^{-2} \int_0^1 \int_0^1 h_1(u)\, h_3(u,v,v)\, dudv + \mathcal{O}(n^{-3}), \text{ as } n \to \infty$$

and also that

$$2\, \mathrm{cov}(I_{1n},I_{4n}) = 2n^{-2} \int_0^1 h_1(u)\, h_4(u)\, du.$$

Finally we remark that it is easy to prove by using similar arguments as above that

$$\sigma^2(I_{3n}) + \sigma^2(I_{4n}) = \mathcal{O}(n^{-3}), \text{ as } n \to \infty$$

and also that

$$|\mathrm{cov}(I_{2n},I_{3n}) + \mathrm{cov}(I_{2n},I_{4n}) + \mathrm{cov}(I_{3n},I_{4n})| = \mathcal{O}(n^{-\frac{5}{2}}), \text{ as } n \to \infty.$$

Combining all these results we have proved (4.14). Note that $\sigma^2 \le A_1$ follows from lemma 3.4 (relation (3.11) with $\ell = 2$). The assertion $\sigma^2|b| \le A_2$ is a simple consequence of lemma 3.4 (with $\ell = 2$) and the formula for b given in (2.18). □

LEMMA 4.2. *Suppose that positive numbers* C_1, C_1', C_1'', C_2, D_2 *and* σ_0^2 *exist such that* $\sigma^2(J_1,F) \ge \sigma_0^2$ *and the assumptions of lemma* 4.1 *are satisfied. Then there exists* $A > 0$ *depending on* n, J_1, J_2 *and* F *only through* C_1, C_1', C_1'', C_2, D_2 *and* σ_0^2 *such that for any fixed real number* m

$$(4.15) \qquad |\sigma^{-m}(S_n) - n^{\frac{m}{2}} \sigma^{-m}| \le A\, n^{\frac{m}{2}-1}$$

where $\sigma^2 = \sigma^2(J_1,F)$ *is as in* (2.7).

PROOF. The statement is immediately from lemma 4.1. □

The next lemma will enable us to show that $T_n^* - S_n^*$ is of negligible order for our purposes. Let τ_n^* denote the c.f. of S_n^*.

LEMMA 4.3. *Suppose that positive numbers* B, C_1, C_1', C_1'', C_2, δ, $D_{2+\delta}$, σ_0^2, K_1, K_2, α_1, α_2 *and a number* $\gamma > 1$ *exist such that* $\beta_{2+\delta} \le D_{2+\delta}$, $\sigma^2(J_1,F) \ge \sigma_0^2$ *and the assumptions* (2.9), (2.10) *and* (2.11) *are satisfied. Then there exists* $A > 0$ *depending on* n, *the* c_{in} *and* F *only through* B, C_1, C_1', C_1'', C_2, δ, $D_{2+\delta}$, σ_0^2, K_1, K_2, α_1, *and* α_2 *such that for every* $\varepsilon > 0$ *and all* $n \ge 1$

$$(4.16) \qquad \int_{|t|\le n^{\varepsilon}} |\rho_n^*(t)-\tau_n^*(t)|\ |t|^{-1}\, dt \le A\, n^{-1-\min(\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)+\varepsilon}$$

PROOF. It follows from lemma X.V. 4.1. of FELLER (1966) that

$$(4.17) \qquad |\rho_n^*(t) - \tau_n^*(t)| \le |t|\ E|T_n^* - S_n^*|$$

for all t and $n \ge 1$. Using (4.5), (4.6), assumption (2.11) and applying Taylor's theorem we see directly that

$$(4.18)\qquad \sigma^2(T_n-S_n) \le 3\,K_1^2\, E\Big(\int_0^1 |\Gamma_n(s)-s|^{3+\alpha_1}\, d\,F^{-1}(s)\Big)^2 +$$

$$+\ 3\,K_2^2\, n^{-2}\, E\Big(\int_0^1 |\Gamma_n(s)-s|^{1+\alpha_2}\, d\,F^{-1}(s)\Big)^2 +$$

$$+\ 3\,\sigma^2\Big(n^{-1}\sum_{i=1}^{n}\Big(c_{in}-n\int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)\,ds\Big)\,F^{-1}(U_{in})\Big).$$

Application of lemma 3.3 with $\ell = 2$ and $p = 3 + \alpha_1$ and $p = 1 + \alpha_2$ respectively implies that the sum of the first two terms on the right of (4.18) is

$$(4.19)\qquad \mathcal{O}(n^{-3-\min(\alpha_1,\alpha_2)}),\ \text{as } n \to \infty.$$

To treat the third term on the right of (4.18) we need the following simple inequality: $\sigma^2(\sum_{i=1}^{n} a_i X_{in}) \le \sigma^2(\sum_{i=1}^{n} b_i X_{in})$, provided $a_i a_j \le b_i b_j$ for all $1 \le i, j \le n$. (The proof of this inequality is immediate from the well-known fact that the covariance of any two order statistics is always non-negative). Using this and assumption (2.9) we see directly that

$$\sigma^2\Big(n^{-1}\sum_{i=1}^{n}\Big(c_{in}-n\int_{\frac{i-1}{n}}^{\frac{i}{n}} J_1(s)\,ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J_2(s)\,ds\Big)\,F^{-1}(U_{in})\Big) \le$$

$$\le B^2\, n^{-1-2\gamma}\,\sigma^2(X_1).$$

Combining this result with (4.19) it is easy to conclude that

$$(4.20)\qquad \sigma^2(T_n-S_n) = \mathcal{O}(n^{-3-\min(\alpha_1,\alpha_2)}) + \mathcal{O}(n^{-1-2\gamma}),\ \text{as } n \to \infty.$$

To complete our proof we remark that it follows from an application of the lemma's 3.1 and 4.2 (with m= -2) that (4.20) implies that

$$\sigma^2(T_n^*-S_n^*) = O(n^{-2-\min(\alpha_1,\alpha_2)}) + O(n^{-2\gamma}), \text{ as } n \to \infty.$$

This combined with (4.17) proves (4.16). □

Next define for real t and n ≥ 1

$$\text{(4.21)} \qquad \tau_{1n}(t) = E\, e^{itJ_{1n}}(1+it(J_{2n}+J_{3n}+J_{4n}) + \frac{(it)^2}{2} J_{2n}^2).$$

In the following lemma we shall approximate τ_n^* by τ_{1n} for all $|t| \leq n^{\varepsilon}$.

LEMMA 4.4. *Suppose that positive numbers* C_1, C_1', C_1'', C_2, δ, $D_{3+\delta}$, *and* σ_0^2 *exist such that* $\beta_{3+\delta} \leq D_{3+\delta}$, $\sigma^2(J_1,F) \geq \sigma_0^2$ *and assumption* (2.10) *is satisfied. Then there exists* $A > 0$ *depending on* n, J_1, J_2 *and* F *only through* C_1, C_1', C_1'', C_2, δ, $D_{3+\delta}$ *and* σ_0^2 *such that for every* $\varepsilon > 0$ *and all* $n \geq 1$

$$\text{(4.22)} \qquad \int_{|t|\leq n^{\varepsilon}} |\tau_n^*(t) - \tau_{1n}(t)|\; |t|^{-1}\, dt \leq A\, n^{-\frac{3}{2}+3\varepsilon}.$$

PROOF. Application of lemma X.V.4.1. of FELLER (1966) yields

$$|\tau_n^*(t) - \tau_{1n}(t)| = |E e^{itJ_{1n}}(e^{it(J_{2n}+J_{3n}+J_{4n})} -1-it(J_{2n}+J_{3n}+J_{4n}) -$$

$$- \frac{(it)^2}{2} J_{2n}^2)| \leq t^2\,(E|J_{2n}J_{3n}|+E|J_{2n}J_{4n}|+E|J_{3n}J_{4n}|+E\,J_{3n}^2+$$

$$+E\,J_{4n}^2) + |t|^3\, E|J_{2n}+J_{3n}+J_{4n}|^3,$$

for all t and n ≥ 1. It is not difficult to verify from the proof of lemma 4.1 and from lemma 4.2 that the coefficient of t^2 in the above inequality is $O(n^{-\frac{3}{2}})$, as $n \to \infty$. An application of the c_r-inequality, lemma 3.3 with $\ell = 3$ and $p = 2,3$ and 4 respectively, and of lemma 4.2 shows that also $E|J_{2n} + J_{3n} + J_{4n}|^3 = O(n^{-\frac{3}{2}})$, as $n \to \infty$. Combining these results we easily check that (4.22) is proved. □

We continue with the analysis of $\tau_{1n}(t)$. For convenience we write σ_n^2 to indicate $n\sigma^2(S_n)$ and we denote the c.f. of $h_1(U_1)$ by ρ. To start with we remark that it follows from (4.21) that

(4.23)
$$\tau_{1n}(t) = \rho^n\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) +$$

$$+ \frac{it}{2n^{\frac{3}{2}}\sigma_n} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1)\, E\, e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} \cdot h_2(U_1,U_2) +$$

$$+ \frac{it}{2n^{\frac{3}{2}}\sigma_n} \rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n\, E\, e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} \hat{h}_2(U_1,U_1) +$$

$$+ \frac{it}{6n^{\frac{5}{2}}\sigma_n} \rho^{n-3}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1)(n-2)\, E\, e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} \cdot$$

$$\cdot h_3(U_1,U_2,U_3) +$$

$$+ \frac{it}{6n^{\frac{5}{2}}\sigma_n} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 3n(n-1)\, E\, e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_3(U_1,U_1,U_2) +$$

$$+ \frac{it}{6n^{\frac{5}{2}}\sigma_n} \rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n\, E\, e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} \hat{h}_3(U_1,U_1,U_1) +$$

$$+ \frac{it}{n^{\frac{3}{2}}\sigma_n} \rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n \, E \, e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} h_4(U_1) \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-4}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1)(n-2)(n-3) \cdot$$

$$\cdot \left(E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2)\right)^2 \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-3}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 4n(n-1)(n-2) \; \cdot$$

$$\cdot \; E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} h_2(U_1,U_2) h_2(U_1,U_3) \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-3}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 2n(n-1)(n-2) \; \cdot$$

$$\cdot \; E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} \hat{h}_2(U_1,U_1) h_2(U_2,U_3) \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 4n(n-1) \; \cdot$$

$$\cdot \; E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} \hat{h}_2(U_1,U_1) \cdot h_2(U_1,U_2) \; +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) 2n(n-1)\ E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} (h_2(U_1,U_2))^2 +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-2}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n(n-1) (E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}h_1(U_1)} \hat{h}_2(U_1,U_1))^2 +$$

$$+ \frac{(it)^2}{8n^3\sigma_n^2} \rho^{n-1}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) n\ E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}h_1(U_1)} (\hat{h}_2(U_1,U_1))^2.$$

In the next lemma we derive an asymptotic expansion for the factors $\rho^{n-m}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right)$ appearing in the terms on the right of (4.23).

LEMMA 4.5. *Suppose that positive numbers* C_1, C_1', C_1'', C_2, D_4 *and* σ_0^2 *exist such that* $\sigma^2(J_1,F) \geq \sigma_0^2$ *and the assumptions* (2.10) *and* (2.13) *are satisfied. Then there exist* $A > 0$ *and* $a > 0$ *depending on* n, J_1, J_2 *and* F *only through* C_1, C_1', C_1'', C_2, D_4 *and* σ_0^2, *a sequence of positive numbers* $\delta_1, \delta_2, \ldots$ *with* δ_n *depending only on* n *and with* $\lim_{n\to\infty} \delta_n = 0$, *and a fixed polynomial* P *in* t, *such that for any fixed* $m \geq 0$ *and all* $|t| \leq an^{\frac{1}{2}}$ *and* $n \geq 1$

$$(4.24)\qquad \left|\rho^{n-m}\left(\frac{t}{n^{\frac{1}{2}}\sigma_n}\right) - e^{-\frac{t^2}{2}}\left(1 - \frac{(it)^2}{n}\left(\frac{m}{2}+b\right) + \frac{(it)^3\int_0^1 h_1^3(u)du}{6n^{\frac{1}{2}}\sigma^3} + \right.\right.$$

$$\left.\left. + \frac{(it)^4\left(\int_0^1 h_1^4(u)du - 3\sigma^4\right)}{24n\sigma^4} + \frac{(it)^6\left(\int_0^1 h_1^3(u)du\right)^3}{72n\sigma^6}\right)\right| \leq$$

$$\leq A\ \delta_n\ n^{-1}\ |t|\ P(t)\ e^{-\frac{t^2}{4}},$$

where $\sigma^2 = \sigma^2(J_1,F)$ *is as in* (2.7) *and* $b = b(J_1,J_2,F)$ *as in* (2.18).

PROOF. Since $(\sigma(n-m))^{-\frac{1}{2}} \sum_{i=1}^{n-m} h_1(U_i)$ is a properly standardized sum of independent identically distributed r.v.'s with expectation zero, variance one, and finite fourth moment, it follows directly from the classical theory of Edgeworth expansions for such sums (see e.g. GNEDENKO-KOLMOGOROV (1954), §41, theorem 2.1, inequality (b)) that there exist $a > 0$ and a sequence of positive numbers $\delta_1, \delta_2, \ldots$ satisfying the assumptions of the lemma such that for all $|t| \leq an^{\frac{1}{2}}$ and $n \geq 1$

$$(4.25)\qquad \Bigg|\rho^{n-m}\left(\frac{t}{(n-m)^{\frac{1}{2}}\sigma}\right) - e^{-\frac{t^2}{2}}\left(1 + \frac{(it)^3\int_0^1 h_1^3(u)du}{6n^{\frac{1}{2}}\sigma^3} + \frac{(it)^4(\int_0^1 h_1^4(u)du-3\sigma^4)}{24n\sigma^4} + \frac{(it)^6(\int_0^1 h_1^3(u)du)^2}{72n\sigma^6}\right)\Bigg| =$$

$$= O(\delta_n\, n^{-1}|t|\ P(t)\ e^{-\frac{t^2}{4}}), \text{ as } n \to \infty,$$

where P is a fixed polynomial in t. We perform now a change of variables $t_n = t\, n^{\frac{1}{2}}\sigma_n/(n-m)^{\frac{1}{2}}\sigma)$. It follows after expanding $e^{-\frac{t_n^2}{2}}$ around t and using the result of lemma 4.1 that we obtain (4.24). □

The expectations appearing on the right of (4.23) are expanded in the following lemma.

LEMMA 4.6. *Suppose that positive numbers* C_1, C_1', C_1'', C_2, D_4 *and* σ_0^2 *exist such that* $\sigma^2(J_1,F) \geq \sigma_0^2$ *and the assumptions* (2.10) *and* (2.13) *are satisfied. Then there exists* $A > 0$ *depending on* n, J_1, J_2, *and* F *only through* C_1, C_1', C_1'', C_2, D_4 *and* σ_0^2 *such that for all* t *and* $n \geq 1$

(4.26) $$\left| E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2) - \frac{(it)^2}{n\sigma^2}\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)\,dudv - \right.$$

$$\left. - \frac{(it)^3}{n^{\frac{3}{2}}\sigma^3}\int_0^1\int_0^1 h_1^2(u)h_1(v)h_2(u,v)\,dudv \right| \le A(n^{-2}(t^2+t^4) + n^{-\frac{5}{2}}|t|^3).$$

(4.27) $$\left| E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} \hat{h}_2(U_1,U_1) - \frac{it}{n^{\frac{1}{2}}\sigma}\int_0^1 h_1(u)h_2(u,u)\,du \right| \le A(n^{-1}t^2+n^{-\frac{3}{2}}|t|).$$

(4.28) $$\left| E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} h_3(U_1,U_2,U_3) - \right.$$

$$\left. - \frac{(it)^3}{n^{\frac{3}{2}}\sigma^3}\int_0^1\int_0^1\int_0^1 h_1(u)h_1(v)h_1(w)h_3(u,v,w)\,dudvdw \right| \le A(n^{-2}t^4+n^{-\frac{5}{2}}|t|^3).$$

(4.29) $$\left| E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} h_3(U_1,U_1,U_2) - \right.$$

$$\left. - \frac{it}{n^{\frac{1}{2}}\sigma}\int_0^1\int_0^1 h_1(u)h_3(u,v,v)\,dudv \right| \le A(n^{-1}t^2+n^{-\frac{3}{2}}|t|).$$

(4.30) $$\left| E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} \hat{h}_3(U_1,U_1,U_1) \right| \le A\, n^{-\frac{1}{2}}|t|.$$

(4.31) $$\left| E e^{\frac{it}{n^{\frac{1}{2}}\sigma_n} h_1(U_1)} h_4(U_1) - \frac{it}{n^{\frac{1}{2}}\sigma}\int_0^1 h_1(u)h_4(u)\,du \right| \le A(n^{-1}t^2+n^{-\frac{3}{2}}|t|).$$

(4.32) $$\left|\left(E e^{\frac{it}{n^{\frac12}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2)\right)^2 - \frac{(it)^4}{n^2\sigma^4}\left(\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)\,du\,dv\right)^2\right| \le A\left(n^{-\frac52}|t|^5+n^{-3}t^4\right).$$

(4.33) $$\left|E e^{\frac{it}{n^{\frac12}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} h_2(U_1,U_2)h_2(U_1,U_3) - \frac{(it)^2}{n\sigma^2}\int_0^1\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,w)h_2(v,w)\,du\,dv\,dw\right| \le A\left(n^{-\frac32}|t|^3+n^{-2}t^2\right).$$

(4.34) $$\left|E e^{\frac{it}{n^{\frac12}\sigma_n}(h_1(U_1)+h_1(U_2)+h_1(U_3))} \hat{h}_2(U_1,U_1)h_2(U_2,U_3)\right| \le A n^{-\frac32}|t|^3.$$

(4.35) $$\left|E e^{\frac{it}{n^{\frac12}\sigma_n}(h_1(U_1)+h_1(U_2))} \hat{h}_2(U_1,U_1)h_2(U_1,U_2)\right| \le A n^{-\frac12}|t|.$$

(4.36) $$\left|E e^{\frac{it}{n^{\frac12}\sigma_n}(h_1(U_1)+h_1(U_2))} (h_2(U_1,U_2))^2 - \int_0^1\int_0^1 h_2^{\,2}(u,v)\,du\,dv\right| \le A n^{-\frac12}|t|$$

(4.37) $$\left|\left(E e^{\frac{it}{n^{\frac12}\sigma_n}h_1(U_1)} \hat{h}_2(U_1,U_1)\right)^2\right| \le A n^{-1}t^2$$

(4.38) $$\left|E e^{\frac{it}{n^{\frac12}\sigma_n}h_1(U_1)} (\hat{h}_2(U_1,U_1))^2\right| \le A.$$

PROOF. Because the statements (4.26) - (4.38) are all proved in essentially the same manner we shall only prove, by way of an example, (4.26). Expanding $\exp\left(\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))\right)$ around $t = 0$ and applying part (i) of the remark made after lemma 3.4 we find that for all t and $n \geq 1$

$$(4.39)\qquad |\mathcal{E}e^{\frac{it}{n^{\frac{1}{2}}\sigma_n}(h_1(U_1)+h_1(U_2))} h_2(U_1,U_2) - \frac{(it)^2}{n\sigma_n^2}\int_0^1\int_0^1 h_1(u)h_1(v)h_2(u,v)dudv -$$

$$-\frac{(it)^3}{n^{\frac{3}{2}}\sigma_n^3}\int_0^1\int_0^1 h_1^2(u)h_1(v)h_2(u,v)dudv\,| \leq$$

$$\leq \frac{t^4}{n^2\sigma_n^4}\mathcal{E}|h_1(U_1)+h_1(U_2)|^4|h_2(U_1,U_2)|.$$

Application of part (ii) of the remark made after lemma 3.4 shows that the term on the right of (4.39) is $\mathcal{O}(n^{-2}\sigma_n^{-4}t^4)$, as $n \to \infty$. Next we remark that lemma 4.2 implies that $\sigma_n^{-1} = \sigma^{-1} + \mathcal{O}(n^{-1})$, as $n \to \infty$. Inserting this result in (4.39) we have proved (4.26). □

We are now in a position to prove (4.1). We first apply lemma 4.3 with $0 < \varepsilon < \min(\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)$ to see that the integral on the left of (4.16) is $o(n^{-1})$, as $n \to \infty$. Secondly we use lemma 4.4 with $0 < \varepsilon < \frac{1}{6}$ to find that the integral on the left of (4.22) is also $o(n^{-1})$, as $n \to \infty$. To proceed let us note that we can write down $\tilde{\rho}_n(t)$ explicitly as

$$(4.40)\qquad \tilde{\rho}_n(t) = e^{-\frac{t^2}{2}}\left(1-\frac{it^3\kappa_3}{6n^{\frac{1}{2}}}+\frac{3\kappa_4t^4-\kappa_3^2t^6}{72\,n}\right).$$

Next we apply (4.40) and the results of the lemma's 4.5 and 4.6 to check that for all $n \geq 1$

$$\int_{|t|\le an^{\frac{1}{2}}} |\tau_{1n}(t) - \tilde{\rho}_n(t)|\ |t|^{-1}dt \le A\ \delta_n\ n^{-1}$$

with A, a en the δ_n as in lemma 4.5. Hence we can conclude that (4.1) holds for $0 < \varepsilon < \min(\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1,\frac{1}{6})$ under the assumptions (2.9), (2.10), (2.11), (2.13) and the assumptions of lemma 3.5.

Next we consider (4.2) and (4.3). Finding sufficient conditions for (4.2) is a problem of an entirely different nature, which was solved by VAN ZWET (1977). In his theorem 4.1 he obtains a bound for the characteristic function for a linear combination of order statistics. This result of VAN ZWET (1977) provides the argument at exactly the same place in our proof where Cramer's condition (C) (see CRAMER (1962)) is used in the classical proof for sums of independent identically distributed r.v.'s.

To prove (4.2) we remark first that application of theorem 4.1 of VAN ZWET (1977) shows that his bound applies to our situation, provided positive numbers C_1, C_2, m, M_1, M_2, B, γ, c, t_1 and t_2 exist such that $\|J_1\| \le C_1$, $\|J_2\| \le C_2$ and the assumptions (2.9), (2.12) and (2.14) are satisfied. It is also clear from VAN ZWET (1977) that the only missing ingredient to complete the proof of (4.2) is the requirement that there exist positive numbers e and E such that $e \le n^{\frac{1}{2}}\ \sigma(T_n) \le E$ for all $n \ge 1$. To see this we first use the lemma's 3.5 and 4.1 to find that $n^{\frac{1}{2}}\ \sigma(S_n)$ is bounded away from zero and infinity and then apply (4.20) (c.f. also (5.10)). Hence (4.2) is shown to hold if we suppose that, for some $\delta > 0$, $\beta_{2+\delta} \le D_{2+\delta} < \infty$ and that the assumptions (2.9), (2.10), (2.11), (2.12) and (2.14) are all satisfied.

To prove (4.3) we simply use (4.40) and note that, under the assumptions (2.10) and (2.13) and the assumptions of lemma 3.5 there exist positive constants A_3 and A_4, depending only on C_1, C_1', C_1'', D_4, M_1, c, t_1, and t_2, such that $|\kappa_3| \le A_3$ and $|\kappa_4| \le A_4$. Since the assumptions of theorem 2.1 imply those of lemma 3.5 this completes the proof of theorem 2.1.

To conclude this section it may be useful to mention that if we suppose that, for some $\delta > 0$, $\beta_{4+\delta} = E|X_1|^{4+\delta} < \infty$ and that the assumptions of theorem 2.1 are all satisfied the expansion K_n established in theorem 2.1 is in fact an Edgeworth expansion; i.e. $\kappa_3 n^{-\frac{1}{2}}$ and $\kappa_4 n^{-1}$ are the first order terms in the asymptotic expansions for the third and fourth cumulant of T_n^*, whereas the higher order terms in these expansions can be proved to be of order $o(n^{-1})$.

5. PROOF OF THEOREM 2.2.

In this section we present a proof of theorem 2.2. To start with we remark that for each $n \geq 1$ and real x

$$(5.1) \qquad G_n(x) = F_n^*(x\sigma n^{-\frac{1}{2}}\, \sigma^{-1}(T_n) + (\mu - E(T_n))\sigma^{-1}(T_n)).$$

Using this identity and applying theorem 2.1 we find that for all $n \geq 1$

$$(5.2) \qquad \sup_x |G_n(x) - K_n(x\sigma n^{-\frac{1}{2}}\, \sigma^{-1}(T_n) + (\mu - E(T_n))\sigma^{-1}(T_n))\ | \leq A\delta_n n^{-1}$$

with A and the δ_n as in theorem 2.1, holds under the assumptions of theorem 2.1. To proceed we shall need expansions for $\sigma n^{-\frac{1}{2}}\, \sigma^{-1}(T_n)$ and $(\mu - E(T_n))\sigma^{-1}(T_n)$. In the following lemma we give these expansions.

<u>LEMMA 5.1</u>. *Suppose that positive numbers* B, C_1, C_1', C_1'', C_2, δ, $D_{2+\delta}$, K_1, K_2, α_1, α_2, σ_0^2 *and a number* $\gamma > 1$ *exist such that* $\beta_{2+\delta} \leq D_{2+\delta}$ *and* $\sigma^2(J_1,F) \geq \sigma_0^2$ *and the assumptions* (2.9), (2.10) *and* (2.11) *are satisfied. Then there exists* $A > 0$ *depending on* n, *the* c_{in} *and* F *only through* B, C_1, C_1', C_1'', C_2. δ, $D_{2+\delta}$, K_1, K_2, α_1, α_2, σ_0^2 *and* γ *such that for all* $n \geq 1$

$$(5.3) \qquad |(\mu - E(T_n))\ \sigma^{-1}(T_n) - an^{-\frac{1}{2}}| \leq A\ n^{-1-\min(\frac{1}{2},\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)}$$

and

$$(5.4) \qquad |\sigma\ n^{-\frac{1}{2}}\ \sigma^{-1}(T_n) - 1 + b\ n^{-1}| \leq A\ n^{-1-\min(\frac{1}{2},\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)}$$

with $a = a(J_1,J_2,F)$ *and* $b = b(J_1,J_2,F)$ *as in* (2.17) *and* (2.18).

PROOF. We first prove (5.4). Application of lemma 4.1, (4.20) and the Cauchy-Schwarz inequality yields

$$(5.5)\qquad \frac{\sigma^2}{n\sigma^2(T_n)} = \frac{\sigma^2}{n\sigma^2(S_n)}\,(1 + O(n^{-1-\min(\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)}),\ \text{as } n \to \infty.$$

Lemma 4.1 implies that

$$(5.6)\qquad \frac{\sigma^2}{n\sigma^2(S_n)} = 1 - 2\frac{b}{n} + O(n^{-\frac{3}{2}}),\ \text{as } n \to \infty.$$

Combining (5.5) and (5.6) we find

$$(5.7)\qquad \frac{\sigma^2}{n\sigma^2(T_n)} = 1 - 2\frac{b}{n} + O(n^{-1-\min(\frac{1}{2},\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)})\ \text{as } n \to \infty.$$

Inequality (5.4) is an immediate consequence of (5.7). To prove (5.3) we first use (4.20) again to see that

$$(5.8)\qquad E\,T_n = E\,S_n + O(E|T_n-S_n|) = E\,S_n + O(n^{-\frac{3}{2}-\min(\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)}),\ \text{as } n \to \infty .$$

Using the definition of S_n (cf. (4.6)) and noting that $E(\Gamma_n(s)-s)^3 = n^{-2}s(1-s)(1-2s)$ we can easily check that

$$E\,S_n = \mu - a\,\sigma\,n^{-1} + O(n^{-2}),\ \text{as } n \to \infty$$

so that (5.8) implies that

$$(5.9)\qquad \mu - E\,T_n = a\,\sigma\,n^{-1} + O(n^{-\frac{3}{2}-\min(\frac{1}{2},\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)}),\ \text{as } n \to \infty.$$

Because (5.7) directly implies that

$$(5.10) \qquad \sigma^{-1}(T_n) = n^{+\frac{1}{2}}\,\sigma^{-1} + \mathcal{O}(n^{-\frac{1}{2}}), \text{ as } n \to \infty,$$

we have proved (5.3). □

To complete the proof of theorem 2.2 we use (2.4), (2.19), (5.3), (5.4) and lemma 3.5 and apply a simple Taylor expansion argument to find that

$$(5.11) \qquad K_n(xn^{-\frac{1}{2}}\,\sigma^{-1}(T_n)\sigma + (\mu - E(T_n))\,\sigma^{-1}(T_n)) = L_n(x) + \\ + \mathcal{O}(n^{-1-\min(\frac{1}{2},\frac{\alpha_1}{2},\frac{\alpha_2}{2},\gamma-1)}), \text{ as } n \to \infty,$$

uniformly in x. Combining this with (5.2) completes the proof of theorem 2.2.

6. EXTENSIONS AND APPLICATIONS

In the theorems 2.1 and 2.2 we have established asymptotic expansions for the d.f.'s of linear combinations of order statistics with remainder $o(n^{-1})$. However, it is clear from the proofs given in this paper as well as from statement and proof of theorem 4.1 of VAN ZWET (1977) that, in principle, asymptotic expansions for the d.f.'s of linear combinations of order statistics to any order can be obtained, of course at cost of stronger conditions, in essentially the same way.

An extension in a different direction is concerned with the order of the remainder of the asymptotic expansions established in this paper. In thoerem 2.1 and 2.2 we have been content with a remainder that is $o(n^{-1})$. However, no new difficulties will be encountered when showing that under somewhat stronger conditions the remainder of our expansions is of order $n^{-\frac{3}{2}}$, which is of course the natural order of the remainder term. We will state the result without further proof.

Suppose that positive numbers B_1, C_1, C_1', C_1'', C_1''', C_2, C_2', D_5, K_1, K_2,

M_1, M_2, α_1, α_2, c, m *and numbers* $\gamma > \frac{3}{2}$ *and* $0 \leq t_1 < t_2 \leq 1$ *exist such that assumption* (2.9) *is fulfilled;* J_1 *is three-times differentiable with first, second and third derivative* J_1', J_1'' *and* J_1''' *on* (0,1) *and* J_2 *is differentiable on* (0,1) *with derivative* J_2' *on* (0,1);

$$\|J_1\| \leq C_1, \ \|J_1'\| \leq C_1', \ \|J_1''\| \leq C_1'', \ \|J_1'''\| \leq C_1''', \ \|J_2\| \leq C_2, \ \|J_2'\| \leq C_2'$$

and

$$|J_1'''(s_1) - J_1'''(s_2)| \leq K_1 |s_1 - s_2|^{\alpha_1} \text{ and}$$

$$|J_2'(s_1) - J_2'(s_2)| \leq K_2 |s_1 - s_2|^{\alpha_2}$$

for $0 < s_1, s_2 < 1$;
F *possesses a finite absolute fifth moment* β_5 *with*

$$\beta_5 \leq D_5$$

and the assumptions (2.12) *and* (2.14) *are satisfied. Then there exists* $A > 0$ *depending on* n, *the* c_{in} *and* F *only through* B, C_1, C_1', C_1'', C_1''', C_2, C_2', D_5, K_1, K_2, M_1, M_2, α_1, α_2, c, m, t_1 *and* t_2 *such that for all* $n \geq 1$

$$\sup_x |F_n^*(x) - K_n(x)| \leq An^{-\frac{3}{2}}$$

and

$$\sup_x |G_n(x) - L_n(x)| \leq An^{-\frac{3}{2}}.$$

Throughout this paper we have considered $T_n = n^{-1} \sum_{i=1}^{n} c_{in} X_{in}$, i.e. a linear combination of the X_{in}. More generally, one may also consider $n^{-1} \sum_{i=1}^{n} c_{in} h(U_{in})$ and under suitable conditions on h obtain parallel results.

An important application of the asymptotic expansions established in this paper lies in the computation of asymptotic deficiencies in the sense of HODGES and LEHMANN (1970) for estimators and tests based on linear combinations of order statistics. These computations will be given in a separate paper.

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