Sample Path Large Deviations for Heavy-Tailed Lévy Processes and Random Walks

Chang-Han Rhee[1,3] Jose Blanchet[2,4] Bert Zwart[1,3]

June 10, 2016

Abstract

Let X be a Lévy process with regularly varying Lévy measure ν. We obtain sample-path large deviations of scaled processes $\bar{X}_n(t) \triangleq X(nt)/n$ and obtain a similar result for random walks. Our results yield detailed asymptotic estimates in scenarios where multiple big jumps in the increment are required to make a rare event happen. In addition, we investigate connections with the classical large-deviations framework. In that setting, we show that a weak large deviations principle (with logarithmic speed) holds, but a full large-deviations principle does not hold.

Keywords Sample Path Large Deviations · Regular Variation · $\mathbb{M}$-convergence · Lévy Processes

Mathematics Subject Classification 60F10 · 60G17 · 60B10

1 Introduction

In this paper, we develop sample-path large deviations for one-dimensional Lévy processes and random walks, assuming the jump sizes are heavy-tailed. Specifically, let $X(t), t \geq 0$, be a centered Lévy process. Assume that $\mathbf{P}(X(1) > x)$ is regularly varying of index $-\alpha$, and that $\mathbf{P}(X(1) < -x)$ is regularly varying of index $-\beta$; i.e. there exist slowly varying functions L_+ and L_- such that

$$\mathbf{P}(X(1) > x) = L_+(x)x^{-\alpha}, \qquad \mathbf{P}(X(1) < -x) = L_-(x)x^{-\beta}. \tag{1.1}$$

Throughout the paper, we assume $\alpha, \beta > 1$. We also consider spectrally one-sided processes; in that case only α plays a role. Define $\bar{X}_n = \{\bar{X}_n(t), t \in [0,1]\}$, with $\bar{X}_n(t) = X(nt)/n, t \geq 0$. We are interested in large deviations of $\bar{X}_n$.

This topic fits well in a branch of limit theory that has a long history, has intimate connections to point processes and extreme value theory, and is still a subject of intense activity. The investigation of tail estimates of the one-dimensional distributions of $\bar{X}_n$ (or random walks with heavy-tailed step size distribution) was initiated in Nagaev (1969, 1977). The state of the art of such results is well summarized in Borovkov and Borovkov (2008); Denisov et al. (2008); Embrechts et al. (1997); Foss et al. (2011). In particular, Denisov et al. (2008) describe in detail how fast x needs to grow with n for the asymptotic relation

$$\mathbf{P}(X(n) > x) = n\mathbf{P}(X(1) > x)(1 + o(1)) \tag{1.2}$$

[1]Stochastics Group, Centrum Wiskunde & Informatica, Science Park 123, Amsterdam, North Holland, 1098 XG, The Netherlands
[2]Industrial Engineering & Operations Research, Columbia University, 500 W. 120 Street, New York, New York 10027, USA
[3]Supported by NWO VICI grant.
[4]Supported by NSF grants DMS-0806145/0902075, CMMI-0846816 and CMMI-1069064.

to hold, as $n \to \infty$, in settings that go beyond (1.1). If (1.2) is valid, the so-called *principle of one big jump* is said to hold. A functional version of this insight has been derived in Hult et al. (2005). A significant number of studies investigate the question if and how the principle of a single big jump is affected by the impact of (various forms of) dependence, and cover stable processes, autoregressive processes, modulated processes, and stochastic differential equations; see Buraczewski et al. (2013); Foss et al. (2007); Hult and Lindskog (2007); Konstantinides and Mikosch (2005); Mikosch and Wintenberger (2013); Mikosch and Samorodnitsky (2000); Samorodnitsky (2004).

The problem we investigate in this paper is markedly different from all of these works. Our aim is to develop asymptotic estimates of $\mathbf{P}(\bar{X}_n \in A)$ for a sufficiently general collection of sets A, so that it is possible to study continuous functionals of $\bar{X}_n$ in a systematic manner. For many of such functionals, and many sets A, the associated rare event will not be caused by a single big jump, but multiple jumps. The results in this domain (e.g. Blanchet and Shi (2012); Foss and Korshunov (2012); Zwart et al. (2004)) are few, each with an ad-hoc approach. As in large-deviations theory for light tails, it is desirable to have more general tools available.

Another aspect of heavy-tailed large deviations we aim to clarify in this paper is the connection with the standard large-deviations approach, which has not been touched upon in any of the above-mentioned references. In our setting, the goal would be to obtain a function I such that

$$-\inf_{\xi \in A^\circ} I(\xi) \le \liminf_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{\log n} \le \limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{\log n} \le -\inf_{\xi \in A^-} I(\xi), \tag{1.3}$$

where A° and A^- are the interior and closure of A; all our large deviations results are derived in the Skorokhod J_1 topology. Equation (1.3) is a classical large deviations principle (LDP) with sub-linear speed (cf. Dembo and Zeitouni (2009)). Using existing results in the literature (e.g. Denisov et al. (2008)), it is not difficult to show that $X(n)/n = \bar{X}_n(1)$ satisfies an LDP with rate function $I_1 = I_1(x)$ which is 0 at 0, equal to $(\alpha - 1)$ if $x > 0$, and $(\beta - 1)$ if $x < 0$. This is a lower-semicontinuous function of which the level sets are not compact. Thus, in large-deviations terminology, I_1 is a rate function, but is not a good one. This implies that techniques such as the projective limit approach cannot be applied. In fact, in Section (4.3), we show that there does not exist an LDP of the form (1.3) for general sets A, by giving a counterexample. A version of (1.3) for compact sets is derived in Section 4.2, as a corollary of our main results. A result similar to (1.3) for random walks with semi-exponential (Weibullian) tails has been derived in Gantert (1998) (see also Gantert (2000); Gantert et al. (2014) for related results). Though an LDP for finite-dimensional distributions can be derived, lack of exponential tightness also persists at the sample-path level. To make the rate function good (i.e., to have compact level sets), a topology chosen in Gantert (1998) is considerably weaker than any of the Skorokhod topologies (but sufficient for the application that is central in that work).

The approach followed in the present paper is based on the recent developments in the theory of regular variation. In particular, in Lindskog et al. (2014), the classical notion of regular variation is re-defined through a new convergence concept called $\mathbb{M}$-convergence (this is in itself a refinement of other reformulations of regular variation in function spaces; see de Haan and Lin (2001); Hult and Lindskog (2005, 2006)). In Section 2, we further investigate the $\mathbb{M}$-convergence framework by deriving a number of general results that facilitate the development of our proofs.

This paves the way towards our main large deviations results, which are presented in Section 3. We actually obtain estimates that are sharper than (1.3), though we impose a condition on A. For one-sided Lévy processes, our result takes the form

$$C_{\mathcal{J}(A)}(A^\circ) \le \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le C_{\mathcal{J}(A)}(A^-). \tag{1.4}$$

Precise definitions can be found in Section 3.1; for now it is important to note that the function $\mathcal{J}(A)$ is defined as $\inf_{\xi \in A \cap \mathbb{D}_s^\uparrow} \mathcal{D}_+(\xi)$, with $\mathcal{D}_+(\xi)$ the number of discontinuites of ξ, and $\mathbb{D}_s^\uparrow$ the set of all non-increasing step functions vanishing at the origin. Throughout the paper, we adopt the convention that the infimum over an empty set is ∞. Letting $\mathbb{D}_j$ and $\mathbb{D}_{\leqslant j}$ be the sets of step functions vanishing at the origin with precisely j and at most j steps respectively, we note that the measure C_j, defined on $\mathbb{D}\setminus\mathbb{D}_{\leqslant j-1}$ has its support on $\mathbb{D}_j$. A crucial assumption for (1.4) to hold is that the Skorokhod J_1 distance between the sets A and $\mathbb{D}_{\leqslant \mathcal{J}(A)-1}$ is strictly positive. For A such that $\mathcal{J}(A)=1$ this result has been shown in Hult et al. (2005). The interpretation of the "rate function" $\mathcal{J}(A)$ is that it provides the number of jumps in the Lévy process that are necessary to make the event A happen. This can be seen as an extension of the principle of a single big jump to multiple jumps. A rigorous statement on when (1.4) holds can be found in Theorem 3.1, which is the first main result of the paper.

The result that comes closest to (1.4) is Theorem 5.1 in Lindskog et al. (2014) which considers the $\mathbb{M}$-convergence of $\nu[n,\infty)^{-j}\mathbf{P}(X/n\in A)$. This result could be used as a starting point to investigate rare events that happen on a time-scale of $O(1)$. However, in the large-deviations scaling we consider rare events happen on a time-scale of $O(n)$. Controlling the Lévy process on this larger time-scale requires more delicate estimates, eventually leading to a an additional factor n^j in the asymptotic estimates. We further show that the choice $j=\mathcal{J}(A)$ is the only choice that leads to a non-trivial limit.

In Section 3.2 we present sample-path large deviations for two-sided Lévy processes. Our main results in this case are Theorems 3.3–3.5. In the two-sided case, we need to resolve significant combinatorial issues which do not appear in the one sided case. The polynomial rate of decay for $\mathbf{P}(\bar{X}_n\in A)$ described by the function $\mathcal{J}(A)$ in the one-sided case has a more complicated description; the corresponding polynomial rate in the two-sided case is

$$\inf_{\xi,\zeta\in\mathbb{D}_s^{\uparrow};\ \xi-\zeta\in A}(\alpha-1)\mathcal{D}_+(\xi)+(\beta-1)\mathcal{D}_+(\zeta). \tag{1.5}$$

Note that this is a result that one could expect from the result for one-sided Lévy processes and a heuristic application of the contraction principle. A rigorous treatment of the two-sided case requires a more delicate argument compared to the one-sided case: in the one-sided case, the argument simplifies since if one takes j largest jumps away from $\bar{X}_n$, then the probability that the residual process is of significant size is $o\big((n\nu[n,\infty))^j\big)$ so that it doesn't contribute in (1.4) while in two-sided case taking j largest upward jumps and k largest downward jumps from $\bar{X}_n$ doesn't guarantee that the residual process remains small with high enough probability—i.e., the probability that the residual process is of significant size cannot be bounded by $o\big((n\nu[n,\infty))^j(n\nu(-\infty,-n])^k\big)$. In addition, it may be the case that multiple pairs (j,k) of jumps lead to optimal solutions of (1.5). One useful notion that we develop and rely on in our setting is a form of asymptotic equivalence which can best be compared with exponential equivalence in classical large deviations theory.

We derive analogous results for random walks in Section 4.1. Random walks cannot be decomposed into independent components with small jumps and large jumps as easily as Lévy processes, making the analysis of random walks more technical if done directly. However, it is possible to follow an indirect approach. Given a random walk $S_k, k\geq 0$, one can study a subordinated version $S_{N(t)}, t\geq 0$ with $N(t), t\geq 0$ an independent unit rate Poisson process. The Skorokhod J_1 distance between rescaled versions of $S_k, k\geq 0$ and $S_{N(t)}, t\geq 0$ can then be bounded in terms of the deviations of $N(t)$ from t, which have been studied thoroughly. We have not seen this generally applicable idea in other studies.

We prove an LDP of the form (1.3) in Section 4.2, where the upper bound requires a compactness assumption. We construct a counterexample showing that the compactness assumption cannot be totally removed, and thus, a full LDP does not hold. Essentially, if a rare event is caused by j big jumps, then the framework developed in this paper applies if each of these jumps is bounded away from below by a strictly positive constant. Our counterexample in Section 4.3 indicates that it is not trivial to remove this condition.

As one may expect, it is not possible to apply classical variational methods to derive an expression for the exponent $\mathcal{J}(A)$, as is often the case in large deviations for light tails. Nevertheless, there seems to be a generic connection with a class of control problems called impulse control problems. Equation (1.5) is a specific deterministic impulse-control problem, which is related to Barles (1985). We expect that techniques similar to those in Barles (1985) will be useful to characterize optimality of solutions to problems like (1.5).

The latter challenge is not taken up in the present study. Instead, in Section 5, we analyse (1.5) directly in a few specific applications. In particular, we consider two applications to financial mathematics, involving the computation of a Value-at-Risk (VaR) measure (Section 5.1), and the valuation of a specific exotic option (Section 5.2). We also demonstrate how to explicitly solve (1.5) in the case where $A = \{\xi : l(t) \le \xi(t) \le u(t)\}$ for some function l and u in Section 5.3. Last, in Section 5.4 we illustrate how to deal with the case where (1.5) has multiple minima.

In each of these examples, a condition needs to be checked to see whether our framework is applicable. We provide a general result that essentially states that we only need to check this condition for step functions in A, which makes this check rather easy in applications. The applications in the present paper mainly serve to illustrate our main results. More involved applications to Lévy-driven stochastic differential equations, stable processes, Markov additive processes, traffic networks, and rare event simulation now seem to be within reach, and will be considered elsewhere.

In summary, this paper is organized as follows. After developing some preliminary results in Section 2, we present our main results in Section 3. Applications to random walks and connections with classical large deviations theory are investigated in Section 4. In Section 5, we consider four applications of our main results. The remaining sections are devoted to proofs. We collect some useful bounds in Appendix A, and Appendix B gives an overview of all notational conventions that are introduced throughout the paper.

2 $\mathbb{M}$-convergence

This section reviews and develops general concepts and tools that are needed to derive our large deviations results. The proofs of the lemmas and corollaries stated throughout this section are deferred until Section 6.1. We start with briefly reviewing the notion of $\mathbb{M}$-convergence, introduced in Lindskog et al. (2014).

Let $(\mathbb{S}, d)$ be a complete separable metric space, and $\mathscr{S}$ be the Borel σ-algebra on $\mathbb{S}$. Given a closed subset $\mathbb{C}$ of $\mathbb{S}$, let $\mathbb{S}\setminus\mathbb{C}$ be equipped with the relative topology as a subspace of $\mathbb{S}$, and consider the associated sub σ-algebra $\mathscr{S}_{\mathbb{S}\setminus\mathbb{C}} \triangleq \{A : A \subseteq \mathbb{S}\setminus\mathbb{C}, A \in \mathscr{S}\}$ on it. Define $\mathbb{C}^r \triangleq \{x \in \mathbb{S} : d(x, \mathbb{C}) < r\}$ for $r \ge 0$, and let $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ be the class of measures defined on $\mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}$ whose restrictions to $\mathbb{S}\setminus\mathbb{C}^r$ are finite for all $r > 0$. Topologize $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ with a sub-basis $\big\{\{\nu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C}) : \nu(f) \in G\}\colon f \in \mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}, G \text{ open in } \mathbb{R}_+\big\}$ where $\mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}$ is the set of real-valued, non-negative, bounded, continuous functions whose support is bounded away from $\mathbb{C}$ (i.e., $f(\mathbb{C}^r) = \{0\}$ for some $r > 0$). A sequence of measures $\mu_n \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ converges to $\mu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ if $\mu_n(f) \to \mu(f)$ for each $f \in \mathcal{C}_{\mathbb{S}\setminus\mathbb{C}}$. Note that this notion of convergence in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ coincides with the classical notion of weak convergence of measures (Billingsley, 2013) if $\mathbb{C}$ is an empty set. An important characterization of $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$-convergence is as follows:

Result 1 (Theorem 2.1 of Lindskog et al., 2014)**.** *Let $\mu, \mu_n \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$. Then $\mu_n \to \mu$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ as $n \to \infty$ if and only if*

$$\limsup_{n\to\infty} \mu_n(F) \le \mu(F) \tag{2.1}$$

for all closed $F \in \mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}$ bounded away from $\mathbb{C}$ and

$$\liminf_{n\to\infty} \mu_n(G) \ge \mu(G) \tag{2.2}$$

for all open $G \in \mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}$ bounded away from $\mathbb{C}$.

We now introduce a new notion of equivalence between two families of random objects, which will prove to be useful in Section 3.1, and Section 3.2. Let $F_\delta \triangleq \{x \in \mathbb{S} : d(x, F) \le \delta\}$ and $G^{-\delta} \triangleq ((G^c)_\delta)^c$. (Compare these notations to $\mathbb{C}^r$; note that we are using the convention that superscript implies open sets and subscript implies closed sets.)

Definition 1. *Suppose that X_n and Y_n are random elements taking values in a complete separable metric space $(\mathbb{S}, d)$. Y_n is said to be asymptotically equivalent to X_n with respect to ϵ_n and $\mathbb{C}$, if, for each $\delta > 0$ and $\gamma > 0$,*

$$\limsup_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(X_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n, Y_n) \geq \delta) = \limsup_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(Y_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n, Y_n) \geq \delta) = 0.$$

Remark 1. *Note that the asymptotic equivalence w.r.t. $\mathbb{C}$ implies the asymptotic equivalence w.r.t. $\mathbb{C}'$ if $\mathbb{C} \subseteq \mathbb{C}'$. In view of this, the strongest notion of asymptotic equivalence w.r.t. a given sequence ϵ_n is the one w.r.t. an empty set. In this case, the conditions for the asymptotic equivalence reduces to a simple condition: $\mathbf{P}(d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$ for any $\delta > 0$. In our context, this simple condition suffices for the case of one-sided Lévy measures in Section 3.1; however, we have to work with the case that $\mathbb{C}$ is not an empty set when we deal with two-sided Lévy processes in Section 3.2.*

The usefulness of this notion of equivalence comes from the following result.

Lemma 2.1. *Suppose that $\epsilon_n^{-1}\mathbf{P}(X_n \in \cdot) \to \mu(\cdot)$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for some sequence ϵ_n and a closed set $\mathbb{C}$. If Y_n is asymptotically equivalent to X_n with respect to ϵ_n and $\mathbb{C}$, then the law of Y_n has the same (normalized) limit, i.e., $\epsilon_n^{-1}\mathbf{P}(Y_n \in \cdot) \to \mu(\cdot)$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$.*

Another useful observation regarding asymptotic equivalence is that one can extend the lower and upper bounds to more general sets, in case there are asymptotically equivalent distributions that are supported on a subspace of the original space.

Lemma 2.2. *Suppose that $\epsilon_n^{-1}\mathbf{P}(X_n \in \cdot) \to \mu(\cdot)$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for some sequence ϵ_n and a closed set $\mathbb{C}$. In addition, suppose that $\mu(\mathbb{S}\setminus\mathbb{S}_0) = 0$ and $\mathbf{P}(X_n \in \mathbb{S}_0) = 1$ for each n. If Y_n is asymptotically equivalent to X_n with respect to ϵ_n and an empty set, then*

$$\liminf_{n\to\infty} \mathbf{P}(X_n \in G) \geq \mu(G)$$

if G is open and $G \cap \mathbb{S}_0$ is bounded away from $\mathbb{C}$;

$$\limsup_{n\to\infty} \mathbf{P}(X_n \in F) \leq \mu(F)$$

if F is closed and there is a $\delta > 0$ such that $F_\delta \cap \mathbb{S}_0$ is bounded away from $\mathbb{C}$.

This lemma is particularly important for the applications in Section 5 of this paper, where it is used multiple times to check the validity of our main results in specific situations.

A version of the continuous mapping principle is satisfied by the convergence in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$. Let $(\mathbb{S}', d')$ be a complete separable metric space, and let $\mathbb{C}'$ be a closed subset of $\mathbb{S}'$.

Result 2 (Mapping theorem; Theorem 2.3 of Lindskog et al. (2014))**.** *Let $h : (\mathbb{S}\setminus\mathbb{C}, \mathscr{S}_{\mathbb{S}\setminus\mathbb{C}}) \to (\mathbb{S}'\setminus\mathbb{C}', \mathscr{S}_{\mathbb{S}'\setminus\mathbb{C}'})$ be a measurable mapping such that $h^{-1}(A')$ is bounded away from $\mathbb{C}$ for any $A' \in \mathscr{S}_{\mathbb{S}'\setminus\mathbb{C}'}$ bounded away from $\mathbb{C}'$. Then $\hat{h} : \mathbb{M}(\mathbb{S}\setminus\mathbb{C}) \to \mathbb{M}(\mathbb{S}'\setminus\mathbb{C}')$ defined by $\hat{h}(\nu) = \nu \circ h^{-1}$ is continuous at μ provided $\mu(D_h) = 0$, where D_h is the set of discontinuity points of h.*

We will need the following slight extension to restrict our attention to cases where the jump times of two-sided Lévy processes do not match.

Lemma 2.3. *Let $\mathbb{S}_0$ be a measurable subset of $\mathbb{S}$, and $h : (\mathbb{S}_0, \mathscr{S}_{\mathbb{S}_0}) \to (\mathbb{S}' \setminus \mathbb{C}', \mathscr{S}'_{\mathbb{S}' \setminus \mathbb{C}'})$ be a measurable mapping such that $h^{-1}(A')$ is bounded away from $\mathbb{C}$ for any $A' \in \mathscr{S}_{\mathbb{S}' \setminus \mathbb{C}'}$ bounded away from $\mathbb{C}'$. Then $\hat{h} : \mathbb{M}(\mathbb{S} \setminus \mathbb{C}) \to \mathbb{M}(\mathbb{S}' \setminus \mathbb{C}')$ defined by $\hat{h}(\nu) = \nu \circ h^{-1}$ is continuous at μ provided that $\mu(\partial \mathbb{S}_0 \setminus \mathbb{C}^r) = 0$ and $\mu(D_h \setminus \mathbb{C}^r) = 0$ for all $r > 0$, where D_h is the set of discontinuity points of h.*

When we focus on Lévy processes, we are specifically interested in the case where $\mathbb{S}$ is $\mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty$ (or $\mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty \times [0,1]^\infty$ in the two-sided case), where $\mathbb{R}_+^{\infty\downarrow} \triangleq \{x \in \mathbb{R}_+^\infty : x_1 \geq x_2 \geq \dots\}$, and $\mathbb{S}'$ is the Skorokhod space $\mathbb{D} = \mathbb{D}([0,1], \mathbb{R})$ — the space of real-valued RCLL functions on $[0,1]$. We use the usual product metrics $d_{\mathbb{R}_+^{\infty\downarrow}}(x,y) = \sum_{i=1}^\infty \frac{|x_i - y_i| \wedge 1}{2^i}$ and $d_{[0,1]^\infty}(x,y) = \sum_{i=1}^\infty \frac{|x_i - y_i|}{2^i}$ for $\mathbb{R}_+^{\infty\downarrow}$ and $[0,1]^\infty$, respectively. For the finite product of metric spaces, we use the maximum metric; i.e., we use $d_{\mathbb{S}_1 \times \dots \times \mathbb{S}_d}((x_1, \dots, x_d), (y_1, \dots, y_d)) \triangleq \max_{i=1,\dots,d} d_{\mathbb{S}_i}(x_i, y_i)$ for the product $\mathbb{S}_1 \times \dots \times \mathbb{S}_d$ of metric spaces $(\mathbb{S}_i, d_{\mathbb{S}_i})$. For $\mathbb{D}$, we use the usual Skorokhod J_1 metric $d(x,y) \triangleq \inf_{\lambda \in \Lambda} \|\lambda - e\| \vee \|x \circ \lambda - y\|$, where Λ denotes the set of all non-decreasing homeomorphisms from $[0,1]$ onto itself, e denotes the identity, and $\|\cdot\|$ denotes the supremum norm. Let

$$S_j \triangleq \{(x,u) \in \mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty : 0, 1, u_1, \dots, u_j \text{ are all distinct}\}.$$

This set will play the role of $\mathbb{S}_0$ of Lemma 2.3. Define $T_j : S_j \to \mathbb{D}$ to be $T_j(x,u) = \sum_{i=1}^j x_i 1_{[u_i,1]}$. Let $\mathbb{D}_j$ be the subspaces of the Skorokhod space consisting of nondecreasing step functions, vanishing at the origin, with exactly j jumps and $\mathbb{D}_{\leqslant j} \triangleq \bigcup_{0 \leq i \leq j} \mathbb{D}_i$—i.e., nondecreasing step functions vanishing at the origin with at most j jumps. Define $\mathbb{H}_j \triangleq \{x \in \mathbb{R}_+^{\infty\downarrow} : x_j > 0, x_{j+1} = 0\}$, and $\mathbb{H}_{\leqslant j} \triangleq \{x \in \mathbb{R}_+^{\infty\downarrow} : x_{j+1} = 0\}$. The continuous mapping principle applies to T_m, as we can see in the following result.

Result 3 (Lemma 5.3 and Lemma 5.4 of Lindskog et al., 2014)**.** *Suppose $A \subset \mathbb{D}$ is bounded away from $\mathbb{D}_{\leqslant j-1}$. Then, $T_j^{-1}(A)$ is bounded away from $\mathbb{H}_{\leqslant j-1} \times [0,1]^\infty$. Moreover, $T_j : S_j \to \mathbb{D}$ is continuous.*

A consequence of Result 3 and Lemma 2.3 is that one can derive a limit theorem in path space from a limit theorem for jump sizes.

Corollary 2.1. *If $\mu_n \to \mu$ in $\mathbb{M}\big((\mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty) \setminus (\mathbb{H}_{\leqslant j-1} \times [0,1]^\infty)\big)$, and $\mu\big(S_j^c \setminus (\mathbb{H}_{\leqslant j-1} \times [0,1]^\infty)^r\big) = 0$ for all $r > 0$, then $\mu_n \circ T_j^{-1} \to \mu \circ T_j^{-1}$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{\leqslant j-1})$.*

To work with two-sided Lévy measures, we need Lemma 2.4 below, which is a two-sided analogue of Corollary 2.1. Let $\mathbb{D}_{l,m}$ denote the subspace of the Skorokhod space consisting of step functions vanishing at the origin with exactly l upward jumps and m downward jumps. Let $\mathbb{H}_{l,m}$ denote the product $\mathbb{H}_l \times \mathbb{H}_m = \{(x,y) \in \mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} : x_l > 0, x_{l+1} = 0, y_m > 0, y_{m+1} = 0\}$. Let $\mathbb{D}_{<j,k} \triangleq \bigcup_{(l,m) \in I_{<j,k}} \mathbb{D}_{l,m}$ and $\mathbb{H}_{<j,k} \triangleq \bigcup_{(l,m) \in I_{<j,k}} \mathbb{H}_{l,m}$, where $I_{<j,k} \triangleq \{(l,m) \in \mathbb{Z}_+^2 \setminus (j,k) : (\alpha - 1)l + (\beta - 1)m \leq (\alpha - 1)j + (\beta - 1)k\}$ and $\mathbb{Z}_+$ denotes the set of non-negative integers. Define $T_{j,k} : S_{j,k} \to \mathbb{D}$ as

$$T_{j,k}(x,y,u,v) = \sum_{i=1}^j x_i 1_{[u_i,1]} - \sum_{i=1}^k y_i 1_{[v_i,1]},$$

where $S_{j,k} \triangleq \{(x,y,u,v) \in \mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty \times [0,1]^\infty : 0, 1, u_1, \dots, u_j, v_1, \dots, v_k \text{ are all distinct}\}$.

Lemma 2.4. *For $j,k \geq 0$, $T_{j,k} : S_{j,k} \to \mathbb{D}$ is continuous. Furthermore, suppose $A \subset \mathbb{D}$ is bounded away from $\mathbb{D}_{<j,k}$. Then $T_{j,k}^{-1}(A)$ is bounded away from $\mathbb{H}_{<j,k} \times [0,1]^\infty \times [0,1]^\infty$. Therefore, if $\mu_n \to \mu$ in $\mathbb{M}\big((\mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty \times [0,1]^\infty) \setminus (\mathbb{H}_{<j,k} \times [0,1]^\infty \times [0,1]^\infty)\big)$ and $\mu\big(S_{j,k}^c \setminus (\mathbb{H}_{<j,k} \times [0,1]^\infty \times [0,1]^\infty)^r\big) = 0$ for all $r > 0$, then $\mu_n \circ T_{j,k}^{-1} \to \mu \circ T_{j,k}^{-1}$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$.*

We next characterize convergence-determining classes for the convergence in $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$.

Lemma 2.5. *Suppose*
(i) $\mathcal{A}_p$ *is a* π*-system;*
(ii) *each open set* $G \subseteq \mathbb{S}$ *bounded away from* $\mathbb{C}$ *is a countable union of sets in* $\mathcal{A}_p$*;*
(iii) *for each closed set* $F \subseteq \mathbb{S}$ *bounded away from* $\mathbb{C}$*, there is a set* $A \in \mathcal{A}_p$ *bounded away from* $\mathbb{C}$ *such that* $F \subseteq A^\circ$ *and* $\mu(A \setminus A^\circ) = 0$.
If, in addition, $\mu \in \mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ *and* $\mu_n(A) \to \mu(A)$ *for every* $A \in \mathcal{A}_p$ *such that* A *is bounded away from* $\mathbb{C}$*, then* $\mu_n \to \mu$ *in* $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$.

Remark 2. *Since* $\mathbb{S}$ *is a separable metric space, the Lindelöf property holds. Therefore, a sufficient condition for assumption (ii) of Lemma 2.5 is that for every* $x \in \mathbb{S} \setminus \mathbb{C}$ *and* $\epsilon > 0$*, there is* $A \in \mathcal{A}_p$ *such that* $x \in A^\circ \subseteq B(x, \epsilon)$*. To see that this implies assumption (ii), note that for any given open set* G*, one can construct a cover* $\{(A_x)^\circ : x \in G\}$ *of* G *by choosing* A_x *so that* $x \in (A_x)^\circ \subseteq G$ *and then extract a countable subcover (due to the Lindelöf property) whose union is equal to* G*. Note also that if* A *in assumption (iii) is open, then* $\mu(A \setminus A^\circ) = \mu(\emptyset) = 0$ *automatically.*

3 Sample-Path Large Deviations

In this section, we present large-deviations results for scaled Lévy processes with heavy-tailed Lévy measures. Section 3.1 studies a special case, where the Lévy measure is concentrated on the positive part of the real line, and Section 3.2 extends this result to Lévy processes with two-sided Lévy measures. In both cases, let $X_n(t) \triangleq X(nt)$ be a scaled process of X, where X is a Lévy process with a Lévy measure ν. Recall that X_n has Itô representation:

$$X_n(s) = nsa + B(ns) + \int_{|x|\le 1} x[N([0,ns] \times dx) - ns\nu(dx)] + \int_{|x|>1} xN([0,ns] \times dx), \tag{3.1}$$

with a a drift parameter, B a Brownian motion, and N a Poisson random measure with mean measure Leb$\times\nu$ on $[0,n] \times (0,\infty)$; Leb denotes the Lebesgue measure.

3.1 One-sided Large Deviations

Let X be a Lévy process with Lévy measure ν. In this section, we assume that ν is a regularly varying (at infinity, with index $-\alpha < -1$) Lévy measure concentrated on $(0,\infty)$. Consider a centered and scaled process

$$\bar{X}_n(s) \triangleq \frac{1}{n} X_n(s) - sa - \mu_1^+ \nu_1^+ s,$$

where $\mu_1^+ \triangleq \frac{1}{\nu_1^+} \int_{[1,\infty)} x\nu(dx)$, and $\nu_1^+ \triangleq \nu[1,\infty)$. Let ν_α^j denote the restriction (to $\mathbb{R}_+^{j\downarrow}$) of the j-fold product measure of ν_α, where $\nu_\alpha(x,\infty) \triangleq x^{-\alpha}$. Let $C_0(\cdot) \triangleq \delta_{\mathbf{0}}(\cdot)$ be the Dirac measure concentrated on the zero function. Additionally, for each $j \ge 1$, define a measure $C_j(\cdot) \triangleq \mathbf{E}\Big[\nu_\alpha^j\{y \in (0,\infty)^j : \sum_{i=1}^j y_i 1_{[U_i,1]} \in \cdot\}\Big]$ concentrated on $\mathbb{D}_j$, where the random variables $U_i, i \ge 1$ are i.i.d. uniform on $[0,1]$. Let $\mathbb{D}_s^\uparrow$ denote the subset of $\mathbb{D}$ consisting of non-decreasing step functions vanishing at the origin, and let $\mathcal{D}_+(\xi)$ denote the number of upward jumps of an element ξ in $\mathbb{D}$. Finally, set

$$\mathcal{J}(A) \triangleq \inf_{\xi \in \mathbb{D}_s^\uparrow \cap A} \mathcal{D}_+(\xi). \tag{3.2}$$

The main result of this section is the following large-deviations theorem for $\bar{X}_n$.

Theorem 3.1. *Suppose that A is a measurable set. If $\mathcal{J}(A) < \infty$, and if A is bounded away from $\mathbb{D}_{\leqslant \mathcal{J}(A)-1}$, then*

$$C_{\mathcal{J}(A)}(A^\circ) \le \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le C_{\mathcal{J}(A)}(A^-). \tag{3.3}$$

Proof. Note first that $\mathcal{J}(A^\circ) > \mathcal{J}(A)$ implies that A° doesn't contain any element of $\mathbb{D}_{\leqslant \mathcal{J}(A)}$. Hence, A° is a $C_{\mathcal{J}(A)}$-null set, since $C_{\mathcal{J}(A)}$ is supported on $\mathbb{D}_{\leqslant \mathcal{J}(A)}$. Therefore, the lower bound holds trivially if $\mathcal{J}(A^\circ) > \mathcal{J}(A)$. On the other hand, $\mathcal{J}(A) = \mathcal{J}(A^-)$, since A is bounded away from $\mathbb{D}_{\leqslant \mathcal{J}(A)}$. In view of these observations, we can assume w.l.o.g. that $\mathcal{J}(A^\circ) = \mathcal{J}(A) = \mathcal{J}(A^-)$. Theorem 3.1 is now an immediate consequence of Theorem 3.2, given below. □

Remark 3. *In the proof of Theorem 3.2, we establish the asymptotic equivalence (w.r.t. an empty set) of $\bar{X}_n$ to a process that is supported on $\mathbb{D}_{\mathcal{J}(A)}$. Therefore, Lemma 2.2 applies, and (3.3) remains valid for all sets A such that $A_\delta \cap \mathbb{D}_{\mathcal{J}(A)}$ is bounded away from $\mathbb{D}_{\leqslant \mathcal{J}(A)-1}$ for some $\delta > 0$.*

Remark 4. *If $\mathcal{J}(A) = \infty$, and A is bounded away from $\mathbb{D}_{\leqslant i-1}$ for some $i \ge 1$, then Theorem 3.2 applies with $j = i$ to give that $(n\nu[n,\infty))^{-i}\mathbf{P}(\bar{X}_n \in A) \to 0$.*

Theorem 3.2. *For each $j \ge 0$,*

$$(n\nu[n,\infty))^{-j}\mathbf{P}(\bar{X}_n \in \cdot) \to C_j(\cdot), \tag{3.4}$$

in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{\leqslant j-1})$, as $n \to \infty$.

Proof Sketch. The proof of Theorem 3.2 is based on establishing the asymptotic equivalence of $\bar{X}_n$ and the process obtained by just keeping its j biggest jumps, which we will denote by $\hat{J}_n^{\leqslant j}$ in Section 6. Such an equivalence is established via Proposition 6.1, and Proposition 6.2. Then, Proposition 6.3 identifies the limit of $\hat{J}_n^{\leqslant j}$, which coincides with the limit in (3.4). The full proof of Theorem 3.2 is provided in Section 6.2. □

Theorem 3.1 dictates the "right" choice of j in Theorem 3.2 for which (3.4) can lead to a limit in $(0,\infty)$. We conclude this section with an investigation of a sufficient condition for C_j-continuity; i.e., we provide a sufficient condition on A which guarantees $C_j(\partial A) = 0$. The latter property implies

$$C_j(A^\circ) = C_j(A) = C_j(A^-), \tag{3.5}$$

implying that the liminf and limsup in our asymptotic estimates yield the same result. Assume that A is a subset of $\mathbb{D}_j$ bounded away from $\mathbb{D}_{\leqslant j-1}$; i.e., $d(A, \mathbb{D}_{\leqslant j-1}) > \gamma$ for some $\gamma > 0$. Consider a path $\xi \in A$. Note that every $\xi \in \mathbb{D}_j$ is determined by the pair of jump sizes and jump times $(x,u) \in (0,\infty)^j \times [0,1]^j$; i.e., $\xi(t) = \sum_{i=1}^j x_i 1_{[u_i,1]}(t)$. Formally, we define a mapping $\hat{T}_j : \hat{S}_j \to \mathbb{D}_j$ by $\hat{T}_j(x,u) = \sum_{i=1}^j x_i 1_{[u_i,1]}$, where $\hat{S}_j \triangleq \{(x,u) \in \mathbb{R}_+^{j\downarrow} \times [0,1]^j : 0, 1, u_1, \ldots, u_j \text{ are all distinct}\}$. Since $d(A, \mathbb{D}_{\leqslant j-1}) > \gamma$, we know that $\hat{T}_j(x,u) \in A$ implies $x \in (\gamma,\infty)^j$; see Lemma 6.4. In view of this, we can see that (3.5) holds if the Lebesgue measure of $\hat{T}_j^{-1}(\partial A)$ is 0 since $C_j(A) = \int_{(x,u)\in \hat{T}_j^{-1}(A)} du d\nu_\alpha^j(x)$. As we will see in Section 5, one of the typical settings that arises in applications is that the set A can be written as a finite combination of unions and intersections of $\phi_1^{-1}(A_1), \ldots, \phi_m^{-1}(A_m)$, where each $\phi_i : \mathbb{D} \to \mathbb{S}_i$ is a continuous function, and all sets A_i are subsets of general topological space $\mathbb{S}_i$. If we denote this operation of taking unions and intersections by Ψ (i.e., $A = \Psi(\phi_1^{-1}(A_1), \ldots, \phi_m^{-1}(A_m))$), then

$$\Psi(\phi_1^{-1}(A_1^\circ), \ldots, \phi_m^{-1}(A_m^\circ)) \subseteq A^\circ \subseteq A \subseteq A^- \subseteq \Psi(\phi_1^{-1}(A_1^-), \ldots, \phi_m^{-1}(A_m^-)).$$

Therefore, (3.5) holds if $\hat{T}_j^{-1}(\Psi(\phi_1^{-1}(A_1^-), \ldots, \phi_m^{-1}(A_m^-))) \setminus \hat{T}_j^{-1}(\Psi(\phi_1^{-1}(A_1^\circ), \ldots, \phi_m^{-1}(A_m^\circ)))$ has Lebesgue measure zero. A similar principle holds for the limit measures $C_{j,k}$, defined in the next section where we deal with two-sided Lévy processes. For more concrete examples, see Section 5.1 and Section 5.2.

3.2 Two-sided Large Deviations

Consider a two-sided Lévy measure ν for which $\nu[x,\infty)$ is regularly varying with index $-\alpha$ and $\nu(-\infty,-x]$ is regularly varying with index $-\beta$. Let

$$\bar{X}_n(s) \triangleq \frac{1}{n}X_n(s) - sa - (\mu_1^+\nu_1^+ - \mu_1^-\nu_1^-)s,$$

where

$$\mu_1^+ \triangleq \frac{1}{\nu_1^+}\int_{[1,\infty)} x\nu(dx), \qquad \nu_1^+ \triangleq \nu[1,\infty), \qquad \mu_1^- \triangleq \frac{-1}{\nu_1^-}\int_{(-\infty,-1]} x\nu(dx), \qquad \nu_1^- \triangleq \nu(-\infty,-1].$$

The limit measures $C_{j,k}$ in the main results of this section are concentrated on $\mathbb{D}_{j,k}$, which we define as the subspace of $\mathbb{D}$, consisting of step functions vanishing at the origin with exactly j upward jumps and k downward jumps.

Let ν_α^j be as defined in Section 3.1. Similarly, let ν_β^k denote the restriction (to $\mathbb{R}_+^{k\downarrow}$) of the k-fold product measure of ν_β, where $\nu_\beta(x,\infty) \triangleq x^{-\beta}$. Let $C_{0,0}(\cdot) \triangleq \delta_{\mathbf{0}}(\cdot)$ be the Dirac measure concentrated on the zero function. For each $(j,k) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$, define a measure $C_{j,k}(\cdot) \triangleq \mathbf{E}\Big[\nu_\alpha^j \times \nu_\beta^k\{(x,y) \in (0,\infty)^j \times (0,\infty)^k : \sum_{i=1}^j x_i 1_{[U_i,1]} - \sum_{i=1}^k y_i 1_{[V_i,1]} \in \cdot\}\Big]$ concentrated on $\mathbb{D}_{j,k}$, where U_i's and V_i's are i.i.d. uniform on $[0,1]$. Recall that $\mathbb{D}_{<j,k} = \bigcup_{(l,m)\in I_{<j,k}} \mathbb{D}_{l,m}$ and $I_{<j,k} = \{(l,m) \in \mathbb{Z}_+^2 \setminus (j,k) : (\alpha-1)l + (\beta-1)m \leq (\alpha-1)j + (\beta-1)k\}$. Let $\mathcal{I}(j,k) \triangleq (\alpha-1)j + (\beta-1)k$, and consider

$$(\mathcal{J}(A), \mathcal{K}(A)) \in \operatorname*{arg\,min}_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A \neq \emptyset}} \mathcal{I}(j,k). \tag{3.6}$$

The next theorem applies to the case where the minimizing argument in (3.6) is a single pair (which is implied by its assumption).

Theorem 3.3. *Suppose that A is a measurable set. If the argument minimum in (3.6) is non-empty and A is bounded away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$, then*

$$\begin{aligned} C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ) &\leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}(n\nu(-\infty,-n])^{\mathcal{K}(A)}} \\ &\leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}(n\nu(-\infty,-n])^{\mathcal{K}(A)}} \leq C_{\mathcal{J}(A),\mathcal{K}(A)}(A^-). \end{aligned} \tag{3.7}$$

Proof. Note that, in general,

$$\min_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A^- \neq \emptyset}} \mathcal{I}(j,k) \leq \mathcal{I}(\mathcal{J}(A),\mathcal{K}(A)) \leq \min_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A^\circ \neq \emptyset}} \mathcal{I}(j,k),$$

and the left inequality cannot be strict since A is bounded away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$. On the other hand, if the right inequality is strict, then $\mathbb{D}_{\mathcal{J}(A),\mathcal{K}(A)} \cap A^\circ = \emptyset$, which in turn implies $C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ) = 0$, since $C_{\mathcal{J}(A),\mathcal{K}(A)}$ is supported on $\mathbb{D}_{\mathcal{J}(A),\mathcal{K}(A)}$. Therefore, the lower bound is trivial if the right inequality is strict. In view of these observations, we can assume w.l.o.g. that $(\mathcal{J}(A),\mathcal{K}(A))$ is also in both $\operatorname*{arg\,min}_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A^\circ\neq\emptyset}} \mathcal{I}(j,k)$ and $\operatorname*{arg\,min}_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A^-\neq\emptyset}} \mathcal{I}(j,k)$. Now, (3.7) is an immediate consequence of Theorem 3.5, given below. ∎

Remark 5. *If the argument minimum in (3.6) is empty and A is bounded away from $\mathbb{D}_{<l,m}$ for some $(l,m) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$, then Theorem 3.5 applies with $(j,k) = (l,m)$ to give $(n\nu[n,\infty))^{-l}(n\nu(-\infty,-n])^{-m}$ $\mathbf{P}(\bar{X}_n \in A) \to 0$ as $n \to \infty$.*

In case one is interested in a set for which the arg min of $\mathcal{I}$ in (3.6) is not unique, a natural approach is to partition A into smaller sets and analyze each element separately. In the next theorem, we show that this strategy can be successfully employed with a minimal requirement on A. However, due to the presence of two different slowly varying functions $n^\alpha\nu[n,\infty)$ and $n^\beta\nu(-\infty,-n]$, the limit behavior may not be dominated by a single $\mathbb{D}_{l,m}$.

To deal with this case, let $I_{=j,k} \triangleq \{(l,m) : (\alpha-1)l + (\beta-1)m = (\alpha-1)j + (\beta-1)k\}$, $I_{\ll j,k} \triangleq \{(l,m) : (\alpha-1)l + (\beta-1)m < (\alpha-1)j + (\beta-1)k\}$, $\mathbb{D}_{=j,k} \triangleq \bigcup_{(l,m)\in I_{=j,k}} \mathbb{D}_{l,m}$, and $\mathbb{D}_{\ll j,k} \triangleq \bigcup_{(l,m)\in I_{\ll j,k}} \mathbb{D}_{l,m}$. Denote the slowly varying functions $n^\alpha\nu[n,\infty)$ and $n^\beta\nu(-\infty,-n]$ by $L_+(n)$ and $L_-(n)$, respectively.

Theorem 3.4. *Suppose that A is a measurable set. If the argument minimum in (3.6) is non-empty and A is bounded away from $\mathbb{D}_{\ll \mathcal{J}(A),\mathcal{K}(A)}$, then for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\frac{\sum_{(l,m)} \big(C_{l,m}(A^\circ \cap \mathbb{D}_{l,m}) - \epsilon\big) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}} \le \mathbf{P}(\bar{X}_n \in A) \le \frac{\sum_{(l,m)} \big(C_{l,m}(A^- \cap \mathbb{D}_{l,m}) + \epsilon\big) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}}$$

for all $n \ge N$, where the summations are over the pairs $(l,m) \in I_{=\mathcal{J}(A),\mathcal{K}(A)}$.

Proof. Let $(l,m) \in I_{=\mathcal{J}(A),\mathcal{K}(A)}$. We first claim that A being bounded away from $\mathbb{D}_{\ll \mathcal{J}(A),\mathcal{K}(A)}$ implies that for any $(j,k) \in I_{=\mathcal{J}(A),\mathcal{K}(A)} \setminus \{(l,m)\}$, there exists $\delta > 0$ such that $A \cap (\mathbb{D}_{l,m})_\delta$ is bounded away from $\mathbb{D}_{j,k}$. We will justify this claim at the end of the proof of this theorem. From that claim, one can choose δ so that $A \cap (\mathbb{D}_{l,m})_\delta$ is bounded away from the entire $\mathbb{D}_{<l,m}$. To derive the lower bound, we first apply Theorem 3.3 to $A^\circ \cap (\mathbb{D}_{l,m})^{-\delta}$ and obtain

$$C_{l,m}(A^\circ \cap (\mathbb{D}_{l,m})^{-\delta}) \le \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A^\circ \cap (\mathbb{D}_{l,m})^{-\delta})}{(n\nu[n,\infty))^l (n\nu(-\infty,-n])^m} \le \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A \cap \mathbb{D}_{l,m})}{(n\nu[n,\infty))^l (n\nu(-\infty,-n])^m}.$$

Taking $\delta \to 0$, we obtain

$$C_{l,m}(A^\circ \cap \mathbb{D}_{l,m}) \le \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A \cap \mathbb{D}_{l,m})}{(n\nu[n,\infty))^l (n\nu(-\infty,-n])^m}.$$

That is, for any given $\epsilon > 0$, there exists an $N_{l,m} \in \mathbb{N}$ such that

$$\frac{\big(C_{l,m}(A^\circ \cap \mathbb{D}_{l,m}) - \epsilon\big) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)l+(\beta-1)m}} \le \mathbf{P}(\bar{X}_n \in A \cap \mathbb{D}_{l,m}), \tag{3.8}$$

for all $n \ge N_{l,m}$. Meanwhile, an obvious bound holds for $A \setminus \bigcup_{(l,m)\in I_{=\mathcal{J}(A),\mathcal{K}(A)}} \mathbb{D}_{l,m}$; i.e.,

$$0 \le \mathbf{P}\left(\bar{X}_n \in A \setminus \textstyle\bigcup_{(l,m)\in I_{=\mathcal{J}(A),\mathcal{K}(A)}} \mathbb{D}_{l,m}\right). \tag{3.9}$$

Since $(\alpha-1)l + (\beta-1)m = (\alpha-1)\mathcal{J}(A) + (\beta-1)\mathcal{K}(A)$ for $(l,m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$, summing (3.8) over $(l,m) \in I_{=\mathcal{J}(A),\mathcal{K}(A)}$ together with (3.9), we arrive at the lower bound of the theorem, with $N = \max_{(l,m)\in I_{=\mathcal{J}(A),\mathcal{K}(A)}} N_{l,m}$. Turning to the upper bound, we apply Theorem 3.3 to $A^- \cap (\mathbb{D}_{l,m})_\delta$ to get

$$\limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A^- \cap (\mathbb{D}_{l,m})_\delta)}{(n\nu[n,\infty))^l (n\nu(-\infty,-n])^m} \le C_{l,m}(A^- \cap (\mathbb{D}_{l,m})_\delta).$$

That is, for any given $\epsilon > 0$, there exists $N'_{l,m} \in \mathbb{N}$ such that

$$\mathbf{P}(\bar{X}_n \in A \cap (\mathbb{D}_{l,m})_\delta) \le \frac{\big(C_{l,m}(A^- \cap (\mathbb{D}_{l,m})_\delta) + \epsilon/2\big) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}}, \tag{3.10}$$

for all $n \ge N'_{l,m}$. On the other hand, since $A^- \setminus \bigcup_{(l,m)\in I_{=\mathcal{J}(A),\mathcal{K}(A)}} (\mathbb{D}_{l,m})^\delta$ is closed and bounded away from $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$,

$$\limsup_{n\to\infty} \frac{\mathbf{P}\left(\bar{X}_n \in A \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\delta\right)}{(n\nu[n,\infty))^{\mathcal{J}(A)}(n\nu(-\infty,-n])^{\mathcal{K}(A)}} \le C_{\mathcal{J}(A),\mathcal{K}(A)}\left(A^- \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\delta\right),$$

where the union is over the pairs $(l,m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$. Therefore, there exists N' such that

$$\begin{aligned}\mathbf{P}\left(\bar{X}_n \in A \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\delta\right) &\le \frac{\left(C_{\mathcal{J}(A),\mathcal{K}(A)}\left(A^- \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\delta\right) + \epsilon/2\right) L_+^{\mathcal{J}(A)}(n) L_-^{\mathcal{K}(A)}(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}} \\ &= \frac{(\epsilon/2)\, L_+^{\mathcal{J}(A)}(n) L_-^{\mathcal{K}(A)}(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}},\end{aligned} \tag{3.11}$$

since $A^- \setminus \bigcup_{(l,m)} (\mathbb{D}_{l,m})^\delta$ is disjoint from the support of $C_{\mathcal{J}(A),\mathcal{K}(A)}$. Summing (3.10) over $(l,m) \in I_{=\mathcal{J}(A),\mathcal{K}(A)}$ and (3.11),

$$\mathbf{P}(\bar{X}_n \in A) \le \frac{\sum_{(l,m)} \big(C_{l,m}\big(A^- \cap (\mathbb{D}_{l,m})_\delta\big) + \epsilon\big) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}}, \tag{3.12}$$

for $n \ge N$, where $N = N' \vee \max_{(l,m)\in I_{=\mathcal{J}(A),\mathcal{K}(A)}} N'_{l,m}$. Taking $\delta \to 0$, we obtain the upper bound of the theorem.

Now, we are left with justifying the claim made at the beginning of this proof. To prove the claim, suppose that (l,m) and (j,k) are two distinct pairs that belong to $I_{=\mathcal{J}(A),\mathcal{K}(A)}$ and assume w.l.o.g. that $j < l$. (If $j > l$, it should be the case that $k < m$, and hence one can proceed similarly by switching the roles of upward jumps and downward jumps in the following argument.) Suppose also that $d\big(A, \mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}\big) > \gamma$ for some $\gamma > 0$, and $\xi \in A \cap (\mathbb{D}_{l,m})_\delta$, where $\gamma = c\delta$ for some large $c > 0$ (we will see how large c has to be later). Then, there exists a $\zeta \in \mathbb{D}_{l,m}$ such that $d(\zeta,\xi) \le 2\delta$. Note that $d\big(\zeta, \mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}\big) \ge (c-2)\delta$; in particular, $d\big(\zeta, \mathbb{D}_{j,m}\big) \ge (c-2)\delta$. If we write $\zeta \triangleq \sum_{i=1}^l x_i 1_{[u_i,1]} - \sum_{i=1}^m y_i 1_{[v_i,1]}$, this implies that $x_{j+1} \ge \frac{(c-2)\delta}{l-j}$. Otherwise, $(c-2)\delta > \sum_{i=j+1}^l x_i = \|\zeta - \zeta'\| \ge d(\zeta,\zeta')$, where $\zeta' \triangleq \zeta - \sum_{i=j+1}^l x_i 1_{[u_i,1]} \in \mathbb{D}_{j,m}$. Therefore, $d(\zeta, \mathbb{D}_{j,k}) \ge \frac{(c-2)\delta}{2(l-j)}$, which in turn implies $d(\xi, \mathbb{D}_{j,k}) \ge \frac{(c-2)\delta}{2(l-j)} - 2\delta$. In Conclusion, by picking a large enough c so that $\frac{(c-2)\delta}{2(l-j)} - 2\delta > 0$, one can make $A \cap (\mathbb{D}_{l,m})_\delta$ bounded away from $\mathbb{D}_{j,k}$. □

Remark 6. *As in the one-sided case, we can relax the condition that A is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$. However, we cannot resort to Lemma 2.2 to attain the most useful form of such an extension, since the asymptotic equivalence established in the proof of Theorem 3.5 is w.r.t. $\mathbb{D}_{<j,k}$, rather than an empty set (apart from the problem of dealing with the slowly varying functions). Instead, we can take a more direct approach, taking advantage of the specific properties of the $\mathbb{D}_{l,m}$'s. Note first that $(\mathbb{D}_{\ll j,k})_\delta \subseteq (\mathbb{D}_{=j,k})_\delta$, and hence $A^- \setminus \bigcup_{(l,m)\in I_{=\mathcal{J}(A),\mathcal{K}(A)}} (\mathbb{D}_{l,m})^\delta$ is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$, for all A. In addition, if $A \cap (\mathbb{D}_{l,m})_\delta$ is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$, then there exists $\delta' > 0$ such that $A \cap (\mathbb{D}_{l,m})_{\delta'}$ is bounded away from $\mathbb{D}_{<l,m}$ by the claim stated at the beginning of the proof. With these two observations, one can check that the proof of Theorem 3.4 is still valid (without assuming that the entire A is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$), as long as there exists a $\delta > 0$ such that $A \cap (\mathbb{D}_{l,m})_\delta$ is bounded away from $\mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$ for each $(l,m) \in I_{=\mathcal{J}(A),\mathcal{K}(A)}$. Now, since the existence of $\delta > 0$ such that $d\big(A \cap (\mathbb{D}_{l,m})_\delta, \mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}\big) > 0$ is implied by the existence of $\delta > 0$ such that $d\big(A_\delta \cap \mathbb{D}_{l,m}, \mathbb{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}\big) > 0$, we can conclude that Theorem 3.4 applies if there exists $\delta > 0$ such that $A_\delta \cap \mathbb{D}_{=\mathcal{J}(A),\mathcal{K}(A)}$ is bounded away from $\mathcal{D}_{\ll\mathcal{J}(A),\mathcal{K}(A)}$.*

Remark 7. *If the argument minimum in (3.6) is empty and A is bounded away from* $\mathbb{D}_{\ll j,k}$ *for some* $(j,k) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$*, then a similar argument as in the proof of Theorem 3.4 leads to*

$$\frac{n^{(\alpha-1)j+(\beta-1)k}}{\max_{(l,m)\in I_{=j,k}} L_+^l(n)L_-^m(n)}\mathbf{P}(\bar{X}_n \in A) \to 0.$$

Theorem 3.5. *For each* $(j,k) \in \mathbb{Z}_+^2$*,*

$$(n\nu[n,\infty))^{-j}(n\nu(-\infty,-n])^{-k}\mathbf{P}(\bar{X}_n \in \cdot) \to C_{j,k}(\cdot) \tag{3.13}$$

in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$ *as* $n \to \infty$*.*

Proof Sketch. In view of Lemma 6.3, our task is to prove that $\bar{X}_n$ is asymptotically equivalent to $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k}$ w.r.t. $(n\nu[n,\infty))^j(n\nu(-\infty,-n])^k$ and $\mathbb{D}_{<j,k}$, where $\hat{J}_n^{\leqslant j}$ and $\hat{K}_n^{\leqslant k}$ are, roughly speaking, the processes obtained by keeping the j and k largest jumps of J_n/n and K_n/n, respectively (their precise definitions are given right above Proposition 6.4 in Section 6). Once we have this equivalence, the conclusion of the theorem is immediate from Lemma 6.3. The argument for asymptotic equivalence is essentially identical to that of Theorem 3.2, except for Proposition 6.4, which corresponds to Proposition 6.2 for one-sided Lévy measures. Therefore, we provide Proposition 6.4 in Section 6.2, and omit the rest of the proof. □

4 Implications

This section explores the implications of the large-deviations results in Section 3, and is organized as follows. Section 4.1 proves a result similar to Theorem 3.3, now focusing on random walks with heavy-tailed increments. Section 4.2 develops a weak large deviation priciple (LDP) of the form (1.3) for the scaled Lévy processes. Finally, Section 4.3 shows that the weak LDP proved in Section 4.2 is the best one can hope for in the presence of regularly varying tails, by showing that a full LDP of the form (1.3) does not exist.

4.1 Random Walks

Let $S_k, k \geq 0$, be a random walk, set $\bar{S}_n(t) = S_{[nt]}/n, t \geq 0$, and define $S_n = \{S_n(t), t \in [0,1]\}$. Let $N(t), t \geq 0$, be an independent unit rate Poisson process. Define the Lévy process $X(t) \triangleq S_{N(t)}, t \geq 0$, and set $\bar{X}_n(t) \triangleq X(nt)/n, t \geq 0$. The goal is to prove an analogue of Theorem 3.3 for the scaled random walk $\bar{S}_n$. Let $\mathcal{J}(\cdot)$, $\mathcal{K}(\cdot)$, and $C_{j,k}(\cdot)$ be defined as in Section 3.2.

Theorem 4.1. *Suppose that* $\mathbf{P}(S_1 \geq x)$ *is regularly varying with index* $-\alpha$ *and* $\mathbf{P}(S_1 \leq -x)$ *is regularly varying with index* $-\beta$*. Let A be a measurable set bounded away from* $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$*. Then*

$$\begin{aligned} C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ) &\leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{(nP(S_1 \geq n)))^{\mathcal{J}(A)}(nP(S_1 \leq -n))^{\mathcal{K}(A)}} \\ &\leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{(nP(S_1 \geq n))^{\mathcal{J}(A)}(nP(S_1 \leq -n))^{\mathcal{K}(A)}} \leq C_{\mathcal{J}(A),\mathcal{K}(A)}(A^-). \end{aligned} \tag{4.1}$$

Proof. The idea is to combine our notion of asymptotic equivalence with Theorem 3.3. First, we need to derive the asymptotic behavior of the Lévy measure of the constructed Lévy process. From Example A3.17 in Embrechts et al. (1997), we obtain $P(X(1) \geq x) \sim P(S_1 \geq x)$. Moreover, Embrechts et al. (1979) implies that $\nu(x,\infty) \sim \mathbf{P}(X(1) \geq x)$. Similarly, it follows that $\nu(-\infty,-x) \sim P(S_1 \leq -x)$.

Second, note that the proof of Theorem 3.3 now carries over without modification if (3.13) holds for $\bar{S}_n$ also. In view of Lemma 2.1, the proof will be completed if we prove the asymptotic equivalence between $\bar{X}_n$ and $\bar{S}_n$ (w.r.t. a geometrically decaying sequence and the empty set). To prove the asymptotic equivalence, we first argue that the Skorokhod distance between $\bar{S}_n$ and $\bar{X}_n$ is bounded by $\sup_{t\in[0,1]}|N(tn)/n - t|$. To see this, define the homeomorphism $\lambda_n(t)$ as the linear interpolation of the jump points of $N(nt)/n$, and observe that $\bar{X}_n(t) = \bar{S}_n(\lambda_n(t))$. Thus, the distance between $\bar{S}_n$ and $\bar{X}_n$ is bounded by $\sup_{t\in[0,1]}|\lambda_n(t) - t|$ which, in itself, is bounded by $\sup_{t\in[0,1]}|N(tn)/n - t|$. From Lemma A.1,

$$\mathbf{P}(\sup_{t\in[0,1]}|N(tn)/n - t|) > \delta) \le 3 \sup_{t\in[0,1]} \mathbf{P}(|N(tn)/n - t|) > \delta/3),$$

where $\mathbf{P}(|N(tn)/n - t|) > \delta/3)$ vanishes at a geometric rate w.r.t. n uniform in $t \in [0,1]$, from which the asymptotic equivalence follows. □

4.2 Large Deviation Principle

In this section, we show that $\bar{X}_n$ satisfies a weak large deviation principle with speed $\log n$, and a rate function which is piece-wise linear in the number of discontinuities. More specifically, define

$$I(\xi) \triangleq \begin{cases} (\alpha-1)\mathcal{D}_+(\xi) + (\beta-1)\mathcal{D}_-(\xi), & \text{if } \xi \text{ is a step function with } \xi(0)=0, \\ \infty, & \text{otherwise.} \end{cases} \quad (4.2)$$

Theorem 4.2. *The scaled process $\bar{X}_n$ satisfies the weak large deviation principle with rate function I and speed $\log n$, i.e.,*

$$-\inf_{x\in G} I(x) \le \liminf_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in G)}{\log n} \quad (4.3)$$

for every open set G, and

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\bar{X}_n \in K)}{\log n} \le -\inf_{x\in K} I(x) \quad (4.4)$$

for every compact set K.

The proof of Theorem 4.2 is provided in Section 6. It is based on Theorem 3.3, and a reduction of the case of general A to open neighborhoods; reminiscent of arguments made in the proof of Cramérs theorem Dembo and Zeitouni (2009).

4.3 Nonexistence of Strong Large Deviation Principle

We conclude the current section by showing that the weak LDP presented in the previous section is the best one can hope for in our setting, in the sense that for any Lévy process X with a regularly varying Lévy measure, $\bar{X}_n$ cannot satisfy a strong LDP; i.e., (4.4) in Theorem 4.2 cannot be extended to all closed sets.

Consider a mapping $\pi : \mathbb{D} \to \mathbb{R}_+^2$ that maps paths in $\mathbb{D}$ to their largest jump sizes, i.e.,

$$\pi(\xi) \triangleq \Big(\sup_{t\in(0,1]} \big(\xi(t) - \xi(t-)\big), \sup_{t\in(0,1]} \big(\xi(t-) - \xi(t)\big) \Big).$$

Note that π is continuous, since each coordinate is continuous: for example, if the first coordinate (the largest upward jump sizes) of $\pi(\xi)$ and $\pi(\zeta)$ differ by ϵ then $d(\xi,\zeta) \ge \epsilon/2$, which implies that the first coordinate is continuous. Now, to derive a contradiction, suppose that $\bar{X}_n$ satisfies a strong LDP. In particular, suppose (4.4) in Theorem 4.2 is true for all closed sets rather than just compact sets. Since π is continuous w.r.t. the J_1 metric, $\pi(\bar{X}_n)$ has to satisfy a strong LDP with rate function $I'(y) = \inf\{I(\xi) : \xi \in \mathbb{D}, y = \pi(x)\}$

by the contraction principle, in case I' is a rate function. (Since I is not a good rate function, I' is not automatically guaranteed to be a rate function per se; see, for example, Theorem 4.2.1 and the subsequent remarks of Dembo and Zeitouni (2009).) From the exact form of I', given by

$$I'(y_1, y_2) = (\alpha - 1)\mathbb{I}(y_1 > 0) + (\beta - 1)\mathbb{I}(y_2 > 0),$$

one can check that I' indeed happens to be a rate function. For the sake of simplicity, suppose that $\alpha = \beta = 2$, and $\nu[x, \infty) = \nu(-\infty, -x] = x^{-2}$. Let $\hat{J}_n^{\leqslant 1} \triangleq \frac{1}{n} Q_n^{\leftarrow}(\Gamma_1) 1_{[U_1,1]}$ and $\hat{K}_n^{\leqslant 1} \triangleq \frac{1}{n} R_n^{\leftarrow}(\Delta_1) 1_{[V_1,1]}$ where $Q_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu[s, \infty) < y\} = (n/y)^{1/2}$ and $R_n^{\leftarrow}(y) \triangleq \inf\{s > 0 : n\nu(-\infty, -s] < y\} = (n/y)^{1/2}$. The random variables Γ_1 and Δ_1 are standard exponential, and U_1, V_1 uniform $[0, 1]$ (see also Section 6 for similar and more general notational conventions). Note that $\bar{Y}_n \triangleq (\hat{J}_n^{\leqslant 1}, \hat{K}_n^{\leqslant 1})$ is exponentially equivalent to $\pi(\bar{X}_n)$ if we couple $\pi(\bar{X}_n)$ and $(\hat{J}_n^{\leqslant 1}, \hat{K}_n^{\leqslant 1})$, using the representation of $\bar{X}_n$ as in (6.4): for any $\delta > 0$, $\mathbf{P}\big(|\bar{Y}_n - \pi(\bar{X}_n)| > \delta\big) \leq \mathbf{P}\big(\bar{Y}_n \neq \pi(\bar{X}_n)\big) = \mathbf{P}\big(Q_n^{\leftarrow}(\Gamma_1) \leq 1 \text{ or } R_n^{\leftarrow}(\Delta_1) \leq 1\big)$, which decays at an exponential rate. Hence,

$$\frac{\log \mathbf{P}\big(|\bar{Y}_n - \pi(\bar{X}_n)| > \delta\big)}{\log n} \to -\infty,$$

as $n \to \infty$, where $|\cdot|$ is the Euclidean distance. As a result, $\bar{Y}_n$ should satisfy the same (strong) LDP as $\pi(\bar{X}_n)$. Now, consider the set $A \triangleq \bigcup_{k=2}^{\infty} [\log k, \infty) \times [k^{-1/2}, \infty)$. Then, since $[\log k, \infty) \times [k^{-1/2}, \infty) \subseteq A$ for $k \geq 2$,

$$\begin{aligned}
\mathbf{P}(\bar{Y}_n \in A) &\geq \mathbf{P}\big((\hat{J}_n^{\leqslant 1}, \hat{K}_n^{\leqslant 1}) \in [\log n, \infty) \times [n^{-1/2}, \infty)\big) \\
&= \mathbf{P}\big(Q_n^{\leftarrow}(\Gamma_1) > n \log n, R_n^{\leftarrow}(\Delta_1) > n^{1/2}\big) \\
&= \mathbf{P}\left(\left(\frac{n}{\Gamma_1}\right)^{1/2} > n \log n, \left(\frac{n}{\Delta_1}\right)^{1/2} > n^{1/2}\right) \\
&= \mathbf{P}\left(\Gamma_1 < \frac{1}{n(\log n)^2}\right) \mathbf{P}(\Delta_1 < 1) \\
&= (1 - e^{-\frac{1}{n(\log n)^2}})(1 - e^{-1}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\limsup_{n\to\infty} \mathbf{P}(\bar{Y}_n \in A) &\geq \limsup_{n\to\infty} \frac{\log(1 - e^{-\frac{1}{n(\log n)^2}})(1 - e^{-1})}{\log n} \\
&\geq \limsup_{n\to\infty} \frac{\log \frac{1}{n(\log n)^2}(1 - \frac{1}{2n(\log n)^2})(1 - e^{-1})}{\log n} = -1.
\end{aligned} \tag{4.5}$$

On the other hand, since $A \subseteq (0, \infty) \times (0, \infty)$,

$$-\inf_{(y_1,y_2)\in A} I'(y_1, y_2) = -2. \tag{4.6}$$

Noting that A is a closed (but not compact) set, we arrive at a contradiction to the large deviation upper bound for $\bar{Y}_n$. This, in turn, proves that $\bar{X}_n$ cannot satisfy a full LDP.

5 Applications

In this section, we illustrate the use of our main results, established in Section 3, in several problem contexts that arise in control, insurance, and finance. In all examples, we assume that $\bar{X}_n(t) = X(nt)/n$, where $X(\cdot)$ is a centered Lévy process satisfying (1.1).

5.1 Crossing High Levels with Moderate Jumps

We are interested in level crossing probabilities of Lévy processes where the jumps are conditioned to be moderate. More precisely, we are interested in probabilities of the form $\mathbf{P}\big(\sup_{t\in[0,1]}[\bar{X}_n(t) - ct] \geq a;\ \sup_{t\in[0,1]}[\bar{X}_n(t) - \bar{X}_n(t-)] \leq b\big)$. We make a technical assumption that a is not a multiple of b and focus on the case where the Lévy process $\bar{X}_n$ is spectrally positive.

The setting of this example is relevant in, for example, insurance, where huge claims may be reinsured and therefore do not play a role in the ruin of an insurance company. Asmussen and Pihlsgård (2005) focus on obtaining various estimates of infinite-time ruin probabilities using analytic methods. Here, we provide complementary sharp asymptotics for the finite-time ruin probability, using probabilistic techniques.

Set $A \triangleq \{\xi \in \mathbb{D} : \sup_{t\in[0,1]}[\xi(t) - ct] \geq a; \sup_{t\in[0,1]}[\xi(t) - \xi(t-)] \leq b\}$ and define $j \triangleq \lceil a/b \rceil$. Intuitively, j should be the key parameter, as it takes at least j jumps of size b to cross level a. Our goal is to make this intuition rigorous by applying Theorem 3.1 and by showing that the upper and lower bounds are tight.

In view of Remark 3, we first check that $A_\delta \cap \mathbb{D}_j$ is bounded away from the closed set $\mathbb{D}_{\leqslant j-1}$ for some $\delta > 0$. To see this, it suffices to show that

1) $\sup_{t\in[0,1]}[\xi(t) - \xi(t-)] \leq b$ and $\sup_{t\in[0,1]}[\zeta(t) - \zeta(t-)] > b'$ imply $d(\xi, \zeta) > \frac{b'-b}{3}$; and

2) $\sup_{t\in[0,1]}[\xi(t) - ct] < a'$ and $\sup_{t\in[0,1]}[\zeta(t) - ct] \geq a$ imply $d(\xi, \zeta) \geq \frac{a-a'}{c+1}$.

It is straightforward to check 1). To see 2), note that for any $\epsilon > 0$, one can find t^* such that $\zeta(t^*) - ct^* \geq a - \epsilon$. Of course, $\xi(\lambda(t^*)) - c\lambda(t^*) < a'$ for any homeomorphism $\lambda(\cdot)$. Subtracting the latter inequality from the former inequality, we obtain

$$\zeta(t^*) - \xi(\lambda(t^*)) \geq a - a' - \epsilon + c(t^* - \lambda(t^*)). \tag{5.1}$$

One can choose λ so that $d(\xi, \zeta) + \epsilon \geq \|\lambda - e\| \geq \lambda(t^*) - t^*$ and $d(\zeta, \xi) + \epsilon \geq \|\zeta - \xi \circ \lambda\| \geq \zeta(t^*) - \xi(\lambda(t^*))$, which together with (5.1) yields

$$d(\xi, \zeta) > a - a' - (c+1)\epsilon - cd(\xi, \zeta).$$

This leads to $d(\xi, \zeta) \geq \frac{a-a'}{c+1}$ by taking $\epsilon \to 0$. With 1) and 2) in hand, it follows that $\phi_1(\xi) \triangleq \sup_{t\in[0,1]}[\xi(t) - \xi(t-)]$ and $\phi_2(\xi) \triangleq \sup_{t\in[0,1]}[\xi(t) - ct]$ are continuous functionals and $A_\delta \subseteq A(\delta)$, where $A(\delta) \triangleq \{\xi \in \mathbb{D} : \sup_{t\in[0,1]}[\xi(t) - ct] \geq a - (c+1)\delta; \sup_{t\in[0,1]}[\xi(t) - \xi(t-)] \leq b + 3\delta\}$. Since $\xi \in A(\delta) \cap \mathbb{D}_j$ implies that the jump size of ξ is bounded from below by $(b+3\delta)j - (a - (c+1)\delta)$, one can choose $\delta > 0$ so that $A(\delta) \cap \mathbb{D}_j$ is bounded away from $\mathbb{D}_{\leqslant j-1}$. This implies that $A_\delta \cap \mathbb{D}_j$ is also bounded away from $\mathbb{D}_{\leqslant j-1}$ for sufficiently small $\delta > 0$. Hence, Theorem 3.1 applies with $\mathcal{J}(A) = j$.

Next, to identify the limit, recall the discussion at the end of Section 3.1. Note that $A = \phi_1^{-1}[a, \infty) \cap \phi_2^{-1}(-\infty, b]$ and

$$\begin{aligned} \hat{T}_j^{-1}(\phi_1^{-1}[a,\infty) \cap \phi_2^{-1}(-\infty, b]) &= \Big\{(x,u) \in \hat{S}_j : \textstyle\sum_{i=1}^j x_i \geq a + c\max_{i=1,\dots,j} u_i,\ \max_{i=1,\dots,j} x_i \leq b\Big\}, \\ \hat{T}_j^{-1}(\phi_1^{-1}(a,\infty) \cap \phi_2^{-1}(-\infty, b)) &= \Big\{(x,u) \in \hat{S}_j : \textstyle\sum_{i=1}^j x_i > a + c\max_{i=1,\dots,j} u_i,\ \max_{i=1,\dots,j} x_i < b\Big\}. \end{aligned} \tag{5.2}$$

We see that $\hat{T}_j^{-1}(\phi_1^{-1}[a,\infty) \cap \phi_2^{-1}(-\infty, b]) \setminus \hat{T}_j^{-1}(\phi_1^{-1}(a,\infty) \cap \phi_2^{-1}(-\infty, b))$ has Lebesgue measure 0, and hence, A is C_j-continuous. Thus, (3.5) holds with

$$C_j(A) = \mathbf{E}\left[\nu_\alpha^j\{(0,\infty)^j : \sum_{i=1}^j x_i 1_{[U_i,1]} \in A\}\right] = \int_{(x,u)\in\hat{T}_j^{-1}(A)} \prod_{i=1}^j [\alpha x_i^{-\alpha-1} dx_i du_i] > 0.$$

Therefore, we conclude that

$$\mathbf{P}\left(\sup_{t\in[0,1]}[\bar{X}_n(t)-ct]\geq a;\ \sup_{t\in[0,1]}[\bar{X}_n(t)-\bar{X}_n(t-)]\leq b\right)\sim C_j\big(A\big)(n\nu[n,\infty))^j. \tag{5.3}$$

In particular, the probability of interest is regularly varying with index $-(\alpha-1)\lceil a/b\rceil$.

5.2 A Two-sided Barrier Crossing Problem

We consider a Lévy-driven Ornstein-Uhlenbeck process of the form

$$d\bar{Y}_n(t)=-\kappa d\bar{Y}_n(t)+d\bar{X}_n(t), \qquad \bar{Y}_n(0)=0.$$

We apply our results to provide sharp large-deviations estimates for

$$b(n)=\mathbf{P}\left(\inf\{\bar{Y}_n(t):0\leq t\leq 1\}\leq -a_-,\bar{Y}_n(1)\geq a_+\right)$$

as $n\to\infty$, where $a_-,a_+>0$. This probability can be interpreted as the price of a barrier digital option (see Cont and Tankov, 2004, Section 11.3). In order to apply our results it is useful to represent $\bar{Y}_n$ as an explicit function of $\bar{X}_n$. In particular, we have that

$$\bar{Y}_n(t)=\exp(-\kappa t)\left(\bar{Y}_n(0)+\int_0^t\exp(\kappa s)\,d\bar{X}_n(s)\right) \tag{5.4}$$

$$=\bar{X}_n(t)-\kappa\exp(-\kappa t)\int_0^t\exp(\kappa s)\,\bar{X}_n(s)\,ds. \tag{5.5}$$

Hence, if $\phi:\mathbb{D}([0,1],\mathbb{R})\to\mathbb{D}([0,1],\mathbb{R})$ is defined via

$$\phi(\xi)(t)=\xi(t)-\kappa\exp(-\kappa t)\int_0^t\exp(\kappa s)\,\xi(s)\,ds,$$

then $\bar{Y}_n=\phi\left(\bar{X}_n\right)$. Moreover, if we let

$$A=\left\{\xi\in\mathbb{D}:\inf_{0\leq t\leq 1}\phi(\xi)(t)\leq -a_-,\phi(\xi)(1)\geq a_+\right\},$$

then we obtain

$$b(n)=\mathbf{P}\left(\bar{X}_n\in A\right).$$

In order to easily verify topological properties of A, let us define $m,\pi_1:\mathbb{D}([0,1],\mathbb{R})\to\mathbb{R}$ by $m(\xi)=\inf_{0\leq t\leq 1}\xi(t)$, and $\pi_1(\xi)=\xi(1)$. Note that π_1 is continuous (see Billingsley, 2013, Theorem 12.5), that m is continuous as well, and so is ϕ. Thus, $m\circ\phi$ and $\pi_1\circ\phi$ are continuous. We can therefore write

$$A=(m\circ\phi)^{-1}(-\infty,-a_-]\cap(\pi_1\circ\phi)^{-1}[a_+,\infty),$$

concluding that A is a closed set. We now apply Theorem 3.3. To show that $\mathbb{D}_{i,0}$ is bounded away from $(m\circ\phi)^{-1}(-\infty,-a_-]$, select θ such that $d(\theta,\mathbb{D}_{i,0})<r$ with $r<a_-/(1+\kappa\exp(\kappa))$. There exists a $\xi\in\mathbb{D}_{i,0}$ such that $d(\theta,\xi)<r$ and ξ satisfies $\xi(t)=\sum_{j=1}^i x_j I_{[u_j,1]}(t)$, with $i\geq 1$. There also exists a homeomorphism $\lambda:[0,1]\to[0,1]$ such that

$$\sup_{t\in[0,1]}|\lambda(t)-t|\vee|(\xi\circ\lambda)(t)-\theta(t)|<r. \tag{5.6}$$

Now, define $\psi = \theta - (\xi \circ \lambda)$. Due to the linearity of ϕ, and representations (5.4) and (5.5), we obtain that

$$\begin{aligned}\phi(\theta)(t) &= \phi((\xi \circ \lambda))(t) + \phi(\psi)(t) \\ &= \exp(-\kappa t) \sum_{j=1}^{i} \exp\left(\kappa \lambda^{-1}(u_j)\right) x_j I_{[\lambda^{-1}(u_j),1]}(t) + \psi(t) - \kappa \exp(-\kappa t) \int_0^t \exp(\kappa s)\, \psi(s)\, ds.\end{aligned}$$

Since $x_j \geq 0$, applying the triangle inequality and inequality (5.6) we conclude (by our choice of r), that

$$\inf_{0 \leq t \leq 1} \phi(\theta)(t) \geq -r(1 + \kappa \exp(\kappa)) > -a_-.$$

A similar argument allows us to conclude that $\mathbb{D}_{0,i}$ is bounded away from $(\pi_1 \circ \phi)^{-1}[a_+, \infty)$. Hence, in addition to being closed, A is bounded away from $\mathbb{D}_{0,i} \cup \mathbb{D}_{i,0}$ for any $i \geq 1$. Moreover, let $\xi \in A \cap \mathbb{D}_{1,1}$, with

$$\xi(t) = x I_{[u,1]}(t) - y I_{[v,1]}(t), \tag{5.7}$$

where $x > 0$ and $y > 0$. Using (5.4), we obtain that $\xi \in A \cap \mathbb{D}_{1,1}$, is equivalent to

$$y \geq a_-, \quad u > v, \text{ and } x \geq a_+ \exp(\kappa(1-u)) + y \exp(-\kappa(u-v)).$$

Now, we claim that

$$\begin{aligned}A^\circ &= \left\{\xi \in \mathbb{D} : \inf_{0 \leq t \leq 1} \phi(\xi)(t) < -a_-, \phi(\xi)(1) > a_+\right\} \\ &= (m \circ \phi)^{-1}(-\infty, -a_-) \cap (\pi_1 \circ \phi)^{-1}(a_+, \infty).\end{aligned} \tag{5.8}$$

It is clear that A° contains the open set in the right hand side. We now argue that such a set is actually maximal, so that equality holds. Suppose that $\phi(\xi)(1) = a_+$, while $\min_{0 \leq t \leq 1} \phi(\xi)(t) < -a_-$. We then consider $\psi = -\delta I_{\{1\}}(t)$ with $\delta > 0$, and note that $d(\xi, \xi + \psi) \leq \delta$, and

$$\phi(\xi + \psi)(t) = \phi(\xi)(t)\, I_{[0,1)}(t) + (a_+ - \delta)\, I_{\{1\}}(t),$$

so that $\xi + \psi \notin A$. Similarly, we can see that the other inequality (involving a_-) must also be strict, hence concluding that (5.8) holds.

We deduce that, if $\xi \in A^\circ \cap \mathbb{D}_{1,1}$ with ξ satisfying (5.7), then

$$y > a_-, \quad u > v,\ x > a_+ \exp(\kappa(1-u)) + y \exp(-\kappa(u-v)).$$

Thus, we can see that A is $C_{1,1}(\cdot)$-continuous, either directly or by invoking our discussion in Section 3.1 regarding continuity of sets. Therefore, applying Theorem 3.3, we conclude that

$$b(n) \sim n\nu[n, \infty) n\nu(-\infty, -n] C_{1,1}(A)$$

as $n \to \infty$, where

$$C_{1,1}(A) = \int_0^1 \int_{a_-}^\infty \int_v^1 \int_{a_+ \exp(\kappa(1-u)) + y \exp(-\kappa(u-v))}^\infty \nu_\alpha(dx)\, du\, \nu_\beta(dy) dv.$$

In particular, the probability of interest is regularly varying with index $2 - \alpha - \beta$.

5.3 Identifying the optimal number of jumps for sets of the form $A = \{\xi : l \leq \xi \leq u\}$

The sets that appeared in the examples in Section 5.1 and Section 5.2 lend themselves to a direct characterization of the optimal numbers of jumps $(\mathcal{J}(A), \mathcal{K}(A))$. However, in more complicated problems, deciding what kind of paths the most probable limit behaviors consist of may not be as obvious. In this section, we show that for sets of a certain form, we can identify an optimal path. Consider continuous real-valued functions l and u, which satisfy $l(t) < u(t)$ for every $t \in [0, 1]$, and suppose that $l(0) < 0 < u(0)$. Define $A = \{\xi : l(t) \leq \xi(t) \leq u(t)\}$. We assume that both $\alpha, \beta < \infty$, which is the most interesting case.

The goal of this section is to construct an algorithm which yields an expression for $\mathcal{J}(A)$ and $\mathcal{K}(A)$. In fact, we can completely identify a function h that solves the optimization problem defining $(\mathcal{J}(A), \mathcal{K}(A))$. This function will be a step function with both positive and negative steps. We first construct such a function, and then verify its optimality. The first step is to identify the times at which this function jumps. Define the sets

$$A_t \triangleq \{x : l(t) \leq x \leq u(t)\}, \qquad A^*_{s,t} \triangleq \cap_{s \leq r \leq t} A_r,$$

and the times $(t_n, n \geq 1)$ by

$$t_{n+1} \triangleq 1 \wedge \inf\{t > t_n : A_{\tau_n, t} = \emptyset\} \quad \text{for} \quad n \geq 2, \qquad t_1 \triangleq 1 \wedge \inf\{t > 0 : 0 \notin A_t\}.$$

Let $n^* = \inf\{n \geq 1 : t_n = 1\}$. Assume that $n^* > 1$, since the zero function is the obvious optimal path in case $n^* = 1$. Due to the construction of the times $t_n, n \geq 1$, we have the following properties:

- Either $l(t_1) = 0$ or $u(t_1) = 0$.
- For every $n = 1, \ldots, n^* - 2$, $\sup_{t \in [t_n, t_{n+1}]} l(t) = \inf_{t \in [t_n, t_{n+1}]} u(t)$.
- $H_{fin} \triangleq [\sup_{t \in [t_{n^*-1}, t_{n^*}]} l(t), \inf_{t \in [t_{n^*-1}, t_{n^*}]} u(t)]$ is nonempty.

Set $h_n \triangleq \sup_{t \in [t_n, t_{n+1}]} l(t)$ for $n = 1, \ldots, n^* - 1$, and set $h_{n^*-1} \triangleq h_{fin}$ for any $h_{fin} \in H_{fin}$. Define now $h(t)$ as 0 on $t \in [0, t_1)$, $h(t) = h_n$ on $t \in [t_n, t_{n+1})$ for $n = 1, \ldots, n^* - 2$, and $h(t) = h_{n^*-1}$ on $t \in [t_{n^*-1}, 1]$. We claim now that $(\mathcal{J}(A), \mathcal{K}(A)) = (\mathcal{J}(\{h\}), \mathcal{K}(\{h\}))$. In fact, we can prove that if $g \in A$ is a step function, $\mathcal{D}_+(g) \geq \mathcal{D}_+(h)$ and $\mathcal{D}_-(g) \geq \mathcal{D}_-(h)$, which implies the optimality of h. The proof is based on the following observation. At each t_{n+1}, either

1) for any $\epsilon > 0$ one can find $t \in [t_{n+1}, t_{n+1} + \epsilon]$ such that $u(t) < h_n$, or
2) for any $\epsilon > 0$ one can find $t \in [t_{n+1}, t_{n+1} + \epsilon]$ such that $l(t) > h_n$.

Otherwise, there exists $\epsilon > 0$ such that $h_n \in A_{t_n, t_{n+1}+\epsilon}$, contradicting the definition of t_n, which requires $A_{t_n, t_{n+1}+\epsilon} = \emptyset$. From this observation, we can prove that on each interval $(t_n, t_{n+1}]$, any feasible path must jump at least once in the same direction as that of the jump of h. To see this, first suppose that 1) is the case at t_{n+1}, and $g \in A$ is a step function. Note that due to its continuity, $l(\cdot)$ should have achieved its supremum at $t_{sup} \in [t_n, t_{n+1}]$, i.e., $l(t_{sup}) = h_n$, and hence, $g(t_{sup}) \geq h_n$. On the other hand, due to the right continuity of g and 1), g has to be strictly less than h_n at t_{n+1}, i.e., $g(t_{n+1}) < h_n$. Therefore, g must have a downward jump on $(t_{sup}, t_{n+1}] \subseteq (t_n, t_{n+1}]$. Note that the direction of the jump of h in the interval $(t_n, t_{n+1}]$ (more specifically at t_{n+1}) also has to be downward. Since g is an arbitrary feasible path, this means that whenever h jumps downward on (t_n, t_{n+1}), any feasible path in A should also jump downward. Hence, any feasible path must have either equal or a greater number of downward jumps as h's on $[0, 1]$. Case 2) leads to a similar conclusion about the number of upward jumps of feasible paths. The number of upward jumps of h is optimal, proving that h is indeed the optimal path.

5.4 Multiple Optima

This section illustrates how to handle a case where we require Theorem 3.4, and consider an illustrative example where a rare event can be caused by two different configurations of big jumps. Suppose that the regularly varying indices $-\alpha$ and $-\beta$ for positive and negative parts of the Lévy measure ν of X are equal, and consider the set $A \triangleq \{\xi \in \mathbb{D} : |\xi(t)| \geq t - 1/2\}$. Then, $\arg\min_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A\neq\emptyset}} \mathcal{I}(j,k) = \{(1,0),(0,1)\}$, and $\mathbb{D}_{\ll 1,0} = \mathbb{D}_{\ll 0,1} = \mathbb{D}_{0,0}$. Since $|\xi(1)| \geq 1/2$ for any $\xi \in A$, $d(A, \mathbb{D}_{\ll 0,0}) = 1/2 > 0$. Theorem 3.4 therefore applies, and for each $\epsilon > 0$, there exists N such that

$$\mathbf{P}(\bar{X}_n \in A) \geq \frac{\big(C_{l,m}(A^\circ \cap \mathbb{D}_{1,0}) - \epsilon\big)L_+(n) + \big(C_{l,m}(A^\circ \cap \mathbb{D}_{0,1}) - \epsilon\big)L_-(n)}{n^{\alpha-1}},$$
$$\mathbf{P}(\bar{X}_n \in A) \leq \frac{\big(C_{l,m}(A^- \cap \mathbb{D}_{1,0}) + \epsilon\big)L_+(n) + \big(C_{l,m}(A^- \cap \mathbb{D}_{0,1}) + \epsilon\big)L_-(n)}{n^{\alpha-1}},$$

for all $n \geq N$. Note that A is closed, since if there is $\xi \in \mathbb{D}$ and $s \in [0,1]$ such that $|\xi(s)| < s - 1/2$, then $B(\xi, \frac{s-1/2-\xi(s)}{2}) \subseteq A^c$. Therefore, $A^- \cap \mathbb{D}_{1,0} = A \cap \mathbb{D}_{1,0} = \{\xi = x1_{[u,1]} : x \geq 1/2, 0 < u \leq 1/2\}$, and hence, $C_{1,0}(A^- \cap \mathbb{D}_{1,0}) = \mathbf{P}(U_1 \in (0,1/2])\nu_\alpha[1/2,\infty) = (1/2)^{1-\alpha}$. Noting that $A^\circ \cap \mathbb{D}_{1,0} \supseteq (A \cap \mathbb{D}_{1,0})^\circ = \{\xi = x1_{[u,1]} : x > 1/2, 0 < u < 1/2\}$, we deduce $C_{1,0}(A^\circ \cap \mathbb{D}_{1,0}) \geq \mathbf{P}(U_1 \in (0,1/2))\nu_\alpha(1/2,\infty) = (1/2)^{1-\alpha}$. Therefore, $C_{1,0}(A^\circ \cap \mathbb{D}_{1,0}) = C_{1,0}(A^- \cap \mathbb{D}_{1,0}) = (1/2)^{1-\alpha}$. Similarly, we can check that $C_{0,1}(A^\circ \cap \mathbb{D}_{0,1}) = C_{0,1}(A^- \cap \mathbb{D}_{0,1}) = (1/2)^{1-\beta}\ (= (1/2)^{1-\alpha})$. Therefore, for $n \geq N$,

$$((1/2)^{1-\alpha} - \epsilon)(L_+(n) + L_-(n))n^{1-\alpha} \leq \mathbf{P}(\bar{X}_n \in A) \leq ((1/2)^{1-\alpha} + \epsilon)(L_+(n) + L_-(n))n^{1-\alpha}.$$

This is equivalent to

$$\left(\frac{1}{2}\right)^{1-\alpha} \leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(L_+(n) + L_-(n))n^{1-\alpha}} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(L_+(n) + L_-(n))n^{1-\alpha}} \leq \left(\frac{1}{2}\right)^{1-\alpha}.$$

Hence,

$$\lim_{n\to\infty} \frac{\mathbf{P}(\bar{X}_n \in A)}{(L_+(n) + L_-(n))n^{1-\alpha}} = \left(\frac{1}{2}\right)^{1-\alpha}.$$

6 Proofs

Section 6.1, Section 6.2, and Section 6.3 provide proofs of the results in Section 2, Section 3, and Section 4, respectively.

6.1 Proofs of Section 2

Recall that $F_\delta = \{x \in \mathbb{S} : d(x,F) \leq \delta\}$ and $G^{-\delta} = ((G^c)_\delta)^c$.

Proof of Lemma 2.1. Let G be an open set bounded away from $\mathbb{C}$ so that $G \subseteq (\mathbb{S} \setminus \mathbb{C})^{-\gamma}$ for some $\gamma > 0$. For a given $\delta > 0$, due to the assumed asymptotic equivalence, $\mathbf{P}(X_n \in (\mathbb{S} \setminus \mathbb{C})^{-\gamma}, d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$. Therefore,

$$\begin{aligned}
\liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(Y_n \in G) &\geq \liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}\left(X_n \in G^{-\delta}, d(X_n,Y_n) < \delta\right) \\
&= \liminf_{n\to\infty} \epsilon_n^{-1}\left\{\mathbf{P}\left(X_n \in G^{-\delta}\right) - \mathbf{P}\left(X_n \in G^{-\delta}, d(X_n,Y_n) \geq \delta\right)\right\} \\
&\geq \liminf_{n\to\infty} \epsilon_n^{-1}\left\{\mathbf{P}\left(X_n \in G^{-\delta}\right) - \mathbf{P}\left(X_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n,Y_n) \geq \delta\right)\right\} \qquad (6.1) \\
&= \liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}\left(X_n \in G^{-\delta}\right) \\
&\geq \mu(G^{-\delta}).
\end{aligned}$$

Since G is an open set, $G = \bigcup_{\delta>0} G^{-\delta}$. Due to the continuity of measures, $\lim_{\delta\to 0}\mu(G^{-\delta}) = \mu(G)$, and hence, we arrive at the lower bound

$$\liminf_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(X_n \in G) \geq \mu(G)$$

by taking $\delta \to 0$. Now, turning to the upper bound, consider a closed set F bounded away from $\mathbb{C}$ so that $F \subseteq (\mathbb{S}\setminus\mathbb{C})^{-\gamma}$ for some $\gamma > 0$. Given a $\delta > 0$, by the equivalence assumption, $\mathbf{P}(Y_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma}, d(X_n, Y_n) \geq \delta) = o(\epsilon_n)$. Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(Y_n \in F) &= \limsup_{n\to\infty} \epsilon_n^{-1}\left\{\mathbf{P}\left(Y_n \in F,\, d(X_n,Y_n) < \delta\right) + \mathbf{P}\left(Y_n \in F,\, d(X_n,Y_n) \geq \delta\right)\right\} \\
&\leq \limsup_{n\to\infty} \epsilon_n^{-1}\left\{\mathbf{P}\left(X_n \in F_\delta\right) + \mathbf{P}\left(Y_n \in (\mathbb{S}\setminus\mathbb{C})^{-\gamma},\, d(X_n,Y_n) \geq \delta\right)\right\} \\
&= \limsup_{n\to\infty} \epsilon_n^{-1}\mathbf{P}\left(X_n \in F_\delta\right) \\
&\leq \mu(F_\delta)
\end{aligned} \tag{6.2}$$

as far as δ is small enough so that F_δ is bounded away from $\mathbb{C}$. Note that $\{F_\delta\}$ is a decreasing sequence of sets, $F = \bigcap_{\delta>0} F_\delta$ (since F is closed), and $\mu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ (and hence μ is a finite measure on $\mathbb{S}\setminus\mathbb{C}^r$ for some $r > 0$ such that $F_\delta \subseteq \mathbb{S}\setminus\mathbb{C}^r$ for some $\delta > 0$). Due to the continuity (from above) of finite measures, $\lim_{\delta\to 0}\mu(F_\delta) = \mu(F)$. Therefore, we arrive at the upper bound

$$\limsup_{n\to\infty} \epsilon_n^{-1}\mathbf{P}(X_n \in F) \leq \mu(F)$$

by taking $\delta \to 0$. ☐

Proof of Lemma 2.2. The argument is identical to the proof of Lemma 2.1, except for the last two inequalities of (6.1) and (6.2). The penultimate equalities of the two displays can be simply ignored since the assumed asymptotic equivalence is w.r.t. an empty set. To see the last inequality of (6.1), note that one can pick $r > 0$ such that $G^{-\delta} \cap \mathbb{S}_0 \cap \mathbb{C}_r = 0$, implying that $\mathbb{G}^{-\delta} \cap \mathbb{C}_r^{\mathsf{c}}$ is an open set bounded away from $\mathbb{C}$. Hence

$$\begin{aligned}
\liminf_{n\to\infty} \mathbf{P}(X_n \in G^{-\delta}) &= \liminf_{n\to\infty} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{S}_0) = \liminf_{n\to\infty} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{S}_0 \cap \mathbb{C}_r^{\mathsf{c}}) \\
&= \liminf_{n\to\infty} \mathbf{P}(X_n \in G^{-\delta} \cap \mathbb{C}_r^{\mathsf{c}}) \geq \mu(G^{-\delta} \cap \mathbb{C}_r^{\mathsf{c}}) = \mu(G^{-\delta} \cap \mathbb{C}_r^{\mathsf{c}} \cap \mathbb{S}_0) \\
&= \mu(G^{-\delta} \cap \mathbb{S}_0) = \mu(G^{-\delta}),
\end{aligned}$$

which validates the last inequality in (6.1). To validate the last inequality of (6.2), since $F_\delta \cap \mathbb{S}_0$ is also bounded away from $\mathbb{C}$,

$$\begin{aligned}
\limsup_{n\to\infty} \mathbf{P}(X_n \in F_\delta) &= \limsup_{n\to\infty} \mathbf{P}(X_n \in F_\delta \cap \mathbb{S}_0) \leq \limsup_{n\to\infty} \mathbf{P}(X_n \in \overline{F_\delta \cap \mathbb{S}_0}\,) \\
&\leq \mu\big(\,\overline{F_\delta \cap \mathbb{S}_0}\,\big) = \mu\big(\,\overline{F_\delta \cap \mathbb{S}_0} \cap \mathbb{S}_0\big) \leq \mu\big(\bar{F}_\delta \cap \mathbb{S}_0\big) = \mu(\bar{F}_\delta) = \mu(F_\delta).
\end{aligned}$$

☐

Proof of Lemma 2.3. The proof is an easy adaptation of the proof of Result 2 based on the fact that $\partial h^{-1}(A') \subseteq \mathbb{S}\setminus\mathbb{C}^r$ for some $r > 0$ due to the assumption, and the fact that $\partial h^{-1}(A') \subseteq h^{-1}(\partial A') \cap D_h \cap \partial\mathbb{S}_0$. ☐

Proof of Lemma 2.4. The first two claims are straightforward analogies of Result 3. As a consequence of those claims, we can apply Lemma 2.3 with $\mathbb{S}_0 = S_{j,k}$ and $h = T_{j,k}$ to conclude the proof of the last claim. ☐

Proof of Lemma 2.5. From (i) and the inclusion-exclusion formula, $\mu_n(\bigcup_{i=1}^m A_i) \to \mu(\bigcup_{i=1}^m A_i)$ as $n \to \infty$ for any finite m if $A_i \in \mathcal{A}_p$ is bounded away from $\mathbb{C}$ for $i = 1, \ldots, m$. If G is open and bounded away from $\mathbb{C}$, there is a sequence of sets $A_i, i \geq 1$ in $\mathcal{A}_p$ such that $G = \bigcup_{i=1}^\infty A_i$; note that since G is bounded away from $\mathbb{C}$, A_i's are also bounded away from $\mathbb{C}$. For any $\epsilon > 0$, one can find M_ϵ such that $\mu(\bigcup_{i=1}^{M_\epsilon} A_i) \geq \mu(G) - \epsilon$, and hence,

$$\liminf_{n\to\infty} \mu_n(G) \geq \liminf_{n\to\infty} \mu_n(\bigcup_{i=1}^{M_\epsilon} A_i) = \mu(\bigcup_{i=1}^{M_\epsilon} A_i) \geq \mu(G) - \epsilon.$$

Taking $\epsilon \to 0$, we arrive at the lower bound (2.2). Turning to the upper bound, given a closed set F, we pick $A \in \mathcal{A}_p$ bounded away from $\mathbb{C}$ such that $F \subseteq A^\circ$. Then,

$$\begin{aligned}
\mu(A) - \limsup_{n\to\infty} \mu_n(F) &= \lim_{n\to\infty} \mu_n(A) + \liminf_{n\to\infty}(-\mu_n(F)) \\
&= \liminf_{n\to\infty}(\mu_n(A) - \mu_n(F)) = \liminf_{n\to\infty} \mu_n(A \setminus F) \\
&\geq \liminf_{n\to\infty} \mu_n(A^\circ \setminus F) \geq \mu(A^\circ \setminus F) \\
&= \mu(A) - \mu(F).
\end{aligned}$$

Note that $\mu(A) < \infty$ since A is bounded away from $\mathbb{C}$, which together with the above inequality establishes the upper bound (2.2). □

6.2 Proofs of Section 3

This section provides the proofs for the limit theorems presented in Section 3 — Theorem 3.2 for one-sided Lévy measures and Theorem 3.5 for two-sided Lévy measures. The theorems are based on

1. The asymptotic equivalence between the target object $\bar{X}_n$ and the process obtained by keeping its j largest upward jumps and k largest downward jumps, which will be denoted as $J_n^{\leqslant j} - K_n^{\leqslant k}$ ($K_n^{\leqslant k} = 0$ in the one-sided case): Proposition 6.1 and Proposition 6.2 prove such asymptotic equivalences in the one-sided case. For two-sided Lévy measures, the argument for the asymptotic equivalence that corresponds to Proposition 6.1 is identical to the one-sided case, and hence omitted. Proposition 6.4 is the two-sided version of Proposition 6.2. Two technical lemmas (Lemma 6.1 and Lemma 6.2) play key roles in Proposition 6.2 and Proposition 6.4.

2. $\mathbb{M}$-convergence of $J_n^{\leqslant j} - K_n^{\leqslant k}$: Lemma 6.3 identifies the convergence of jump size sequences, and Proposition 6.3 deduces the convergence of $J_n^{\leqslant j} - K_n^{\leqslant k}$ from the convergence of the jump size sequences via the mapping theorem established in Section 2.

Lemma 6.4 collects properties of $\mathbb{D}_j$ and $\mathbb{D}_{j,k}$ useful throughout this section.

Recall that $X_n(t) \triangleq X(nt)$ is a scaled process of X, where X is a Lévy process with a Lévy measure ν. Also recall that X_n has Itô representation

$$X_n(s) = nsa + B(ns) + \int_{|x|\leq 1} x[N([0, ns] \times dx) - ns\nu(dx)] + \int_{|x|>1} xN([0, ns] \times dx), \tag{6.3}$$

where N is the Poisson random measure with mean measure Leb$\times\nu$ on $[0,n]\times(0,\infty)$ and Leb denotes the Lebesgue measure. It is easy to see that

$$
\begin{aligned}
J_n(s) &\triangleq \sum_{l=1}^{\tilde N_n} Q_n^{\leftarrow}(\Gamma_l)1_{[U_l,1]}(s) \stackrel{\mathcal{D}}{=} \int_{x>1} xN([0,ns]\times dx),\\
K_n(s) &\triangleq \sum_{l=1}^{\tilde M_n} R_n^{\leftarrow}(\Delta_l)1_{[V_l,1]}(s) \stackrel{\mathcal{D}}{=} \int_{x<-1} -xN([0,ns]\times dx),
\end{aligned}
$$

where $\Gamma_l = E_1+E_2+...+E_l$ and $\Delta_l = F_1+F_2+...+F_l$; E_i's and F_i's are i.i.d. and standard exponential random variables; U_l's and V_l's are i.i.d. and uniform variables in $[0,1]$; $\tilde N_n = N_n\big([0,1]\times[1,\infty)\big)$ and $\tilde M_n = M_n\big([0,1]\times[1,\infty)\big)$; $N_n = \sum_{l=1}^{\infty}\delta_{(U_l,Q_n^{\leftarrow}(\Gamma_l))}$ and $M_n = \sum_{l=1}^{\infty}\delta_{(V_l,Q_n^{\leftarrow}(\Gamma_l))}$, where $\delta_{(x,y)}$ is the Dirac measure concentrated on (x,y); $Q_n(x)\triangleq n\nu[x,\infty)$, $Q_n^{\leftarrow}(y)\triangleq \inf\{s>0: n\nu[s,\infty)<y\}$ and $R_n(x)\triangleq n\nu(-\infty,-x]$, $R_n^{\leftarrow}(y)\triangleq\inf\{s>0: n\nu(-\infty,-s]<y\}$. Note that $\tilde N_n$ is the number of Γ_l's such that $\Gamma_l\le n\nu_1^+$, where $\nu_1^+\triangleq\nu[1,\infty)$, and hence, $\tilde N_n\sim\text{Poisson}(n\nu_1^+)$. For the same reason, $\tilde M_n\sim\text{Poisson}(n\nu_1^-)$, where $\nu_1^-\triangleq\nu(-\infty,-1]$. Throughout the rest of this section, we use the following representation for the centered and scaled process $\bar X_n\triangleq\frac{1}{n}X_n$:

$$
\bar X_n(s) \stackrel{\mathcal{D}}{=} \frac{1}{n}J_n(s) - \frac{1}{n}K_n(s) + \frac{1}{n}B(ns) + \frac{1}{n}\int_{|x|\le1} x[N([0,ns]\times dx) - ns\nu(dx)] - (\mu_1^+\nu_1^+ - \mu_1^-\nu_1^-)s. \tag{6.4}
$$

Proof of Theorem 3.2. Since ν is concentrated on $(0,\infty)$, we can set $K_n\equiv 0$ and $\mu_1^-\nu_1^-=0$ in (6.4). We decompose $\bar X_n$ into a centered compound Poisson process, a centered Lévy process with small jumps, and a residual process that arises due to centering. After that, we will show that the compound Poisson process determines the limit. More specifically, consider the following decomposition:

$$
\begin{aligned}
\bar X_n(s) &\stackrel{\mathcal{D}}{=} \bar Y_n(s) + \bar J_n(s) + \bar Z_n(s),\\
\bar Y_n(s) &\triangleq \frac{1}{n}B(ns) + \frac{1}{n}\int_{|x|\le1} x[N([0,ns]\times dx) - ns\nu(dx)],\\
\bar J_n(s) &\triangleq \frac{1}{n}\sum_{l=1}^{\tilde N_n}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1^+)1_{[U_l,1]}(s),\\
\bar Z_n(s) &\triangleq \frac{1}{n}\sum_{l=1}^{\tilde N_n}\mu_1^+1_{[U_l,1]}(s) - \mu_1^+\nu_1^+ s,
\end{aligned} \tag{6.5}
$$

where $\mu_1^+\triangleq\frac{1}{\nu_1^+}\int_{[1,\infty)}x\nu(dx)$. Let $\hat J_n^{\leqslant j}\triangleq\frac{1}{n}\sum_{l=1}^{j}Q_n^{\leftarrow}(\Gamma_l)1_{[U_l,1]}$ be, roughly speaking, the process obtained by just keeping the j largest (un-centered) jumps of $\bar J_n$. In view of Lemma 2.1 and Proposition 6.3, it suffices to show that $\bar X_n$ and $\hat J_n^{\leqslant j}$ are asymptotically equivalent. Proposition 6.1 along with Proposition 6.2 prove the desired asymptotic equivalence, and hence, conclude the proof of the Theorem 3.2. ☐

Proposition 6.1. *Let $\bar X_n$ and $\bar J_n$ be as in the proof of Theorem 3.2. Then, $\bar X_n$ and $\bar J_n$ are asymptotically equivalent w.r.t. $\big(n\nu[n,\infty)\big)^j$ and an empty set (for any $j\ge0$).*

Proof. In view of the decomposition (6.5), we are done if we show that $\mathbf{P}(\|\bar Y_n\|>\delta) = o\big((n\nu[n,\infty))^{-j}\big)$

and $\mathbf{P}(\|\bar{Z}_n\| > \delta) = o\big((n\nu[n,\infty))^{-j}\big)$. For the tail probability of $\|\bar{Y}_n\|$,

$$\begin{aligned}&\mathbf{P}\Big[\sup_{t\in[0,1]}|\bar{Y}_n(t)| > \delta\Big]\\ &\le \mathbf{P}\Big[\sup_{t\in[0,n]}\big|B(t)\big| > n\delta/2\Big] + \mathbf{P}\Big[\sup_{t\in[0,n]}\Big|\int_{|x|\le 1} x[N((0,t]\times dx) - t\nu(dx)]\Big| > n\delta/2\Big].\end{aligned}$$

We have an explicit expression for the first term by the reflection principle, and in particular, it decays at a geometric rate w.r.t. n. For the second term, let $Y'(t) \triangleq \int_{|x|\le 1} x[N((0,t]\times dx) - t\nu(dx)]$. Using Etemadi's bound for Lévy processes (see Lemma A.1), we obtain

$$\begin{aligned}&\mathbf{P}\Big[\sup_{t\in[0,n]}\Big|\int_{|x|\le 1} x[N([0,t]\times dx) - t\nu(dx)]\Big| > n\delta/2\Big]\\ &\le 3\sup_{t\in[0,n]}\mathbf{P}\Big[|Y'(t)| > n\delta/6\Big]\\ &\le 3\sup_{t\in[0,n]}\Big\{\mathbf{P}\Big[|Y'(\lfloor t\rfloor)| > n\delta/12\Big] + \mathbf{P}\Big[|Y'(t) - Y'(\lfloor t\rfloor)| > n\delta/12\Big]\Big\}\\ &\le 3\sup_{t\in[0,n]}\mathbf{P}\Big[|Y'(\lfloor t\rfloor)| > n\delta/12\Big] + 3\sup_{t\in[0,n]}\mathbf{P}\Big[|Y'(t) - Y'(\lfloor t\rfloor)| > n\delta/12\Big]\\ &= 3\sup_{1\le k\le n}\mathbf{P}\Big[|Y'(k)| > n\delta/12\Big] + 3\sup_{t\in[0,1]}\mathbf{P}\Big[|Y'(t)| > n\delta/12\Big]\\ &= 3\sup_{1\le k\le n}\mathbf{P}\Big[\Big|\sum_{i=1}^{k}\{Y'(i) - Y'(i-1)\}\Big| > n\delta/12\Big] + 3\sup_{t\in[0,1]}\mathbf{P}\Big[|Y'(t)| > n\delta/12\Big].\end{aligned}$$

Since $Y'(i) - Y'(i-1)$ are i.i.d. with $Y'(i) - Y'(i-1) \stackrel{\mathcal{D}}{=} Y'(1) = \int_{|x|\le 1} x[N((0,1]\times dx) - \nu(dx)]$ and $Y'(1)$ has exponential moments, the first term decreases at a geometric rate w.r.t. n due to the Chernoff bound; on the other hand, in the proof of Lemma 5.1 of Lindskog et al. (2014), the second term is proved to be bounded by $\frac{\mathbf{E}|\tilde{X}(1)|^m}{n^m(\delta/6)^m}$ for any m. Therefore, by choosing m large enough, this term can be discarded. For the tail probability of $\|\bar{Z}_n\|$, note that $\bar{Z}_n$ is a mean zero Lévy process with the same distribution as $\mu_1^+(N(ns)/n - \nu_1^+ s)$, where N is the Poisson process with rate ν_1^+. Therefore, again from the continuous-time version of Etemadi's bound, we see that $\mathbf{P}(\|\bar{Z}_n\| > \delta)$ decays at a geometric rate w.r.t. n for any $\delta > 0$. ☐

Proposition 6.2. *For each $j \ge 0$, let $\bar{J}_n$ and $\hat{J}_n^{\leqslant j}$ be defined as in the proof of Theorem 3.2. Then, $\bar{J}_n$ and $\hat{J}_n^{\leqslant j}$ are asymptotically equivalent w.r.t. $\big(n\nu[n,\infty)\big)^j$ and an empty set.*

Proof. With the convention that the summation is 0 in case the superscript is strictly smaller than the subscript, consider the following decomposition of $\bar{J}_n$:

$$\hat{J}_n^{\leqslant j} \triangleq \frac{1}{n}\sum_{l=1}^{j} Q_n^{\leftarrow}(\Gamma_l)1_{[U_l,1]}, \qquad \check{J}_n^{\leqslant j} \triangleq \frac{1}{n}\sum_{l=1}^{j} -\mu_1^+ 1_{[U_l,1]},$$

$$\bar{J}_n^{>j} \triangleq \frac{1}{n}\sum_{l=j+1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]}, \qquad \bar{R}_n \triangleq \frac{1}{n}\mathbb{I}(\tilde{N}_n < j)\sum_{l=\tilde{N}_n+1}^{j}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]},$$

so that

$$\bar{J}_n = \hat{J}_n^{\leqslant j} + \check{J}_n^{\leqslant j} + \bar{J}_n^{>j} - \bar{R}_n.$$

Note that $\mathbf{P}(\|\check{J}_n^{\leqslant j}\| \geq \delta) = 0$ for sufficiently large n since $\|\check{J}_n^{\leqslant j}\| = j\mu_1/n$. On the other hand, $\mathbf{P}(\|\bar{R}_n\| \geq \delta)$ decays at a geometric rate since $\{\|\bar{R}_n\| \geq \delta\} \subseteq \{\tilde{N}_n < j\}$ and $\mathbf{P}(\tilde{N}_n < j)$ decays at a geometric rate. Since $\mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta) \leq \mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma) + \mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma)$, Lemma 6.1 and Lemma 6.2 given below imply $\mathbf{P}(\|\bar{J}_n^{>j}\| \geq \delta) = o\big((n\nu[n,\infty))^j\big)$ by choosing γ small enough. Therefore, $\hat{J}_n^{\leqslant j}$ and $\bar{J}_n$ are asymptotically equivalent w.r.t. $(n\nu[n,\infty))^j$ and an empty set. □

Define measures $\mu_\alpha^{(j)}$ and $\mu_\beta^{(j)}$ on $\mathbb{R}_+^{\infty\downarrow}$ by

$$\mu_\alpha^{(j)}(dx_1, dx_2, \cdots) \triangleq \prod_{i=1}^{j} \nu_\alpha(dx_i)\mathbb{I}_{[x_1 \geq x_2 \geq \cdots \geq x_j > 0]} \prod_{i=j+1}^{\infty} \delta_0(dx_i), \quad \nu_\alpha(x,\infty) = x^{-\alpha},$$

$$\mu_\beta^{(j)}(dx_1, dx_2, \cdots) \triangleq \prod_{i=1}^{j} \nu_\beta(dx_i)\mathbb{I}_{[x_1 \geq x_2 \geq \cdots \geq x_j > 0]} \prod_{i=j+1}^{\infty} \delta_0(dx_i), \quad \nu_\beta(x,\infty) = x^{-\beta},$$

where δ_0 is the Dirac measure concentrated at 0.

Proposition 6.3. *For each $j \geq 0$,*

$$\big(n\nu[n,\infty)\big)^{-j}\mathbf{P}(\hat{J}_n^{\leqslant j} \in \cdot) \to C_j(\cdot)$$

in $\mathbb{M}\big(\mathbb{D} \setminus \mathbb{D}_{\leqslant j-1}\big)$ as $n \to \infty$, and for each $(j,k) \in \mathbb{Z}_+^2$,

$$\big(n\nu[n,\infty)\big)^{-j}\big(n\nu(-\infty,-n]\big)^{-k}\mathbf{P}(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \cdot) \to C_{j,k}(\cdot)$$

in $\mathbb{M}\big(\mathbb{D} \setminus \mathbb{D}_{<j,k}\big)$ as $n \to \infty$.

Proof. Noting that $(\mu_\alpha^{(j)} \times \text{Leb}) \circ T_j^{-1} = C_j$ and $\mathbf{P}(\hat{J}_n^{\leqslant j} \in \cdot) = \mathbf{P}\Big(\big((Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1), (U_l, l \geq 1)\big) \in T_j^{-1}(\cdot)\Big)$, Lemma 6.3 and Corollary 2.1 prove the first claim. Likewise, since $(\mu_\alpha^{(j)} \times \mu_\beta^{(k)} \times \text{Leb} \times \text{Leb}) \circ T_{j,k}^{-1} = C_{j,k}$ and $\mathbf{P}(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \cdot) = \mathbf{P}\Big(\big((Q_n^{\leftarrow}(\Gamma_l)/n, l \geq 1), (R_n^{\leftarrow}(\Delta_l)/n, l \geq 1), (U_l, l \geq 1), (V_l, l \geq 1)\big) \in T_{j,k}^{-1}(\cdot)\Big)$, Lemma 6.3 and Lemma 2.4 prove the second claim. □

Lemma 6.1. *For any fixed $\gamma > 0$, $\delta > 0$, and $j \geq 0$,*

$$\mathbf{P}\left\{\|\bar{J}_n^{>j}\| \geq \delta, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma\right\} = o\left((n\nu[n,\infty))^j\right). \tag{6.6}$$

Proof. (Throughout the proof of this lemma, we use μ_1 and ν_1 in place of μ_1^+ and ν_1^+ respectively.) We start with the following decomposition of $\bar{J}_n^{>j}$: for any fixed $\lambda \in \Big(0, \frac{\delta}{3\nu_1\mu_1}\Big)$,

$$\begin{aligned}\bar{J}_n^{>j} &= \frac{1}{n}\sum_{l=j+1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_1,1]} \\ &= \tilde{J}_n^{[j+1, n\nu_1(1+\lambda)]} - \tilde{J}_n^{[\tilde{N}_n+1, n\nu_1(1+\lambda)]}\mathbb{I}(\tilde{N}_n < n\nu_1(1+\lambda)) + \tilde{J}_n^{[n\nu_1(1+\lambda)+1, \tilde{N}_n]}\mathbb{I}(\tilde{N}_n > n\nu_1(1+\lambda)),\end{aligned}$$

where

$$\tilde{J}_n^{[a,b]} \triangleq \frac{1}{n}\sum_{l=\lceil a\rceil}^{\lfloor b\rfloor}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1)1_{[U_l,1]}.$$

Therefore,

$$\begin{aligned}
&\mathbf{P}\left\{\|\bar{J}_n^{>j}\| \ge \delta, Q_n^{\leftarrow}(\Gamma_j) \ge n\gamma\right\} \\
&\le \mathbf{P}\left(\left\|\tilde{J}_n^{[j+1,n\nu_1(1+\lambda)]}\right\| \ge \delta/3, Q_n^{\leftarrow}(\Gamma_j) \ge n\gamma\right) + \mathbf{P}\left(\left\|\tilde{J}_n^{[\tilde{N}_n+1,n\nu_1(1+\lambda)]}\right\| \ge \delta/3\right) + \mathbf{P}\left(\tilde{N}_n > n\nu_1(1+\lambda)\right) \\
&= \text{(i)} + \text{(ii)} + \text{(iii)}.
\end{aligned}$$

Noting that $\left\|\tilde{J}_n^{[\tilde{N}_n+1,n\nu_1(1+\lambda)]}\right\| \le (\nu_1(1+\lambda) - \tilde{N}_n/n)\mu_1$ — recall that $\tilde{N}_n$ is defined to be the number of l's such that $Q_n^{\leftarrow}(\Gamma_l) \ge 1$, and hence, $0 \le Q_n^{\leftarrow}(\Gamma_l) < 1$ for $l > \tilde{N}_n$ — we see that (ii) is bounded by

$$\mathbf{P}((\nu_1(1+\lambda) - \tilde{N}_n/n)\mu_1 \ge \delta/3) = \mathbf{P}\left(\frac{\tilde{N}_n}{n\nu_1} \le 1 + \lambda - \frac{\delta}{3\nu_1\mu_1}\right),$$

which decays at a geometric rate w.r.t. n since $\tilde{N}_n$ is Poisson with rate $n\nu_1$. For the same reason, (iii) decays at a geometric rate w.r.t. n. We are done if we prove that (i) is $o\left((n\nu[n,\infty))^j\right)$. Note that $Q_n^{\leftarrow}(\Gamma_j) \ge n\gamma$ implies $Q_n(n\gamma) \ge \Gamma_j$, and hence,

$$\begin{aligned}
\sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1\right)1_{[U_l,1]} &\le \sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\right)1_{[U_l,1]} \\
&\le \sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1\right)1_{[U_l,1]}.
\end{aligned}$$

Therefore, if we define

$$\begin{aligned}
A_n &\triangleq \{Q_n^{\leftarrow}(\Gamma_j) \ge n\gamma\}, \\
B_n' &\triangleq \left\{\inf_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1\right)1_{[U_l,1]}(t) \le -n\delta\right\}, \\
B_n'' &\triangleq \left\{\sup_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1\right)1_{[U_l,1]}(t) \ge n\delta\right\},
\end{aligned}$$

then we have that

$$\text{(i)} \le \mathbf{P}(A_n \cap (B_n' \cup B_n'')) \le \mathbf{P}(A_n \cap B_n') + \mathbf{P}(A_n \cap B_n'') = \mathbf{P}(A_n)(\mathbf{P}(B_n') + \mathbf{P}(B_n''))$$

Due to Lemma 6.4 (c) and Proposition 6.3, $\mathbf{P}(A_n) = \mathbf{P}(\hat{J}_n^{\leqslant j} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant j-1})^{-\gamma/2}) = O\left((n\nu[n,\infty))^j\right)$, and hence, it suffices to show that

$$\mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1\right)1_{[U_l,1]}(t) \le n\delta\right\} \to 1, \tag{6.7}$$

and

$$\mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1\right)1_{[U_l,1]}(t) \ge -n\delta\right\} \to 1, \tag{6.8}$$

for any fixed $\gamma > 0$. Starting with (6.7)

$$
\begin{aligned}
&\mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} \big(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j) - \mu_1\big)1_{[U_l,1]}(t) \le n\delta\right\}\\
&= \mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1 - j} \big(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\big)1_{[U_l,1]}(t) \le n\delta\right\}\\
&\ge \mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1 - j} \big(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\big)1_{[U_l,1]}(t) \le n\delta, \tilde{N}_n \le (1+\lambda)n\nu_1 - j\right\}\\
&\ge \mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n} \big(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\big)1_{[U_l,1]}(t) \le n\delta, \tilde{N}_n \le (1+\lambda)n\nu_1 - j\right\}\\
&\ge \mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n} \big(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\big)1_{[U_l,1]}(t) \le n\delta)\right\} - \mathbf{P}\left\{\tilde{N}_n > (1+\lambda)n\nu_1 - j\right\}.
\end{aligned}
$$

The second inequality is due to the definition of $Q_n^{\leftarrow}$ and that $\mu_1 \ge 1$ (and hence $Q_n^{\leftarrow}(\Gamma_l) - \mu_1 \le 0$ on $l \ge \tilde{N}_n$), while the last inequality comes from the generic inequality $\mathbf{P}(A\cap B) \ge \mathbf{P}(A) - \mathbf{P}(B^c)$. The second probability converges to 0 since $\tilde{N}$ is Poisson with rate $n\nu_1$. Moving on to the first probability in the last expression, observe that $\sum_{l=1}^{\tilde{N}_n} \big(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\big)1_{[U_l,1]}(\cdot)$ has the same distribution as the compound Poisson process $\sum_{i=1}^{J(n\cdot)}(D_i - \mu_1)$, where J is a Poisson process with rate ν_1 and D_i's are i.i.d. random variables with the distribution ν conditioned (and normalized) on $[1,\infty)$, i.e., $\mathbf{P}\{D_i \ge s\} = 1 \wedge \nu[s,\infty)/\nu[1,\infty)$. Using this, we obtain

$$
\begin{aligned}
&\mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n} \big(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\big)1_{[U_l,1]}(t) \le n\delta\right\}\\
&= \mathbf{P}\left\{\sup_{1\le m\le J(n)} \sum_{l=1}^{m}(D_l - \mu_1) \le n\delta\right\}\\
&\ge \mathbf{P}\left\{\sup_{1\le m\le 2n\nu_1} \sum_{l=1}^{m}(D_l - \mu_1) \le n\delta, J(n) \le 2n\nu_1\right\}\\
&\ge \mathbf{P}\left\{\sup_{1\le m\le 2n\nu_1} \sum_{l=1}^{m}(D_l - \mu_1) \le n\delta\right\} - \mathbf{P}\big\{J(n) > 2n\nu_1\big\}
\end{aligned} \tag{6.9}
$$

The second probability vanishes at a geometric rate w.r.t. n because $J(n)$ is Poisson with rate $n\nu_1$. The first term can be investigated by the generalized Kolmogorov inequality, cf. Shneer and Wachtel (2009) (given as Result 4 in Appendix A):

$$
\mathbf{P}\left(\max_{1\le m\le 2n\nu_1} \sum_{l=1}^{m}(D_l - \mu_1) \ge n\delta/2\right) \le C\frac{2n\nu_1 V(n\delta/2)}{(n\delta/2)^2},
$$

where $V(x) = \mathbf{E}[(D_l - \mu_1)^2; \mu_1 - x \le D_l \le \mu_1 + x] \le \mu_1^2 + \mathbf{E}[D_l^2; D_l \le \mu_1 + x]$. Note that

$$\begin{aligned}\mathbf{E}[D_l^2; D_l \le \mu_1 + x] &= \int_0^1 2s ds + \int_1^{\mu_1+x} 2s \frac{\nu[s,\infty)}{\nu[1,\infty)} ds \\ &= 1 + \frac{2}{\nu_1}(\mu_1 + x)^{2-\alpha} L(\mu_1 + x),\end{aligned}$$

for some slowly varying L. Hence,

$$\mathbf{P}\left(\max_{1\le m\le 2n\nu_1} \sum_{l=1}^{m} (D_l - \mu_1) \le n\delta\right) \ge 1 - \mathbf{P}\left(\max_{1\le m\le 2n\nu_1} \sum_{l=1}^{m} (D_l - \mu_1) \ge n\delta/2\right) \to 1,$$

as $n \to \infty$.

Now, turning to (6.8), let $\gamma_n \triangleq Q_n(n\gamma)$.

$$\begin{aligned}&\mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=j+1}^{(1+\lambda)n\nu_1} \left(Q_n^{\leftarrow}(\Gamma_l - \Gamma_j + Q_n(n\gamma)) - \mu_1\right) 1_{[U_l,1]}(t) \ge -n\delta\right\} \\ &= \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1 - j} \left(Q_n^{\leftarrow}(\Gamma_l + \gamma_n) - \mu_1\right) 1_{[U_l,1]}(t) \ge -n\delta\right\} \\ &\ge \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1 - j} \left(Q_n^{\leftarrow}(\Gamma_l + \gamma_n) - \mu_1\right) 1_{[U_l,1]}(t) \ge -n\delta, E_0 \ge \gamma_n\right\} \\ &\ge \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=1}^{(1+\lambda)n\nu_1 - j} \left(Q_n^{\leftarrow}(\Gamma_l + E_0) - \mu_1\right) 1_{[U_l,1]}(t) \ge -n\delta, E_0 \ge \gamma_n\right\} \\ &= \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1 - j+1} \left(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\right) 1_{[U_l,1]}(t) \ge -n\delta, \Gamma_1 \ge \gamma_n\right\} \\ &\ge \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1 - j+1} \left(Q_n^{\leftarrow}(\Gamma_l) - \mu_1\right) 1_{[U_l,1]}(t) \ge -n\delta\right\} - \mathbf{P}\left\{\Gamma_1 < \gamma_n\right\} \\ &= (A) - (B),\end{aligned}$$

where E_0 is a standard exponential random variable. (Recall that $\Gamma_l \triangleq E_1 + E_2 + \cdots + E_l$, and hence $(\Gamma_l + E_0, U_l) \stackrel{\mathcal{D}}{=} (\Gamma_{l+1}, U_l) \stackrel{\mathcal{D}}{=} (\Gamma_{l+1}, U_{l+1})$.) Since $(B) = \mathbf{P}\{\Gamma_1 < \gamma_n\} \to 0$ (recall that $\gamma_n = n\nu[n\gamma, \infty)$ and

ν is regularly varying with index $-\alpha < -1$), we focus on proving that the first term (A) converges to 1:

$$\begin{aligned}
(A) &= \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} \big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t) \ge -n\delta\right\}\\
&\ge \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=2}^{(1+\lambda)n\nu_1-j+1} \big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t) \ge -n\delta, \tilde{N}_n \le (1+\lambda)n\nu_1 - j + 1\right\}\\
&\ge \mathbf{P}\Bigg\{\inf_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n} \big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t) \ge -n\delta/3, \inf_{t\in[0,1]} -\big(Q_n^{\leftarrow}(\Gamma_1)-\mu_1\big)1_{[U_1,1]}(t) \ge -n\delta/3,\\
&\qquad \inf_{t\in[0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} \big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t) \ge -n\delta/3, \qquad \tilde{N}_n \le (1+\lambda)n\nu_1 - j + 1\Bigg\}\\
&\ge \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n} \big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t) \ge -n\delta/3,\right\} + \mathbf{P}\left\{Q_n^{\leftarrow}(\Gamma_1)-\mu_1 \le n\delta/3\right\}\\
&\quad + \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=\tilde{N}_n+1}^{(1+\lambda)n\nu_1-j+1} \big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t) \ge -n\delta/3\right\} + \mathbf{P}\left\{\tilde{N}_n \le (1+\lambda)n\nu_1 - j + 1\right\} - 3\\
&= \text{(AI)} + \text{(AII)} + \text{(AIII)} + \text{(AIV)} - 3.
\end{aligned}$$

The third inequality comes from applying the generic inequality $\mathbf{P}(A\cap B) \ge \mathbf{P}(A) + \mathbf{P}(B) - 1$ three times. Since $\tilde{N}_n$ is Poisson with rate $n\nu_1$,

$$\text{(AIV)} = \mathbf{P}\left\{\tilde{N}_n \le (1+\lambda)n\nu_1 - j + 1\right\} = \mathbf{P}\left\{\frac{\tilde{N}_n}{n\nu_1} \le 1 + \lambda - \frac{j-1}{n\nu_1}\right\} \to 1.$$

For the first term (AI),

$$\begin{aligned}
\text{(AI)} &= \mathbf{P}\left\{\inf_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n} \big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t) \ge -n\delta/3\right\}\\
&= \mathbf{P}\left\{\sup_{t\in[0,1]} \sum_{l=1}^{\tilde{N}_n} \big(\mu_1 - Q_n^{\leftarrow}(\Gamma_l)\big)1_{[U_l,1]}(t) \le n\delta/3\right\}\\
&= \mathbf{P}\left\{\sup_{1\le m\le J(n)} \sum_{l=1}^{m} (\mu_1 - D_l) \le n\delta/3\right\},
\end{aligned}$$

where D_i is defined as before. Note that this is of exactly same form as (6.9) except for the sign of D_l, and hence, we can proceed exactly the same way using the generalized Kolmogorov inequality to prove that this quantity converges to 1 — recall that the formula only involves the square of the increments, and hence, the change of the sign has no effect. For the second term (AII),

$$\text{(AII)} \ge \mathbf{P}\big\{\Gamma_1 > Q_n(n\delta/3 + \mu_1)\big\} \to 1,$$

since $Q_n(n\delta/3+\mu_1)\to 0$. For the third term (AIII),

$$
\begin{aligned}
\text{(AIII)} &= \mathbf{P}\left\{\inf_{t\in[0,1]}\sum_{l=\tilde N_n+1}^{(1+\lambda)n\nu_1-j+1}\big(Q_n^{\leftarrow}(\Gamma_l)-\mu_1\big)1_{[U_l,1]}(t)\ge -n\delta/3\right\}\\
&\ge \mathbf{P}\left\{\inf_{t\in[0,1]}\sum_{l=\tilde N_n+1}^{(1+\lambda)n\nu_1-j+1}(1-\mu_1)1_{[U_l,1]}(t)\ge -n\delta/3\right\}\\
&\ge \mathbf{P}\left\{\sum_{l=\tilde N_n+1}^{(1+\lambda)n\nu_1-j+1}(\mu_1-1)\le n\delta/3\right\}\\
&\ge \mathbf{P}\Big\{(\mu_1-1)\big((1+\lambda)n\nu_1-j-\tilde N_n+1\big)\le n\delta/3\Big\}\\
&\ge \mathbf{P}\left\{1+\lambda-\frac{\delta}{3\nu_1(\mu_1-1)}\le\frac{\tilde N_n}{n\nu_1}+\frac{j-1}{n\nu_1}\right\}\\
&\to 1,
\end{aligned}
$$

since $\lambda<\frac{\delta}{3\nu_1(\mu_1-1)}$. This concludes the proof of the lemma. □

Lemma 6.2. *For any $j\ge 0$, $\delta>0$, and $m<\infty$, there is $\gamma_0>0$ such that*

$$\mathbf{P}\left\{\|\bar J_n^{>j}\|>\delta, Q_n^{\leftarrow}(\Gamma_j)\le n\gamma_0\right\}=o(n^{-m}).$$

Proof. (Throughout the proof of this lemma, we use μ_1 and ν_1 in place of μ_1^+ and ν_1^+ respectively, for the sake of notational simplicity.) Note first that $Q_n^{\leftarrow}(\Gamma_j)=\infty$ if $j=0$ and hence the claim of the lemma is trivial. Therefore, we assume $j\ge 1$ throughout the rest of the proof. Since for any $\lambda>0$

$$
\begin{aligned}
&\mathbf{P}\left\{\|\bar J_n^{>j}\|>\delta, Q_n^{\leftarrow}(\Gamma_j)\le n\gamma\right\}\\
&\le \mathbf{P}\left\{\left\|\sum_{l=j+1}^{\tilde N_n}(Q_n^{\leftarrow}(\Gamma_l)-\mu_1)1_{[U_l,1]}\right\|>n\delta, Q_n^{\leftarrow}(\Gamma_j)\le n\gamma, \frac{\tilde N_n}{n\nu_1}\in\left[\frac{j}{n\nu_1},1+\lambda\right]\right\} \qquad (6.10)\\
&\quad+\mathbf{P}\left\{\frac{\tilde N_n}{n\nu_1}\notin\left[\frac{j}{n\nu_1},1+\lambda\right]\right\},
\end{aligned}
$$

and $\mathbf{P}\left\{\frac{\tilde N_n}{n\nu_1}\notin\left[\frac{j}{n\nu_1},1+\lambda\right]\right\}$ decays at a geometric rate w.r.t. n, it suffices to show that (6.10) is $o(n^{-m})$ for small enough $\gamma>0$. First, recall that by the definition of $Q_n^{\leftarrow}(\cdot)$,

$$Q_n^{\leftarrow}(x)\ge s \qquad\Longleftrightarrow\qquad x\le Q_n(s),$$

and

$$n\nu(Q_n^{\leftarrow}(x),\infty)\le x\le n\nu[Q_n^{\leftarrow}(x),\infty).$$

Let L be a random variable conditionally (on $\tilde N_n$) independent of everything else and uniformly sampled on $\{j+1,j+2,\dots,\tilde N_n\}$. Recall that given $\tilde N_n$ and Γ_j, the distribution of $\{\Gamma_{j+1},\Gamma_{j+2},\dots,\Gamma_{\tilde N_n}\}$ is same as that of the order statistics of $\tilde N_n-j$ uniform random variables on $[\Gamma_j, n\nu[1,\infty)]$. Let $D_l, l\ge 1$, be i.i.d. random variables whose conditional distribution is the same as the conditional distribution of $Q_n^{\leftarrow}(\Gamma_L)$ given $\tilde N_n$ and Γ_j. Then the conditional distribution of $\sum_{l=j+1}^{\tilde N_n}(Q_n(\Gamma_l)-\mu_1)1_{[U_l,1]}$ is the same as that of

$\sum_{l=1}^{\tilde{N}_n-j}(D_l-\mu_1)1_{[U_l,1]}$. Therefore, the conditional distribution of $\left\|\sum_{l=j+1}^{\tilde{N}_n}(Q_n(\Gamma_l)-\mu_1)1_{[U_l,1]}\right\|_\infty$ is the same as the corresponding conditional distribution of $\sup_{1\le m\le \tilde{N}_n-j}\left|\sum_{l=1}^{m}(D_l-\mu_1)\right|$. To make use of this in the analysis what follows, we make a few observations on the conditional distribution of $Q_n^{\leftarrow}(\Gamma_L)$ given Γ_j and $\tilde{N}_n$.

(a) *The conditional distribution of $Q_n^{\leftarrow}(\Gamma_L)$:*
Let $q \triangleq Q_n^{\leftarrow}(\Gamma_j)$. Since Γ_L is uniformly distributed on $[\Gamma_j, Q_n(1)] = [\Gamma_j, n\nu[1,\infty)]$, the tail probability is

$$\begin{aligned}\mathbf{P}\{Q_n^{\leftarrow}(\Gamma_L)\ge s|\Gamma_j,\tilde{N}_n\} &= \mathbf{P}\{\Gamma_L\le Q_n(s)|\Gamma_j,\tilde{N}_n\} = \mathbf{P}\{\Gamma_L\le n\nu[s,\infty)|\Gamma_j,\tilde{N}_n\}\\ &= \mathbf{P}\left\{\frac{\Gamma_L-\Gamma_j}{n\nu[1,\infty)-\Gamma_j}\le \frac{n\nu[s,\infty)-\Gamma_j}{n\nu[1,\infty)-\Gamma_j}\middle|\Gamma_j,\tilde{N}_n\right\}\\ &= \frac{n\nu[s,\infty)-\Gamma_j}{n\nu[1,\infty)-\Gamma_j}\end{aligned}$$

for $s\in[1,q]$; since this is non-increasing w.r.t. Γ_j and $n\nu(q,\infty)\le\Gamma_j\le n\nu[q,\infty)$, we have that

$$\frac{\nu[s,q)}{\nu[1,q)}\le \mathbf{P}\{Q_n^{\leftarrow}(\Gamma_L)\ge s|\Gamma_j,\tilde{N}_n\}\le\frac{\nu[s,q]}{\nu[1,q]}.$$

(b) *Difference in mean between conditional and unconditional distribution:*
From (a), we obtain

$$\tilde{\mu}_n \triangleq \mathbf{E}[Q_n^{\leftarrow}(\Gamma_L)|\Gamma_j,\tilde{N}_n]\in\left[1+\int_1^q\frac{\nu[s,q)}{\nu[1,q)}ds,\,1+\int_1^q\frac{\nu[s,q]}{\nu[1,q]}ds\right],$$

and hence,

$$\begin{aligned}|\mu_1-\tilde{\mu}_n|\le&\left|\frac{\nu[1,q)\int_1^\infty\nu[s,\infty)ds-\nu[1,\infty)\int_1^q\nu[s,q)ds}{\nu[1,\infty)\nu[1,q)}\right|\\ &\vee\left|\frac{\nu[1,q]\int_1^\infty\nu[s,\infty)ds-\nu[1,\infty)\int_1^q\nu[s,q]ds}{\nu[1,\infty)\nu[1,q]}\right|.\end{aligned}$$

Since

$$\begin{aligned}&\frac{\nu[1,q)\int_1^\infty\nu[s,\infty)ds-\nu[1,\infty)\int_1^q\nu[s,q)ds}{\nu[1,\infty)\nu[1,q)}\\ &=\frac{\nu[q,\infty)}{\nu[1,q)}(q-1)+\frac{\int_q^\infty\nu[s,\infty)ds}{\nu[1,\infty)}-\frac{\nu[q,\infty)\int_1^q\nu[s,\infty)ds}{\nu[1,\infty)\nu[1,q)},\end{aligned}$$

and

$$\begin{aligned}&\frac{\nu[1,q)\int_1^\infty\nu[s,\infty)ds-\nu[1,\infty)\int_1^q\nu[s,q)ds}{\nu[1,\infty)\nu[1,q)}-\frac{\nu[1,q]\int_1^\infty\nu[s,\infty)ds-\nu[1,\infty)\int_1^q\nu[s,q]ds}{\nu[1,\infty)\nu[1,q]}\\ &=\frac{\nu\{q\}\left((q-1)\nu[1,\infty)+\int_q^\infty\nu[s,\infty)ds+\int_1^q\nu[s,\infty)ds\right)}{\nu[1,\infty)(\nu[1,q)+\nu\{q\})},\end{aligned}$$

we see that $|\mu_1-\tilde{\mu}_n|$ is bounded by a regularly varying function with index $1-\alpha$ (w.r.t. q) from Karamata's theorem.

(c) *Variance of $Q_n^{\leftarrow}(\Gamma_L)$:* Turning to the variance, we observe that, if $\alpha \le 2$,

$$\mathbf{E}[Q_n^{\leftarrow}(\Gamma_L)^2 | \Gamma_j, \tilde{N}_n] \le \int_0^1 2s ds + 2\int_1^q s\frac{\nu[s,q]}{\nu[1,q]} ds \le 1 + \frac{2}{\nu[1,q]}\int_1^q s\nu[s,\infty) ds = 1 + q^{2-\alpha}L(q) \quad (6.11)$$

for some slowly varying function $L(\cdot)$. If $\alpha > 2$, the variance is bounded w.r.t. q.

Now, with (b) and (c) in hand, we can proceed with an explicit bound since all the randomness is contained in q. Namely, we infer

$$\begin{aligned}
&\mathbf{P}\left(\left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]} \right\|_\infty > n\delta, Q_n^{\leftarrow}(\Gamma_j) \le n\gamma, \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right] \right) \\
&= \mathbf{P}\left(\left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]} \right\|_\infty > n\delta, \Gamma_j \ge Q_n(n\gamma), \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right] \right) \\
&= \mathbf{E}\left[\mathbf{P}\left(\left\| \sum_{l=j+1}^{\tilde{N}_n} (Q_n^{\leftarrow}(\Gamma_l) - \mu_1) 1_{[U_l,1]} \right\|_\infty > n\delta \,\middle|\, \Gamma_j, \tilde{N}_n \right); \Gamma_j \ge Q_n(n\gamma), \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right] \right] \\
&= \mathbf{E}\left[\mathbf{P}\left(\max_{1\le m \le \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right| > n\delta \,\middle|\, \Gamma_j, \tilde{N}_n \right); \Gamma_j \ge Q_n(n\gamma), \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right] \right].
\end{aligned}$$

By Etemadi's bound (Result 5 in Appendix),

$$\begin{aligned}
&\mathbf{P}\left(\max_{1\le m \le \tilde{N}_n - j} \left| \sum_{l=1}^m (D_l - \mu_1) \right| \ge n\delta \,\middle|\, \Gamma_j, \tilde{N}_n \right) \\
&\le 3 \max_{1\le m\le \tilde{N}_n} \mathbf{P}\left(\left| \sum_{l=1}^m (D_l - \mu_1) \right| \ge n\delta \,\middle|\, \Gamma_j, \tilde{N}_n \right) \qquad (6.12) \\
&\le 3 \max_{1\le m\le \tilde{N}_n} \left\{ \mathbf{P}\left(\sum_{l=1}^m (D_l - \mu_1) \ge n\delta \,\middle|\, \Gamma_j, \tilde{N}_n \right) + \mathbf{P}\left(\sum_{l=1}^m (\mu_1 - D_l) \ge n\delta \,\middle|\, \Gamma_j, \tilde{N}_n \right) \right\}
\end{aligned}$$

and as $|D_l - \tilde{\mu}_n|$ is bounded by q, we can apply Prokhorov's bound (Result 6 in Appendix) to get

$$\begin{aligned}
&\mathbf{P}\left(\sum_{l=1}^m (\mu_1 - D_l) \ge n\delta \,\middle|\, \Gamma_j, \tilde{N}_n \right) \\
&= \mathbf{P}\left(\sum_{l=1}^m (\tilde{\mu}_n - D_l) \ge n\delta - m(\mu_1 - \tilde{\mu}_n) \,\middle|\, \Gamma_j, \tilde{N}_n \right) \\
&\le \mathbf{P}\left(\sum_{l=1}^m (\tilde{\mu}_n - D_l) \ge n\delta - n\nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n) \,\middle|\, \Gamma_j, \tilde{N}_n \right) \\
&\le \left(\frac{qn(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{m\mathbf{var}\,(Q_n^{\leftarrow}(\Gamma_L))} \right)^{-\frac{n(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{2q}} \\
&\le \left(\frac{n\nu_1(1+\lambda)\mathbf{var}\,(Q_n^{\leftarrow}(\Gamma_L))}{qn(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))} \right)^{\frac{n(\delta - \nu_1(1+\lambda)(\mu_1 - \tilde{\mu}_n))}{2q}}
\end{aligned}$$

$$= \begin{cases} \left(\frac{\nu_1(1+\lambda)(1+q^{2-\alpha}L_1(q))}{q(\delta-\nu_1(1+\lambda)q^{1-\alpha}L_2(q))}\right)^{\frac{n(\delta-\nu_1(1+\lambda)q^{1-\alpha}L_2(q))}{2q}} & \text{if } \alpha \leq 2, \\ \left(\frac{\nu_1(1+\lambda)C}{q(\delta-\nu_1(1+\lambda)q^{1-\alpha}L_2(q))}\right)^{\frac{n(\delta-\nu_1(1+\lambda)q^{1-\alpha}L_2(q))}{2q}} & \text{otherwise,} \end{cases}$$

for some $C > 0$ if $m \leq (1+\lambda)n\nu_1$. Therefore, there exist constants M and c such that $q \geq M$ (i.e., $\Gamma_j \leq Q_n(M)$) implies

$$\mathbf{P}\left(\sum_{l=1}^{m}(\mu_1 - D_l) \geq n\delta \middle| \Gamma_j\right) \leq c(q^{1-\alpha\wedge 2})^{\frac{n\delta}{8q}},$$

and since we are conditioning on $q = Q_n^{\leftarrow}(\Gamma_j) \leq n\gamma$,

$$c(q^{1-\alpha\wedge 2})^{\frac{n\delta}{8q}} \leq c(q^{1-\alpha\wedge 2})^{\frac{\delta}{8\gamma}}.$$

Hence,

$$\mathbf{P}\left(\sum_{l=1}^{m}(\mu_1 - D_l) \geq n\delta \middle| \Gamma_j\right) \leq c\left(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\right)^{\frac{\delta}{8\gamma}}.$$

With the same argument, we also get

$$\mathbf{P}\left(\sum_{l=1}^{m}(D_l - \mu_1) \geq n\delta \middle| \Gamma_j\right) \leq c\left(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\right)^{\frac{\delta}{8\gamma}}.$$

Combining (6.12) with the two previous estimates, we obtain

$$\mathbf{P}\left(\max_{1\leq m\leq \tilde{N}_n - j}\left|\sum_{l=1}^{m}(D_l - \mu_1)\right| \geq n\delta \middle| \Gamma_j, \tilde{N}_n\right) \leq 6c\left(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\right)^{\frac{\delta}{8\gamma}},$$

on $\Gamma_j \geq Q_n(n\gamma)$, $\tilde{N}_n - j \leq n\nu_1(1+\lambda)$, and $\Gamma_j \leq Q_n(M)$. Now,

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{P}\left(\max_{1\leq m\leq \tilde{N}_n - j}\left|\sum_{l=1}^{m}(D_l - \mu_1)\right| > n\delta \middle| \Gamma_j, \tilde{N}_n\right); \Gamma_j \geq Q_n(n\gamma) \ \&\ \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right]\right] \\
&\leq \mathbf{E}\left[\mathbf{P}\left(\max_{1\leq m\leq \tilde{N}_n - j}\left|\sum_{l=1}^{m}(D_l - \mu_1)\right| > n\delta \middle| \Gamma_j, \tilde{N}_n\right); \Gamma_j \geq Q_n(n\gamma); \frac{\tilde{N}_n}{n\nu_1} \in \left[\frac{j}{n\nu_1}, 1+\lambda\right]; \Gamma_j \leq Q_n(M)\right] \\
&\qquad + \mathbf{P}(\Gamma_j > Q_n(M)) \\
&\leq \mathbf{E}\left[6c\left(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\right)^{\frac{\delta}{8\gamma}}\right] + \mathbf{P}(\Gamma_j > Q_n(M)) \\
&\leq \mathbf{E}\left[6c\left(Q_n^{\leftarrow}(\Gamma_j)^{1-\alpha\wedge 2}\right)^{\frac{\delta}{8\gamma}}; Q_n^{\leftarrow}(\Gamma_j) \geq n^{\beta}\right] + \mathbf{P}\left(Q_n^{\leftarrow}(\Gamma_j) < n^{\beta}\right) + \mathbf{P}\big(\Gamma_j > Q_n(M)\big) \\
&\leq 6c\left(n^{\beta(1-\alpha\wedge 2)}\right)^{\frac{\delta}{8\gamma}} + \mathbf{P}\left(\Gamma_j > Q_n(n^{\beta})\right) + \mathbf{P}\big(\Gamma_j > Q_n(M)\big) \\
&\leq 6c\left(n^{\beta(1-\alpha\wedge 2)}\right)^{\frac{\delta}{8\gamma}} + \mathbf{P}\left(\Gamma_j > (n^{1-\alpha\beta}L(n))\right) + \mathbf{P}\big(\Gamma_j > Q_n(M)\big),
\end{aligned}$$

for any $\beta > 0$. If one chooses β so that $1-\alpha\beta > 0$ (for example, $\beta = \frac{1}{2\alpha}$), the second and third terms vanish at a geometric rate w.r.t. n. On the other hand, we can pick γ small enough compared to δ, so that the first term is decreasing at an arbitrarily fast polynomial rate. This concludes the proof of the lemma. □

Recall that we denote the Lebesgue measure on $[0,1]^\infty$ with Leb and defined measures $\mu_\alpha^{(j)}$ and $\mu_\beta^{(j)}$ on $\mathbb{R}_+^{\infty\downarrow}$ as

$$\mu_\alpha^{(j)}(dx_1, dx_2, \ldots) \triangleq \prod_{i=1}^{j} \nu_\alpha(dx_i)\mathbb{I}_{[x_1\ge x_2\ge\cdots\ge x_j>0]} \prod_{i=j+1}^{\infty} \delta_0(dx_i), \quad \nu_\alpha(x,\infty) = x^{-\alpha},$$
$$\mu_\beta^{(k)}(dx_1, dx_2, \ldots) \triangleq \prod_{i=1}^{k} \nu_\beta(dx_i)\mathbb{I}_{[x_1\ge x_2\ge\cdots\ge x_k>0]} \prod_{i=k+1}^{\infty} \delta_0(dx_i), \quad \nu_\beta(x,\infty) = x^{-\beta},$$

where δ_0 is the Dirac measure concentrated at 0.

Lemma 6.3. *For each $j \ge 0$,*

$$\big(n\nu[n,\infty)\big)^{-j}\mathbf{P}[((Q_n^{\leftarrow}(\Gamma_l)/n, l\ge 1), (U_l, l\ge 1)) \in \cdot\,] \to (\mu_\alpha^{(j)} \times Leb)(\cdot)$$

in $\mathbb{M}\big((\mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty) \setminus (\mathbb{H}_{\leqslant j-1} \times [0,1]^\infty)\big)$ as $n\to\infty$. Also, for each $(j,k) \in \mathbb{Z}_+^2$,

$$\frac{\mathbf{P}\left[\big((Q_n^{\leftarrow}(\Gamma_l)/n, l\ge 1), (R_n^{\leftarrow}(\Delta_l)/n, l\ge 1), (U_l, l\ge 1), (V_l, l\ge 1)\big) \in \cdot\,\right]}{\big(n\nu[n,\infty)\big)^j\big(n\nu(-\infty,-n]\big)^k} \to (\mu_\alpha^{(j)} \times \mu_\beta^{(k)} \times Leb \times Leb)(\cdot)$$

in $\mathbb{M}\big((\mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} \times [0,1]^\infty \times [0,1]^\infty) \setminus (\mathbb{H}_{<j,k} \times [0,1]^\infty \times [0,1]^\infty)\big)$ as $n\to\infty$.

Proof. We first prove that

$$\big(n\nu[n,\infty)\big)^{-j}\mathbf{P}[(Q_n^{\leftarrow}(\Gamma_l)/n, l\ge 1) \in \cdot\,] \to \mu_\alpha^{(j)}(\cdot) \tag{6.13}$$

in $\mathbb{M}(\mathbb{R}_+^{\infty\downarrow} \setminus \mathbb{H}_{\leqslant j-1})$ as $n\to\infty$. To show this, we check the cases $j=0$, $j=1$, and $j=2$ for convergence-determining class of sets $\mathcal{A}_j \triangleq \big\{\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 < z_1, \ldots, x_l < z_l\} : l \ge j, x_1,\ldots,x_l > 0\big\}$, instead of checking for all j's. To see that $\mathcal{A}_j$ is a convergence-determining class for $\mathbb{M}(\mathbb{R}_+^{\infty\downarrow} \setminus \mathbb{H}_{\leqslant j-1})$-convergence, note that $\mathcal{A}_j' \triangleq \big\{\{z \in \mathbb{R}_+^{\infty\downarrow} : x_1 < z_1 \le y_1, \ldots, x_l < z_l \le y_l\} : l \ge j, x_1,\ldots,x_l > 0\big\}$ (where y_i's are allowed to assume ∞) satisfies conditions (i), (ii), and (iii) of Lemma 2.5, and hence, is a convergence-determining class. Now define $\mathcal{A}_j(i)$'s recursively as $\mathcal{A}_j(i+1) \triangleq \{B \setminus A : A, B \in \mathcal{A}_j(i), A \subseteq B\}$, $\mathcal{A}_j(0) = \mathcal{A}_j$. Since we restrict the set-difference operation between nested sets, the limit associated with the sets in $\mathcal{A}_j(i+1)$ is determined by the sets in $\mathcal{A}_j(i)$, and eventually, $\mathcal{A}_j$. Noting that $\mathcal{A}_j' \subseteq \bigcup_{i=0}^\infty \mathcal{A}_j(i)$, we see that $\mathcal{A}_j$ is indeed a convergence-determining class.

Now starting from $j=0$, since $\mu_\alpha^{(0)}(dx_1, dx_2, \ldots) = \prod_{i=1}^\infty \delta_0(dx_i)$, we see that $\mathbf{P}[(Q_n^{\leftarrow}(\Gamma_l)/n, l \ge 1) \in \{z \in \mathbb{R}_+^{\infty\downarrow} : x < z_1\}] = \mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x] = \mathbf{P}[\Gamma_1 \le Q_n(nx)] = (1 - e^{-Q_n(nx)}) \to 0 = \mu_\alpha^{(0)}(\{z \in \mathbb{R}_+^{\infty\downarrow} : x < z_1\})$ for $x > 0$, and $\mathbf{P}[(Q_n^{\leftarrow}(\Gamma_l)/n, l\ge 1) \in \mathbb{R}_+^{\infty\downarrow}] = 1 \to 1 = \mu_\alpha^{(0)}(\mathbb{R}_+^{\infty\downarrow})$ confirming that the limit for the sets in $\mathcal{A}_0$ indeed coincide with the limit in (6.13). For $j=1$,

$$\begin{aligned}(n\nu[n,\infty))^{-1}\mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x] &= (n\nu[n,\infty))^{-1}\mathbf{P}[\Gamma_1 \le Q_n(nx)] = (n\nu[n,\infty))^{-1}(1-e^{-Q_n(nx)}) \\ &\sim (n\nu[n,\infty))^{-1}Q_n(nx) = \frac{n\nu[nx,\infty)}{n\nu[n,\infty)} \to x^{-\alpha} = \nu_\alpha(x,\infty).\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&(n\nu[n,\infty))^{-1}\mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x, Q_n^{\leftarrow}(\Gamma_2)/n > y]\\
&= (n\nu[n,\infty))^{-1}\mathbf{P}[\Gamma_1 \le Q_n(nx), \Gamma_2 \le Q_n(ny)]\\
&\le (n\nu[n,\infty))^{-1}\mathbf{P}[\Gamma_2 \le Q_n(nx) \vee Q_n(ny)]\\
&\le (n\nu[n,\infty))^{-1}\mathbf{P}[\Gamma_2 \le Q_n(nx \wedge ny)]\\
&= (n\nu[n,\infty))^{-1}(1 - e^{-Q_n(nx\wedge ny)} - Q_n(nx \wedge ny)e^{-Q_n(nx\wedge ny)})\\
&\le (n\nu[n,\infty))^{-1}(Q_n(nx \wedge ny) - Q_n(nx \wedge ny)(e^{-Q_n(nx\wedge ny)}))\\
&\le (n\nu[n,\infty))^{-1}(Q_n(nx \wedge ny))^2\\
&= (n\nu[n,\infty))^{-1}(n\nu[n(x \wedge y),\infty))^2 \to 0.
\end{aligned}
$$

For $j = 2$, first consider the case $x > y > 0$:

$$
\begin{aligned}
&\mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x, Q_n^{\leftarrow}(\Gamma_2)/n > y]\\
&= \mathbf{P}[\Gamma_1 \le Q_n(nx), \Gamma_2 \le Q_n(ny)] = 1 - e^{-Q_n(nx)} - Q_n(nx)e^{-Q_n(ny)}\\
&\sim Q_n(nx) - Q_n(nx)^2/2 + O(Q_n(nx)^3) - Q_n(nx)\big(1 - Q_n(ny) + O(Q_n(ny)^2)\big)\\
&= Q_n(nx)Q_n(ny) - Q_n(nx)^2/2 + O(Q_n(nx)^3) + Q_n(nx)O(Q_n(ny)^2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(n\nu[n,\infty))^{-2}\mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x, Q_n^{\leftarrow}(\Gamma_2)/n > y]\\
&\sim (n\nu[n,\infty))^{-2}(n\nu[nx,\infty))(n\nu[ny,\infty)) - (n\nu[n,\infty))^{-2}(n\nu[nx,\infty))^2/2\\
&\quad + O((n\nu[n,\infty))^{-2}(n\nu[nx,\infty))^3) - (n\nu[n,\infty))^{-2}(n\nu[nx,\infty))O((n\nu[ny,\infty))^2)\big)\\
&\to x^{-\alpha}y^{-\alpha} - x^{-2\alpha}/2 = \mu^{(2)}(z \in \mathbb{R}^{\infty\downarrow} : z_1 > x,\ z_2 > y).
\end{aligned}
$$

Similarly for $y > x > 0$,

$$
(n\nu[n,\infty))^{-2}\mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x, Q_n^{\leftarrow}(\Gamma_2)/n > y] \to y^{-2\alpha}/2 = \mu^{(2)}(z \in \mathbb{R}^{\infty\downarrow} : z_1 > x,\ z_2 > y).
$$

For $x > 0$, $y > 0$, $z > 0$,

$$
\begin{aligned}
&(n\nu[n,\infty))^{-2}\mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x, Q_n^{\leftarrow}(\Gamma_2)/n > y, Q_n^{\leftarrow}(\Gamma_3) > z]\\
&\le (n\nu[n,\infty))^{-2}\mathbf{P}[\Gamma_3 \le Q_n(nx) \vee Q_n(ny) \vee Q_n(nz)]\\
&= (n\nu[n,\infty))^{-2}\mathbf{P}[\Gamma_3 \le Q_n(n(x \wedge y \wedge z))]\\
&= (n\nu[n,\infty))^{-2}(1 - e^{-Q_n(n(x\wedge y\wedge z))} - Q_n(n(x \wedge y \wedge z))e^{-Q_n(n(x\wedge y\wedge z))}\\
&\qquad\qquad - Q_n(n(x \wedge y \wedge z))^2 e^{-Q_n(n(x\wedge y\wedge z))}/2)
\end{aligned}
$$

From the Taylor expansion of e^{-x}, i.e., $e^{-x} = 1 - x + x^2/2 + O(x^3)$,

$$
\begin{aligned}
&(n\nu[n,\infty))^{-2}\mathbf{P}[Q_n^{\leftarrow}(\Gamma_1)/n > x, Q_n^{\leftarrow}(\Gamma_2)/n > y, Q_n^{\leftarrow}(\Gamma_3) > z]\\
&= (n\nu[n,\infty))^{-2}O(Q_n(n(x \wedge y \wedge z))^3)\\
&= (n\nu[n,\infty))^{-2}O((n\nu[n(x \wedge y \wedge z),\infty))^3) \to 0.
\end{aligned}
$$

By similar calculation but considering slightly more involved convergence determining class $\mathcal{A}_{j,k} \triangleq \big\{A_1 \times A_2 : A_1 \in \mathcal{A}_l, A_2 \in \mathcal{A}_m, (l,m) \notin I_{<j,k}\big\}$, it is straightforward to check that

$$
\big(n\nu[n,\infty)\big)^{-j}\big(n\nu(-\infty,-n]\big)^{-k}\mathbf{P}\left[\big((Q_n^{\leftarrow}(\Gamma_l)/n, l \ge 1), (R_n^{\leftarrow}(\Delta_l)/n, l \ge 1)\big) \in \cdot\right] \to (\mu_\alpha^{(j)} \times \mu_\beta^{(k)})(\cdot) \quad (6.14)
$$

in $\mathbb{M}\big(\mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} \setminus \mathbb{H}_{<j,k}\big)$ as $n \to \infty$. The conclusions of the lemma are immediate from (6.13), (6.14), and the independence of Γ_i's, U_i's, Δ_i's, and V_i's. □

Next, we prove the asymptotic equivalence for the two-sided Poisson jump process. Let

$$\hat{J}_n^{\leqslant j} \triangleq \frac{1}{n}\sum_{l=1}^{j} Q_n^{\leftarrow}(\Gamma_l)1_{[U_l,1]}, \qquad \bar{J}_n \triangleq \frac{1}{n}\sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]}, \qquad \mu_1^+ \triangleq \frac{1}{\nu_1^+}\int_1^{\infty} x\nu(dx).$$

$$\hat{K}_n^{\leqslant k} \triangleq \frac{1}{n}\sum_{l=1}^{k} R_n^{\leftarrow}(\Delta_l)1_{[V_l,1]}, \qquad \bar{K}_n \triangleq \frac{1}{n}\sum_{l=1}^{\tilde{M}_n}(R_n^{\leftarrow}(\Delta_l) - \mu_1^-)1_{[V_l,1]}, \qquad \mu_1^- \triangleq \frac{-1}{\nu_1^-}\int_{-\infty}^{-1} x\nu(dx).$$

Roughly speaking, as in the case of one-sided Lévy measures, the compound Poisson process $J_n - K_n$ determines the limit behavior of $\bar{X}_n$ while the rest of the Lévy process doesn't contribute to the asymptotic behavior. However, in contrast to the one-sided case, we also have to take out the set that includes the step functions with more than j upward jumps or more than k downward jumps to establish appropriate $\mathbb{M}$-convergences.

Proposition 6.4. *For each $(j,k) \in \mathbb{Z}_+^2$, $\bar{J}_n - \bar{K}_n$ is asymptotically equivalent to $\hat{J}_n^{\leqslant n} - \hat{K}_n^{\leqslant n}$ with respect to $\big(n\nu[n,\infty)\big)^j\big(n\nu(-\infty,-n]\big)^k$ and $\mathbb{D}_{<j,k}$.*

Proof. We decompose $\bar{J}_n - \bar{K}_n$ in a similar way as in Proposition 6.2. Here, however, the asymptotic equivalence w.r.t. an empty set is impossible to achieve. Therefore, we work with the asymptotic equivalence with respect to $\mathbb{D}_{<j,k}$. Let

$$\check{J}_n^{\leqslant j} \triangleq \frac{1}{n}\sum_{l=1}^{j} -\mu_1^+ 1_{[U_l,1]}, \quad \bar{J}_n^{>j} \triangleq \frac{1}{n}\sum_{l=j+1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]},$$

$$\check{K}_n^{\leqslant k} \triangleq \frac{1}{n}\sum_{l=1}^{k} -\mu_1^- 1_{[V_l,1]}, \quad \bar{K}_n^{>k} \triangleq \frac{1}{n}\sum_{l=k+1}^{\tilde{M}_n}(R_n^{\leftarrow}(\Delta_l) - \mu_1^-)1_{[V_l,1]},$$

$$\bar{R}_n^+ \triangleq \frac{1}{n}\mathbb{I}(\tilde{N}_n < j)\sum_{l=\tilde{N}_n+1}^{j}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]}, \quad \bar{R}_n^- \triangleq \frac{1}{n}\mathbb{I}(\tilde{M}_n < j)\sum_{l=\tilde{M}_n+1}^{j}(R_n^{\leftarrow}(\Delta_l) - \mu_1^-)1_{[V_l,1]}$$

so that

$$\bar{J}_n - \bar{K}_n = \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \check{J}_n^{\leqslant j} - \check{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} - \bar{R}_n^+ + \bar{R}_n^-.$$

Note that it is straightforward to see that $\mathbf{P}(\|\check{J}_n^{\leqslant j}\| \ge \delta)$, $\mathbf{P}(\|\check{K}_n^{\leqslant k}\| \ge \delta)$, $\mathbf{P}(\|\bar{R}_n^+\| \ge \delta)$, and $\mathbf{P}(\|\bar{R}_n^-\| \ge \delta)$ are all $o\big((n\nu[n,\infty))^j(n\nu(-\infty,-n])^k\big)$. Therefore, $\bar{J}_n - \bar{K}_n$ is asymptotically equivalent to $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k}$ w.r.t. $(n\nu[n,\infty))^j(n\nu(-\infty,-n])^k)$ and an empty set. If we also show that $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k}$ is asymptotically equivalent to $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k}$ w.r.t. $(n\nu[n,\infty))^j(n\nu(-\infty,-n])^k$ and $\mathbb{D}_{<j,k}$, the conclusion of the proposition follows. In view of this, it suffices to prove that for any given $\delta > 0$, $\gamma > 0$, and $\rho > 0$,

(i) $\mathbf{P}\left(\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\| \ge \delta, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\gamma}\right) = o\big((n\nu[n,\infty))^j(n\nu(-\infty,-n])^k\big)$;

(ii) $\mathbf{P}\left(\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\| \ge \delta, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\rho}\right) = o\big((n\nu[n,\infty))^j(n\nu(-\infty,-n])^k\big)$.

Starting with (i), we decompose the event as follows:

$$\begin{aligned}&\mathbf{P}\left(\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\| \ge \delta, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\gamma}\right)\\ &\le \mathbf{P}\left(\|\bar{J}_n^{>j}\| \ge \frac{\delta}{2}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\gamma}\right) + \mathbf{P}\left(\|\bar{K}_n^{>k}\| \ge \frac{\delta}{2}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\gamma}\right).\end{aligned}$$

Since the two events above can be dealt with in an identical way, we only work out the first probability. Noting that $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\gamma}$ implies $Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma$ and $R_n^{\leftarrow}(\Delta_k) \geq n\gamma$ by Lemma 6.4 (d),

$$\begin{aligned}
&\mathbf{P}\left(\|\bar{J}_n^{>j}\| \geq \frac{\delta}{2}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\gamma}\right) \\
&\leq \mathbf{P}\left(\|\bar{J}_n^{>j}\| \geq \frac{\delta}{2}, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma, R_n^{\leftarrow}(\Delta_k) \geq n\gamma\right) \\
&= \mathbf{P}\left(\|\bar{J}_n^{>j}\| \geq \frac{\delta}{2}, Q_n^{\leftarrow}(\Gamma_j) \geq n\gamma\right) \mathbf{P}\left(R_n^{\leftarrow}(\Delta_k) \geq n\gamma\right) \\
&= o\Big((n\nu[n,\infty))^j\Big) O\Big((n\nu(-\infty,-n])^k\Big) \\
&= o\Big((n\nu[n,\infty))^j (n\nu(-\infty,-n])^k\Big),
\end{aligned}$$

since $\mathbf{P}(R_n^{\leftarrow}(\Delta_k) \geq n\gamma) \leq \mathbf{P}(\hat{K}_n^{\leqslant k} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant k-1})^{-\gamma/2}) = O\big((n\nu(-\infty,-n])^k\big)$ (analogous to Proposition 6.3). Turning to (ii), note that since we already have (i), it is enough to prove that for any given $\delta > 0$ and $\rho > 0$, one can find $\gamma > 0$ such that

$$\begin{aligned}
&\mathbf{P}\left(\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\| \geq \delta, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\rho}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \notin (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\gamma}\right) \\
&= o\left((n\nu[n,\infty))^j (n\nu(-\infty,-n])^k\right). \qquad (6.15)
\end{aligned}$$

To prove (6.15), our strategy is to divide the event $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \notin (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\gamma}$ into, roughly speaking, $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in (\mathbb{D}_{l,m})_\gamma$ for each $(l,m) \in I_{<j,k}$ and conquer each of them separately with the tools we have developed for the one-sided Poisson processes, namely Proposition 6.3, Lemma 6.1, and Lemma 6.2. Note that our one-sided tools are most useful when we have a good control over the behavior of $\hat{J}_n^{\leqslant j}$ near the boundaries of $\mathbb{D}_j$. As one can imagine, these constraints translate into the need for the control over the behavior of $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k}$ near boundaries of $\mathbb{D}_{j,k}$. In view of this, we consider a slightly more involved cover than $\{(\mathbb{D}_{l,m})_\gamma : (l,m) \in I_{<j,k}\}$. More specifically, we construct a cover $\{\mathbb{W}_{l,m} : (l,m) \in I_{<j,k}\}$ of $((\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\gamma})^c$ as follows:

$$\mathbb{W}_{l,m} \triangleq \hat{\mathbb{D}}_{l,m} \setminus (\hat{\mathbb{D}}_{l,m-1} \cup \hat{\mathbb{D}}_{l-1,m}),$$

where

$$\hat{\mathbb{D}}_{l,m} \triangleq (\mathbb{D}_{l,m})_{\gamma_{l,m}}, \qquad \gamma_{l,m} \triangleq \bar{\gamma}(4k)^{-l}(4j)^{-m}, \qquad \bar{\gamma} \triangleq \gamma \max_{(l,m)\in I_{<j,k}} (4k)^l (4j)^m.$$

In this way, we make sure that in the event $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{l,m}$, the l^{th} upward jump and the m^{th} downward jump are bounded from below while the higher order jumps are bounded from above; see Lemma 6.4 (e). The rest of the proof critically hinges on these properties. Before proceeding, note that $\{\mathbb{W}_{l,m} : (l,m) \in I_{<j,k}\}$ indeed covers $((\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\gamma})^c$ because $((\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\gamma})^c = (\mathbb{D}_{<j,k})_\gamma \subseteq \bigcup_{(l,m)\in I_{<j,k}} \hat{\mathbb{D}}_{l,m} \subseteq \bigcup_{(l,m)\in I_{<j,k}} \mathbb{W}_{l,m}$. To see the last inclusion, let $\iota : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ be a one-to-one mapping $\iota(l,m) \triangleq l + (l+m)(l+m+1)/2$, and $M_{<j,k} \triangleq \{i : i = \iota(l,m) \text{ for some } l,m \in I_{<j,k}\}$. Let $\hat{\mathbb{D}}_n \triangleq \hat{\mathbb{D}}_{\iota^{-1}(n)}$ (i.e., $\hat{\mathbb{D}}_{\iota(l,m)} \triangleq \hat{\mathbb{D}}_{l,m}$ for each l,m) and define $\mathbb{W}'_n$ for $n \in M_{<j,k}$ as follows:

$$\mathbb{W}'_n \triangleq \hat{\mathbb{D}}_n \setminus \bigcup_{i\in[0,n)\cap M_{<j,k}} \hat{\mathbb{D}}_i.$$

Note that $\iota(l-1,m) < \iota(l,m)$ and $\iota(l,m-1) < \iota(l,m)$, and hence, $(\hat{\mathbb{D}}_{l,m-1} \cup \hat{\mathbb{D}}_{l-1,m}) \subseteq \bigcup_{i\in[0,\iota(l,m))\cap M_{<j,k}} \hat{\mathbb{D}}_i$. Therefore, $\mathbb{W}'_{\iota(l,m)} \subseteq \mathbb{W}_{l,m}$, and consequently,

$$\bigcup_{(l,m)\in I_{<j,k}} \hat{\mathbb{D}}_{l,m} = \bigcup_{n\in M_{<j,k}} \hat{\mathbb{D}}_n = \bigcup_{n\in M_{<j,k}} \mathbb{W}'_n = \bigcup_{(l,m)\in I_{<j,k}} \mathbb{W}'_{\iota(l,m)} \subseteq \bigcup_{(l,m)\in I_{<j,k}} \mathbb{W}_{l,m}.$$

Now we decompose the probability in (6.15) w.r.t. $\{\mathbb{W}_{l,m}\}_{(l,m)\in I_{<j,k}}$.

$$\begin{aligned}
&\mathbf{P}\left(\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\| \geq \delta, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\rho}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \notin (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\gamma}\right)\\
&\leq \sum_{(l,m)\in I_{<j,k}} \mathbf{P}\left(\left\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\right\| \geq \delta, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\rho}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{l,m}\right)\\
&\triangleq \sum_{(l,m)\in I_{<j,k}} p_{l,m}(n).
\end{aligned}$$

The proof of the proposition is complete if we show that $p_{l,m}(n) = o\left((n\nu[n,\infty)^j(n\nu(-\infty,-n])^k\right)$ for each $(l,m) \in I_{<j,k}$. In view of Lemma 6.4 (e-v), $p_{l,m}(n) = 0$ for $l > j$ or $m > k$, and hence, we only have to consider the following two cases:

a) the case $l < j$ and $m < k$.

b) the case $l = j$ or $m = k$;

For case (a), $p_{l,m}(n)$ decreases at an arbitrarily fast polynomial rate because

$$\begin{aligned}
p_{l,m}(n) &\leq \mathbf{P}\left(\left\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\right\| \geq \delta, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{l,m}\right)\\
&\leq \mathbf{P}\left(\left\|\bar{J}_n^{>j} - \bar{K}_n^{>k}\right\| > \delta, Q_n^{\leftarrow}(\Gamma_{l+1}) \leq 2n\gamma_{l,m}, R_n^{\leftarrow}(\Delta_{l+1}) \leq 2n\gamma_{l,m}\right)\\
&\leq \mathbf{P}\left(\left\|\bar{J}_n^{>j}\right\| > \frac{\delta}{2}, Q_n^{\leftarrow}(\Gamma_{m+1}) \leq 2n\gamma_{l,m}\right) + \mathbf{P}\left(\left\|\bar{K}_n^{>k}\right\| > \frac{\delta}{2}, R_n^{\leftarrow}(\Delta_{m+1}) \leq 2n\gamma_{l,m}\right),
\end{aligned}$$

where the second inequality is due to Proposition 6.4 (e-i) and (e-ii). Due to Lemma 6.2, the first term decreases at an arbitrarily fast polynomial rate if we choose γ small enough compared to δ, and so does the second term (note that it is of the same form as the first term). Therefore, the whole upper bound vanishes at an arbitrarily fast polynomial rate. Turning to case (b), since the treatment for $m = k$ is identical, we assume $l = j$. Note first that $l = j$ and $(l,m) \in I_{<j,k}$ implies that $0 \leq m < k$, and $p_{l,m}(n)$ is bounded by

$$\begin{aligned}
&p_{l,m}(n)\\
&\leq \mathbf{P}\left(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\rho}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{j,m}\right).\\
&\leq \mathbf{P}\left(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\rho}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{j,m}, \|\bar{K}_n^{>k}\| > \frac{\rho}{2}\right) &(6.16)\\
&\quad + \mathbf{P}\left(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\rho}, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{j,m}, \|\bar{K}_n^{>k}\| \leq \frac{\rho}{2}\right). &(6.17)
\end{aligned}$$

Since $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{j,m}$ implies $R_n^{\leftarrow}(\Delta_k) \leq 2n\gamma_{j,m} \leq 2n\bar{\gamma}$, we can apply Lemma 6.2 to show that (6.16) vanishes at an arbitrarily fast polynomial rate by taking γ small enough. Moving on to (6.17), note that if we choose γ small enough so that $4\bar{\gamma} \leq \rho/2$, this event implies $\bar{J}_n^{>j} \notin (\mathbb{D}_{\leqslant L_m})_{\bar{\gamma}}$, where $L_m \triangleq \lfloor \frac{\beta-1}{\alpha-1}(k-m)\rfloor$. To see this, suppose the opposite, i.e., $\bar{J}_n^{>j} \in (\mathbb{D}_{\leqslant L_m})_{\bar{\gamma}}$. Then there exists $\xi \in \mathbb{D}_{\leqslant L_m}$

such that $\|\bar{J}_n^{>j} - \xi\| \le 2\bar{\gamma}$. Note also that $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{j,m} \subseteq (\mathbb{D}_{j,m})_{\bar{\gamma}}$ implies that there exists $\eta \in \mathbb{D}_{j,m}$ such that $\left\|\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} - \eta\right\| \le 2\bar{\gamma}$. Then,

$$\begin{aligned} d_{sk}\left(\eta + \xi, \hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k}\right) &\le \left\|\eta + \xi - \left(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k}\right)\right\| \\ &\le \left\|\eta - \left(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k}\right)\right\| + \left\|\xi - \bar{J}_n^{>j}\right\| + \left\|\bar{K}_n^{>k}\right\| \\ &\le 2\bar{\gamma} + 2\bar{\gamma} + \rho/2 \le \rho, \end{aligned}$$

while $\eta + \xi \in \mathbb{D}_{l',m'}$, where $l' \le j + L_m$ and $m' \le m$. This implies that $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}_{l',m'})_\rho$. Since

$$\begin{aligned} (\alpha-1)l' + (\beta-1)m' &\le (\alpha-1)(j + L_m) + (\beta-1)m \le (\alpha-1)(j + \frac{\beta-1}{\alpha-1}(k-m)) + (\beta-1)m \\ &= (\alpha-1)j + (\beta-1)k, \end{aligned}$$

and $m' < k$, the index pair (l', m') has to be in $I_{<j,k}$. This, in turn, implies that $(\mathbb{D}_{l',m'})_\rho \subseteq (\mathbb{D}_{<j,k})_\rho$, and hence, $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D}_{<j,k})_\rho$, which of course is contradictory to $\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} + \bar{J}_n^{>j} - \bar{K}_n^{>k} \in (\mathbb{D} \setminus \mathbb{D}_{<j,k})^{-\rho}$ proving that $\bar{J}_n^{>j} \notin (\mathbb{D}_{\leqslant L_m})_{\bar{\gamma}}$. In view of this, (6.17) is bounded by $\mathbf{P}\big(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{j,m}, \bar{J}_n^{>j} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant L_m})^{-\bar{\gamma}}\big)$, and hence, the proof of the proposition is complete if we show that this probability is $o\big((n\nu[n,\infty))^j(n\nu(-\infty,-n])^k\big)$. We only consider the cases $m > 0$ and $j > 0$. The other cases ($m > 0$ and $j = 0$; $m = 0$ and $j > 0$; $m = 0$ and $j = 0$) are similar but easier. Thus, write

$$\begin{aligned} &\mathbf{P}\left(\hat{J}_n^{\leqslant j} - \hat{K}_n^{\leqslant k} \in \mathbb{W}_{j,m}, \bar{J}_n^{>j} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant L_m})^{-\bar{\gamma}}\right) \\ &\le \mathbf{P}\left(R_n^{\leftarrow}(\Delta_m) > n\frac{\gamma_{j,m-1}}{2(k+1)}, \bar{J}_n^{>j} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant L_m})^{-\bar{\gamma}}\right) \\ &\le \mathbf{P}\left(\hat{K}_n^{\leqslant m} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant m-1})^{-\frac{\gamma_{l,m-1}}{4(k+1)}}, \bar{J}_n^{>j} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant L_m})^{-\bar{\gamma}}\right) \\ &= \mathbf{P}\left(\hat{K}_n^{\leqslant m} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant m-1})^{-\frac{\gamma_{l,m-1}}{4(k+1)}}\right)\mathbf{P}\left(\bar{J}_n^{>j} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant L_m})^{-\bar{\gamma}}\right) \\ &= P_1(n) \cdot P_2(n), \end{aligned}$$

where the first inequality is due to Lemma 6.4 (e-iv), and the second inequality is due to Lemma 6.4 (c). Since Proposition 6.3 guarantees that $P_1(n)$ is $O\left((n\nu(-\infty,-n])^m\right)$, and

$$\begin{aligned} (\alpha-1)(j + L_m + 1) + (\beta-1)m &= (\alpha-1)(j + \lfloor\frac{\beta-1}{\alpha-1}(k-m)\rfloor + 1) \\ &\quad + (\beta-1)m > (\alpha-1)(j + \frac{\beta-1}{\alpha-1}(k-m)) + (\beta-1)m \\ &= (\alpha-1)k + (\beta-1)j. \end{aligned}$$

Thus, if we show that $P_2(n)$ is $O\left((n\nu[n,\infty))^{j+L_m+1}\right)$, then

$$P_1(n) \cdot P_2(n) = O\left((n\nu[n,\infty))^{j+L_m+1}(n\nu(-\infty,-n])^m\right) = o\left((n\nu[n,\infty)^j(n\nu(-\infty,-n])^k\right).$$

Therefore, we can conclude the proof by showing that for any $\bar{\gamma} > 0$,

$$P_2(n) = \mathbf{P}\left(\bar{J}_n^{>j} \in (\mathbb{D} \setminus \mathbb{D}_{\leqslant L_m})^{-\bar{\gamma}}\right) = O\left((n\nu[n,\infty))^{j+L_m+1}\right).$$

To see this, we introduce yet another parameter $\epsilon \in (0, \bar\gamma)$ and proceed as follows:

$$
\begin{aligned}
&\mathbf{P}\left(\bar J_n^{>j} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant L_m})^{-\bar\gamma}\right)\\
&= \mathbf{P}\left(\bar J_n^{>j} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant L_m})^{-\bar\gamma}, \hat J_n^{\leqslant j+L_m+1} - \hat J_n^{\leqslant j} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant L_m})^{-\epsilon}\right)\\
&\quad + \mathbf{P}\left(\bar J_n^{>j} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant L_m})^{-\bar\gamma}, \hat J_n^{\leqslant j+L_m+1} - \hat J_n^{\leqslant j} \notin (\mathbb{D}\setminus\mathbb{D}_{\leqslant L_m})^{-\epsilon}\right)\\
&\le \mathbf{P}\left(\bar J_n^{>j} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant L_m})^{-\bar\gamma}, Q_n^{\leftarrow}(\Gamma_{j+L_m+1}) \ge n\epsilon\right)\\
&\quad + \mathbf{P}\left(\left\|\bar J_n^{>j} - (\hat J_n^{\leqslant j+L_m+1} - \hat J_n^{\leqslant j})\right\| \ge (\bar\gamma-\epsilon), Q_n^{\leftarrow}(\Gamma_{j+L_m+1}) \le 2n\epsilon\right)\\
&\le \mathbf{P}\left(Q_n^{\leftarrow}(\Gamma_{j+L_m+1}) \ge n\epsilon\right)\\
&\quad + \mathbf{P}\left(\left\|\bar J_n^{>j+L_m+1} + \frac{1}{n}\sum_{l=j+1}^{j+L_m+1} -\mu_1^+ 1_{[U_l,1]}\right\| \ge (\bar\gamma-\epsilon), Q_n^{\leftarrow}(\Gamma_{j+L_m+1}) \le 2n\epsilon\right)\\
&\le \mathbf{P}\left(\hat J_n^{\leqslant j+L_m+1} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant j+L_m})^{-\epsilon/2}\right)\\
&\quad + \mathbf{P}\left(\left\|\sum_{l=j+1}^{j+L_m+1} \mu_1^+ 1_{[U_l,1]}\right\| \ge n\frac{\bar\gamma-\epsilon}{2}\right)\\
&\quad + \mathbf{P}\left(\left\|\bar J_n^{>j+L_m+1}\right\| \ge \frac{\bar\gamma-\epsilon}{2}, Q_n^{\leftarrow}(\Gamma_{j+L_m+1}) \le 2n\epsilon\right),
\end{aligned}
$$

where the first inequality is from Lemma 6.4 (b) and (c). The first term is $O\left((n\nu[n,\infty))^{j+L_m+1}\right)$ by Proposition 6.3, the second term is 0 for sufficiently large n, and the third term vanishes at an arbitrarily fast polynomial rate if we choose ϵ small enough compared to γ (Lemma 6.2). This concludes the proof. □

Recall that $\mathbb{W}_{l,m} = \hat{\mathbb{D}}_{l,m} \setminus (\hat{\mathbb{D}}_{l,m-1} \cup \hat{\mathbb{D}}_{l-1,m})$, where $\hat{\mathbb{D}}_{l,m} = (\mathbb{D}_{l,m})_{\gamma_{l,m}}$, $\gamma_{l,m} = \bar\gamma(4k)^{-l}(4j)^{-m}$, and $\bar\gamma = \gamma \max_{(l,m)\in I_{<j,k}} (4k)^l(4j)^m$.

Lemma 6.4. *Suppose that $x_1 \ge \cdots \ge x_j \ge 0$; $u_i \in (0,1)$ for $i = 1,\dots,j$; $y_1 \ge \cdots \ge y_k \ge 0$; $v_i \in (0,1)$ for $i = 1,\dots,k$; $u_1,\dots,u_j, v_1,\dots,v_k$ are all distinct.*

(a) For any $\epsilon > 0$,

$$\{x \in G : d(x,y) < (1+\epsilon)\delta \text{ implies } y \in G\} \subseteq G^{-\delta} \subseteq \{x \in G : d(x,y) < \delta \text{ implies } y \in G\}.$$

Also, $(A\cap B)_\delta \subseteq A_\delta \cap B_\delta$ and $A^{-\delta} \cup B^{-\delta} \subseteq (A\cup B)^{-\delta}$ for any A and B.

(b) $\sum_{i=1}^j x_i 1_{[u_i,1]} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant j-1})^{-\delta}$ implies $x_j \ge \delta$.

(c) $\sum_{i=1}^j x_i 1_{[u_i,1]} \notin (\mathbb{D}\setminus\mathbb{D}_{\leqslant j-1})^{-\delta}$ implies $x_j \le 2\delta$.

(d) $\sum_{i=1}^j x_i 1_{[u_i,1]} - \sum_{i=1}^k y_i 1_{[v_i,1]} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\delta}$ implies $x_j \ge \delta$ and $y_k \ge \delta$.

(e) Suppose that $\sum_{i=1}^j x_i 1_{[u_i,1]} - \sum_{i=1}^k y_i 1_{[v_i,1]} \in \mathbb{W}_{l,m}$.

(e-i) If $l \in [0, j-1]$ then $x_{l+1} \le 2\gamma_{l,m}$.

(e-ii) If $m \in [0, k-1]$ then $y_{m+1} \le 2\gamma_{l,m}$.

(e-iii) If $l \in [1, j]$ then $x_l > \frac{\gamma_{l-1,m}}{2(j-l+1)}$.

(e-iv) If $m \in [1,k]$ then $y_m > \frac{\gamma_{l,m-1}}{2(k-m+1)}$.

(e-v) $l \le j$ and $m \le k$.

(f) Suppose that $\xi \in \mathbb{D}_{j,k}$. If $l < j$ or $m < k$, then ξ is bounded away from $\mathbb{D}_{l,m}$.

(g) If $I(\xi) > (\alpha-1)j + (\beta-1)k$, then ξ is bounded away from $\mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}$.

Proof. (a) Immediate consequences of the definition.

(b) From (a), we see that $\sum_{i=1}^{j} x_i 1_{[u_i,1]} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant j-1})^{-\delta}$ and $\sum_{i=1}^{j-1} x_i 1_{[u_i,1]} \in \mathbb{D}_{\leqslant j-1}$ implies $d\Big(\sum_{i=1}^{j} x_i 1_{[u_i,1]}, \sum_{i=1}^{j-1} x_i 1_{[u_i,1]}\Big) \ge \delta$, which is not possible if $x_j < \delta$.

(c) We prove that for any $\epsilon > 0$, $\sum_{i=1}^{j} x_i 1_{[u_i,1]} \notin (\mathbb{D}\setminus\mathbb{D}_{\leqslant j-1})^{-\delta}$ implies $x_j < (2+\epsilon)\delta$. To show this, in turn, we work with the contrapositive. Suppose that $x_j > (2+\epsilon)\delta$. If $d(\sum_{i=1}^{j} x_i 1_{[u_i,1]}, \zeta) < (1+\epsilon/2)\delta$, by the definition of the Skorokhod metric, there exists a non-decreasing homeomorphism ϕ of $[0,1]$ onto itself such that $\|\sum_{i=1}^{j} x_i 1_{[u_i,1]} - \zeta\circ\phi\|_\infty < (1+\epsilon/2)\delta$. Note that at each discontinuity point of $\sum_{i=1}^{j} x_i 1_{[y_i,1]}$, $\zeta\circ\phi$ should also be discontinuous. Otherwise, the supremum distance between $\sum_{i=1}^{j} x_i 1_{[u_i,1]}$ and $\zeta\circ\phi$ has to be greater than $(1+\epsilon/2)\delta$, since the smallest jump size of $\sum_{i=1}^{j} x_i 1_{[u_i,1]}$ is greater than $(2+\epsilon)\delta$. Hence, there has to be at least j discontinuities in the path of ζ; i.e., $\zeta \in \mathbb{D}\setminus\mathbb{D}_{\leqslant j-1}$. We have shown that $d(\sum_{i=1}^{j} x_i 1_{[u_i,1]}, \zeta) < (1+\epsilon/2)\delta$ implies $\zeta \in \mathbb{D}\setminus\mathbb{D}_{\leqslant j-1}$, which in turn, along with (a), shows that $\sum_{i=1}^{j} x_i 1_{[u_i,1]} \in (\mathbb{D}\setminus\mathbb{D}_{\leqslant j-1})^{-\delta}$.

(d) Suppose that $\sum_{i=1}^{j} x_i 1_{[u_i,1]} - \sum_{i=1}^{k} y_i 1_{[v_i,1]} \in (\mathbb{D}\setminus\mathbb{D}_{<j,k})^{-\delta}$. Since $\sum_{i=1}^{j-1} x_i 1_{[u_i,1]} - \sum_{i=1}^{k} y_i 1_{[v_i,1]} \notin \mathbb{D}\setminus\mathbb{D}_{<j,k}$,

$$x_j \ge d_{sk}\left(\sum_{i=1}^{j} x_i 1_{[u_i,1]} - \sum_{i=1}^{k} y_i 1_{[v_i,1]},\ \sum_{i=1}^{j-1} x_i 1_{[u_i,1]} - \sum_{i=1}^{k} y_i 1_{[v_i,1]}\right) \ge \delta.$$

Similarly, we get $y_k \ge \delta$.

(e) Let $\zeta \triangleq \sum_{i=1}^{j} x_i 1_{[u_i,1]} - \sum_{i=1}^{k} y_i 1_{[v_i,1]}$. We skip the proofs of (e-ii), (e-iv), and the second claim of (e-v), since they are essentially identical to the proofs of (e-i), (e-iii), and the first claim of (e-v). For (e-i), suppose that $x_{l+1} > 2\gamma_{l,m}$. Then, for any $\zeta' \in \mathbb{D}_{l,m}$, $d(\zeta,\zeta') \ge x_{l+1}/2 > \gamma_{l,m}$, since the number of upward jump doesn't match—ζ has more—and at least one of ζ's unmatched jumps has size greater than x_l. This means that $\inf_{\zeta'\in\mathbb{D}_{l,m}} d(\zeta,\zeta') > \gamma_{l,m}$, i.e., $\zeta \notin \hat{\mathbb{D}}_{l,m}$, which is contradictory to $\zeta \in \mathbb{W}_{l,m}$. For (e-iii), suppose that $x_l \le \frac{\gamma_{l-1,m}}{2(j-l+1)}$. We will derive a contradiction. Note first that if $m < k$, (e-ii) guarantees that $y_{m+1} \le 2\gamma_{l,m}$, and hence, $\sum_{i=m+1}^{k} y_i \le (k-m)2\gamma_{l,m}$. For arbitrary m, therefore, $\sum_{i=m+1}^{k} y_i \le (k-k\wedge m)2\gamma_{l,m}$. Let $\zeta' \triangleq \sum_{i=1}^{l-1} x_i 1_{[u_i,1]} - \sum_{i=1}^{k} y_i 1_{[v_i,1]}$, and $\zeta'' \triangleq \sum_{i=1}^{l-1} x_i 1_{[u_i,1]} - \sum_{i=1}^{m\wedge k} y_i 1_{[v_i,1]}$. Then $d(\zeta,\zeta') \le (j-l+1)x_l$, and $d(\zeta',\zeta'') \le (k-k\wedge m)2\gamma_{l,m}$. From these and the assumption, $d(\zeta,\zeta'') \le \gamma_{l-1,m}/2 + (k-k\wedge m)2\gamma_{l,m} \le \gamma_{l-1,m}$, which is contradictory to the assumption that $\zeta \in \mathbb{W}_{l,m} \subseteq \Big(\hat{\mathbb{D}}_{l-1,m}\Big)^c = \Big((\mathbb{D}_{l-1,m})_{\gamma_{l-1,m}}\Big)^c$. For the first claim of (e-v), suppose that $l > j$. Again, we will derive a contradiction. Note that $l > j$ and $(l,m) \in I$ implies that $m < k$, and as in (e-iii), $\sum_{i=m+1}^{k} y_i \le (k-m)2\gamma_{l,m}$. Let $\zeta' \triangleq \sum_{i=1}^{j} x_i 1_{[u_i,1]} + \sum_{i=j+1}^{l-1} \epsilon 1_{[u_i,1]} - \sum_{j=1}^{k} y_i 1_{[v_i,1]}$ and $\zeta'' \triangleq \sum_{i=1}^{j} x_i 1_{[u_i,1]} + \sum_{i=j+1}^{l-1} \epsilon 1_{[u_i,1]} - \sum_{j=1}^{m} y_i 1_{[v_i,1]}$. Then, $d(\zeta,\zeta'') \le d(\zeta,\zeta') + d(\zeta',\zeta'') \le \epsilon(l-1-j) + (k-m)2\gamma_{l,m}$. By choosing ϵ small enough, we can make $d(\zeta,\zeta'') \le \gamma_{l-1,m}$, which is contradictory to $\zeta \in \mathbb{W}_{l,m}$.

(f) We omit a detailed proof of (f) since it is almost identical to the proof of (c); roughly speaking, the distance between ξ and any element in $\mathbb{D}_{l,m}$ is at least half of i) the smallest upward jump size of ξ in case $l < j$, or ii) the smallest downward jump size of ξ in case $m < k$.

(g) Note that in case $I(\xi)$ is finite, $\mathcal{D}_+(\xi) > j$ or $\mathcal{D}_-(\xi) > k$. In this case, the conclusion is immediate from (f). In case $I(\xi) = \infty$, either $\mathcal{D}_+(\zeta) = \infty$, $\mathcal{D}_-(\zeta) = \infty$, $\xi(0) \ne 0$, or ξ contains a continuous non-constant piece. By containing a continuous non-constant piece, we refer to the case that there exist t_1 and

t_2 such that $t_1 < t_2$, $\xi(t_1) \neq \xi(t_2-)$ and ξ is continuous on (t_1, t_2). For the first two cases where the number of jumps is infinite, the conclusion is an immediate consequence of (f). The case $\xi(0) \neq 0$ is also obvious. Now we are left with dealing with the last case, where ξ has a continuous non-constant piece. To discuss this case, assume w.l.o.g. that $\xi(t_1) < \xi(t_2-)$. We claim that $d(\xi, \mathbb{D}_{j,k}) \geq \frac{\xi(t_2-)-\xi(t_1)}{2(j+1)}$. Note that for any step function ζ,

$$\begin{aligned}\|\xi - \zeta\| &\geq |\xi(t_2-) - \zeta(t_2-)| \vee |\xi(t_1) - \zeta(t_1)| \\ &\geq (\xi(t_2-) - \zeta(t_2-)) \vee (\zeta(t_1) - \xi(t_1)) \\ &\geq \frac{1}{2}\Big\{(\xi(t_2-) - \xi(t_1)) - (\zeta(t_2-) - \zeta(t_1))\Big\} \\ &\geq \frac{1}{2}\Big\{(\xi(t_2-) - \xi(t_1)) - \sum_{t\in(t_1,t_2)} \big(\zeta(t) - \zeta(t-)\big)\Big\} \\ &\geq \frac{1}{2}\Big\{(\xi(t_2-) - \xi(t_1)) - 2\mathcal{D}_+(\zeta)\|\xi - \zeta\|\Big\},\end{aligned}$$

where the fourth inequality is due to the fact that $\|\xi - \zeta\| \geq \frac{\zeta(t)-\zeta(t-)}{2}$ for all $t \in (t_1, t_2)$. From this, we get

$$\|\xi - \zeta\| \geq \frac{\xi(t_2-) - \xi(t_1)}{2(\mathcal{D}_+(\zeta) + 1)} \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j + 1)},$$

for $\zeta \in \mathbb{D}_{j,k}$. Now, suppose that $\zeta \in \mathbb{D}_{j,k}$. Since $\zeta \circ \phi$ is again in $\mathbb{D}_{j,k}$ for any non-decreasing homeomorphism ϕ of $[0, 1]$ onto itself,

$$d(\xi, \zeta) \geq \frac{\xi(t_2-) - \xi(t_1)}{2(j + 1)},$$

which proves the claim. ☐

6.3 Proofs for Section 4

Recall that

$$I(\xi) \triangleq \begin{cases} (\alpha - 1)\mathcal{D}_+(\xi) + (\beta - 1)\mathcal{D}_-(\xi) & \text{if } \xi \text{ is a step function with } \xi(0) = 0 \\ \infty & \text{otherwise} \end{cases}.$$

Proof of Theorem 4.2. Observe first that $I(\cdot)$ is a rate function. The level sets $\{\xi : I(\xi) \leq x\}$ equal $\bigcup_{\substack{(l,m)\in\mathbb{Z}_+^2 \\ (\alpha-1)l+(\beta-1)m\leq\lfloor x\rfloor}} \mathbb{D}_{l,m}$ and are therefore closed—note the level sets are not compact so $I(\cdot)$ is not a good rate function (see, for example, Dembo and Zeitouni (2009) for the definition and properties of good rate functions).

Starting with the lower bound, suppose that G is an open set. We assume w.l.o.g. that $\inf_{\xi\in G} I(\xi) < \infty$, since the inequality is trivial otherwise. Due to the discrete nature of $I(\cdot)$, there exists a $\xi^* \in G$ such that $I(\xi^*) = \inf_{\xi\in G} I(\xi)$. Set $j \triangleq \mathcal{D}_+(\xi^*)$ and $k \triangleq \mathcal{D}_-(\xi^*)$. Let $u_1^+, \ldots, u_j^+$ be the sorted (from the earliest to the latest) upward jump times of ξ^*; $x_1^+, \ldots, x_j^+$ be the sorted (from the largest to the smallest) upward jump sizes of ξ^*; $u_1^-, \ldots, u_k^-$ be the sorted downward jump times of ξ^*; $x_1^-, \ldots, x_k^-$ be the sorted downward jump sizes of ξ^*. Also, let $x_{j+1}^+ = x_{k+1}^- = 0$, $u_0^+ = u_0^- = 0$, and $u_{j+1}^+ = u_{k+1}^- = 1$. Note that if $\zeta \in \mathbb{D}_{l,m}$ for $l < j$, then $d(\xi^*, \zeta) \geq x_j^+/2$ via a similar argument as in the proof of Lemma 6.4 (a). Likewise, if $\zeta \in \mathbb{D}_{l,m}$ for $m < k$, then $d(\xi^*, \zeta) \geq x_k^-/2$. Therefore, $d(\mathbb{D}_{<j,k}, \xi^*) \geq (x_j^+ \wedge x_k^-)/2$. On the other hand, since G is an open set, we can pick $\delta_0 > 0$ so that the open ball $B_{\xi^*,\delta_0} \triangleq \{\zeta \in \mathbb{D} : d(\zeta, \xi) < \delta_0\}$ centered at ξ^* with radius δ_0 is a subset of G—i.e., $B_{\xi^*,\delta_0} \subset G$. Let $\delta = (\delta_0 \wedge x_j^+ \wedge x_k^-)/4$. If $j = k = 0$, then $\xi^* \equiv 0$, and hence, $\{\bar{X}_n \in G\}$ contains $\{\|\bar{X}_n\| \le \delta\}$ which is a subset of $B_{\xi^*,\delta}$. One can apply Lemma A.1 to show that $\mathbf{P}(X_n \in G)$ converges to 1, which, in turn, proves the inequality. Now, suppose that either $j \ge 1$ or $k \ge 1$. Then, $d(B_{\xi^*,\delta}, \mathbb{D}_{<j,k}) \ge \delta$. As $d(B_{\xi^*,\delta}, \mathbb{D}_{<j,k}) > 0$ and $B_{\xi^*,\delta}$ is open, we see from our sharp asymptotics (Theorem 3.2) that

$$C_{j,k}(B_{\xi^*,\delta}) \le \liminf_{n\to\infty} (n\nu[n,\infty))^{-j}(n\nu(-\infty,-n])^{-k} P(\bar{X}_n \in B_{\xi^*,\delta}).$$

From the definition of $C_{j,k}$, it follows that $C_j(B_{\xi^*,\delta}) > 0$. To see this, note first that we can assume w.l.o.g. that $x_i^{\pm}$'s are all distinct since G is open (because, if some of the jump sizes are identical, we can pick ϵ such that $B_{\xi^*,\epsilon} \subseteq G$, and then perturb those jump sizes by ϵ to get a new ξ^* which still belongs to G while whose jump sizes are all distinct.) Suppose that $\xi^* = \sum_{l=1}^{j} x^+_{i_l^+} 1_{[u_i^+,1]} - \sum_{l=1}^{k} x^-_{i_l^-} 1_{[u_i^-,1]}$, where $\{i_1^{\pm}, \dots, i_j^{\pm}\}$ are permutations of $\{1,\dots,j\}$. Let $2\delta' \triangleq \delta \wedge \underline{\Delta}_u^+ \wedge \underline{\Delta}_x^+ \wedge \underline{\Delta}_u^- \wedge \underline{\Delta}_x^-$, where $\underline{\Delta}_u^+ = \min_{i=1,\dots,j+1}(u_i^+ - u_{i-1}^+)$, $\underline{\Delta}_x^+ = \min_{i=1,\dots,j}(x_{i-1}^+ - x_i^+)$, $\underline{\Delta}_u^- = \min_{i=1,\dots,k+1}(u_i^- - u_{i-1}^-)$, and $\underline{\Delta}_x^- = \min_{i=1,\dots,k}(x_{i-1}^- - x_i^-)$. Consider a subset B' of $B_{\xi^*,\delta}$:

$$B' \triangleq \bigg\{ \sum_{l=1}^{j} y^+_{i_l^+} 1_{[v_l^+,1]} - \sum_{l=1}^{k} y^-_{i_l^-} 1_{[v_l^-,1]} : \\ v_i^+ \in (u_i^+ - \delta', u_i^+ + \delta'), y_i^+ \in (x_i^+ - \delta', x_i^+ + \delta'), i = 1,\dots,j; \\ v_i^- \in (u_i^- - \delta', u_i^- + \delta'), y_i^- \in (x_i^- - \delta', x_i^- + \delta'), i = 1,\dots,k \bigg\}.$$

Then,

$$\begin{aligned}
&C_{j,k}(B_{\xi^*,\delta}) \\
&\ge C_{j,k}(B') = (\mu_\alpha \times \mu_\alpha \times Leb \times Leb) \circ T_{j,k}^{-1}(B') \\
&= \int_{(u_1^+-\delta',u_1^++\delta')\times\cdots\times(u_j^+-\delta',u_j^++\delta')} dLeb \cdot \int_{(x_1^+-\delta',x_1^++\delta')\times\cdots\times(x_j^+-\delta',x_j^++\delta')} d\nu_\alpha \\
&\quad \cdot \int_{(u_1^--\delta',u_1^-+\delta')\times\cdots\times(u_k^--\delta',u_k^-+\delta')} dLeb \cdot \int_{(x_1^--\delta',x_1^-+\delta')\times\cdots\times(x_k^--\delta',x_k^-+\delta')} d\nu_\beta \\
&\ge (2\delta')^j (2\delta'(x_1^+)^\alpha)^j (2\delta')^k (2\delta'(x_1^-)^\beta)^k > 0.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\log P(\bar{X}_n \in G)}{\log n} &\ge \liminf_{n\to\infty} \frac{\log P(\bar{X}_n \in B_{\xi^*,\delta})}{\log n} \\
&\ge \liminf_{n\to\infty} \frac{\log(C_{j,k}(B_{\xi^*,\delta})(n\nu[n,\infty))^j(n\nu(-\infty,-n])^k(1+o(1)))}{\log n} \qquad (6.18) \\
&= -\big((\alpha-1)j + (\beta-1)k\big),
\end{aligned}$$

which is the lower bound. Turning to the upper bound, suppose that K is a compact set. We first consider the case where $\inf_{\xi\in K} I(\xi) < \infty$. Pick ξ^*, j and k as in the lower bound, i.e., $I(\xi^*) \triangleq \inf_{\xi\in K} I(\xi)$, $j \triangleq \mathcal{D}_+(\xi^*)$, and $k \triangleq \mathcal{D}_-(\xi^*)$. Here we can assume w.l.o.g. either $j \ge 1$ or $k \ge 1$ since the inequality is trivial in case $j = k = 0$. For each $\zeta \in K$, either $I(\zeta) > I(\xi^*)$, or $I(\zeta) = I(\xi^*)$. We construct an open cover of K by considering these two cases separately:

- If $I(\zeta) > I(\xi^*)$, ζ is bounded away from $\mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}$ (Lemma 6.4 (g)). For each such ζ's, pick a $\delta_\zeta > 0$ in such a way that $d(\zeta, \mathbb{D}_{<j,k} \cup \mathbb{D}_{j,k}) > \delta_\zeta$. Set $j_\zeta \triangleq j$ and $k_\zeta \triangleq k$. Note that in this case $C_{j_\zeta,k_\zeta}(\bar{B}_{\zeta,\delta_\zeta}) = 0$.

- If $I(\zeta) = I(\xi^*)$, set $j_\zeta \triangleq \mathcal{D}_+(\zeta)$ and $k_\zeta \triangleq \mathcal{D}_-(\zeta)$. Since they are bounded away from $\mathbb{D}_{<j_\zeta,k_\zeta}$ (Lemma 6.4 (f)), we can choose $\delta_\zeta > 0$ such that $d(\zeta, \mathbb{D}_{<j_\zeta,k_\zeta}) > \delta_\zeta$ and $C_{j_\zeta,k_\zeta}(\bar{B}_{\zeta,\delta_\zeta}) < \infty$.

Consider an open cover $\{B_{\zeta;\delta_\zeta} : \zeta \in K\}$ of K and its finite subcover $\{B_{\zeta_i;\delta_{\zeta_i}}\}_{i=1,\dots,m}$. For each ζ_i, we apply the sharp asymptotics (Theorem 3.5) to $\bar{B}_{\zeta_i;\delta_{\zeta_i}}$ and repeat a similar argument to (6.18) to get

$$\limsup_{n\to\infty} \frac{\log P(\bar{X}_n \in \bar{B}_{\zeta_i;\delta_{\zeta_i}})}{\log n} \le (\alpha-1)j_{\zeta_i} + (\beta-1)k_{\zeta_i} = -I(\xi^*).$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log P(\bar{X}_n \in \bar{F})}{\log n} &\le \limsup_{n\to\infty} \frac{\log \sum_{i=1}^m P(\bar{X}_n \in \bar{B}_{\zeta_i;\delta_{\zeta_i}})}{\log n} \\ &= \max_{i=1,\dots,m} \limsup_{n\to\infty} \frac{\log P(\bar{X}_n \in \bar{B}_{\zeta_i;\delta_{\zeta_i}})}{\log n} \\ &\le -I(\xi^*) = -\inf_{\xi\in K} I(\xi),\end{aligned}$$

completing the proof of the upper bound in case the right-hand side is finite.

Now, turning to the case $\inf_{\xi\in K} I(\xi) = \infty$, fix an arbitrary finite real number m. Then, $\mathbb{D}_{<m,m}$ is bounded away from each $\zeta \in K$. A similar but simpler argument as for the case of finite infimum, one can show that

$$\limsup_{n\to\infty} \frac{\log P(\bar{X}_n \in K)}{\log n} \le -m.$$

Taking $m \to \infty$, we arrive at the upper bound. □

A Inequalities

Result 4 (Generalized Kolmogorov inequality; Shneer and Wachtel (2009))**.** *Let $S_n = X_1 + \cdots + X_n$ be a random walk with mean zero increments, i.e., $\mathbf{E}X_i = 0$. Then,*

$$\mathbf{P}(\max_{k\le n} S_k \ge x) \le C\frac{nV(x)}{x^2},$$

where $V(x) = \mathbf{E}(X_1^2; |X_1| \le x)$, for all $x > 0$.

Result 5 (Etemadi's inequality)**.** *Let $X_1, ..., X_n$ be independent real-valued random variables defined on some common probability space, and let $\alpha \ge 0$. Let S_k denote the partial sum $S_k = X_1 + \cdots + X_k$. Then*

$$\mathbb{P}\Big(\max_{1\le k\le n} |S_k| \ge 3\alpha\Big) \le 3 \max_{1\le k\le n} \mathbb{P}\big(|S_k| \ge \alpha\big).$$

Result 6 (Prokhorov's inequality; Prokhorov (1959))**.** *Suppose that ξ_i, $i = 1, \dots, n$ are independent, zero-mean random variables such that there exists a constant c for which $|\xi_i| \le c$ for $i = 1, \dots, n$, and $\sum_{i=1}^n \mathbf{var}\,\xi_i < \infty$. Then*

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i > x\right) \le \exp\left\{-\frac{x}{2c}\operatorname{arcsinh}\frac{xc}{2\sum_{i=1}^n \mathbf{var}\,\xi_i}\right\},$$

which, in turn, implies

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i > x\right) \le \left(\frac{cx}{\sum_{i=1}^n \mathbf{var}\,\xi_i}\right)^{-\frac{x}{2c}}.$$

Lemma A.1 (Etemadi's inequality for Lévy processes)**.** *Let Z be a Lévy process. Then,*

$$\mathbb{P}\Big(\sup_{t\in[0,n]} |Z(t)| \ge \delta\Big) \le 3 \sup_{t\in[0,n]} \mathbf{P}\big(|Z(t)| \ge \delta/3\big).$$

Proof. Since Z (and hence $|Z|$ also) is in $\mathbb{D}$, $\sup_{0\le k\le 2^m} |Z(\frac{nk}{2^m})|$ converges to $\sup_{t\in[0,n]} |Z(t)|$ almost surely as $m \to \infty$. To see this, note that one can choose t_i's such that $|Z(t_i)| \ge \sup_{t\in[0,n]} |Z(t)| - i^{-1}$. Since $\{t_i\}$'s are in a compact set $[0, n]$, there is a subsequence, say, t_i', such that $t_i' \to t_0$ for some $t_0 \in [0, n]$. The supremum has to be achieved at either t_0^- or t_0. Either way, with large enough m, $\sup_{0\le k\le 2^m} |Z(\frac{nk}{2^m})|$ becomes arbitrarily close to the supremum. Now, by bounded convergence,

$$\begin{aligned}
\mathbf{P}\left\{\sup_{t\in[0,n]} |Z(t)| > \delta\right\} &= \lim_{m\to\infty} \mathbf{P}\left\{\sup_{0\le k\le 2^m} \left|Z(\frac{nk}{2^m})\right| > \delta\right\} \\
&= \lim_{m\to\infty} \mathbf{P}\left\{\sup_{0\le k\le 2^m} \left|\sum_{i=0}^k \left(Z(\frac{ni}{2^m}) - Z(\frac{n(i-1)}{2^m})\right)\right| > \delta\right\} \\
&\le \lim_{m\to\infty} 3 \sup_{0\le k\le 2^m} \mathbf{P}\left\{\left|\sum_{i=0}^k \left(Z(\frac{ni}{2^m}) - Z(\frac{n(i-1)}{2^m})\right)\right| > \delta/3\right\} \\
&= \lim_{m\to\infty} 3 \sup_{0\le k\le 2^m} \mathbf{P}\left\{\left|Z(\frac{nk}{2^m})\right| > \delta/3\right\} \\
&\le 3 \sup_{t\in[0,n]} \mathbf{P}\left\{|Z(t)| > \delta/3\right\}, \qquad \text{(A.1)}
\end{aligned}$$

where $Z(t) \triangleq 0$ for $t < 0$. ∎

B List of Notations

- $(\mathbb{S}, d)$: complete separable metric space
- $F_\delta \triangleq \{x \in \mathbb{D} : d(x, F) \leq \delta\}$
- $G^{-\delta} \triangleq ((G^c)_\delta)^c$
- A°: interior of A
  A^-: closure of A
  $\partial A = A^- \setminus A^\circ$: boundary of A
- ν: regularly varying Lévy measure with index $-\alpha$ and $-\beta$
  i.e., $\nu[n, \infty) = n^{-\alpha} L_+(n)$ and $\nu(-\infty, -n] = n^{-\beta} L_-(n)$
  $L_+(n) = n^\alpha \nu[n, \infty)$
  $L_-(n) = n^\beta \nu(-\infty, -n]$
- X: Lévy process with Lévy measure ν
  $X_n(t) = X(nt)$
  $\bar{X}_n(t) = \frac{1}{n} X_n(t) - ta - \mu_1^+ \nu_1^+ t$ or $\bar{X}_n(t) = \frac{1}{n} X_n(t) - ta - (\mu_1^+ \nu_1^+ - \mu_1^- \nu_1^-) t$
- $I_{<j,k} = \{(l, m) \in \mathbb{Z}_+^2 \setminus (j, k) : (\alpha - 1)l + (\beta - 1)m \leq (\alpha - 1)j + (\beta - 1)k\}$
  $I_{=j,k} = \{(l, m) \in \mathbb{Z}_+^2 : (\alpha - 1)l + (\beta - 1)m = (\alpha - 1)j + (\beta - 1)k\}$
  $I_{\ll j,k} = \{(l, m) \in \mathbb{Z}_+^2 : (\alpha - 1)l + (\beta - 1)m < (\alpha - 1)j + (\beta - 1)k\}$
- $\mathbb{R}_+$: set of non-negative real numbers
  $\mathbb{Z}_+$: set of non-negative integers
- $\mathbb{R}_+^{\infty\downarrow} = \{x \in \mathbb{R}_+^\infty : x_1 \geq x_2 \geq \ldots\}$
  $\mathbb{R}_+^{j\downarrow} = \{x \in \mathbb{R}_+^j : x_1 \geq x_2 \geq \ldots \geq x_j\}$
  $\mathbb{H}_j = \{x \in \mathbb{R}_+^{\infty\downarrow} : x_j > 0, x_{j+1} = 0\}$
  $\mathbb{H}_{\leqslant j} = \{x \in \mathbb{R}_+^{\infty\downarrow} : x_{j+1} = 0\}$
  $\mathbb{H}_{j,k} = \{(x, y) \in \mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} : x_j > 0, x_{j+1} = 0, y_k > 0, y_{k+1} = 0\}$
  $\mathbb{H}_{<j,k} = \bigcup_{(l,m) \in I_{<j,k}} \mathbb{H}_{l,m}$
- $\mathbb{D} = \mathbb{D}([0, 1], \mathbb{R})$: real-valued RCLL functions on $[0, 1]$
  $\mathbb{D}_s^\uparrow$: subspace of $\mathbb{D}$ consisting of non-decreasing step functions vanishing at 0
  $\mathbb{D}_j$: subspace of $\mathbb{D}$ consisting of non-decreasing step functions vanishing at 0 with j jumps
  $\mathbb{D}_{j,k}$: subspace of $\mathbb{D}$ consisting of step functions vanishing at 0 with j upward jumps and k downward jumps
  $\mathbb{D}_{\leqslant j} = \bigcup_{0 \leqslant l \leqslant j} \mathbb{D}_l$
  $\mathbb{D}_{<j,k} = \bigcup_{(l,m) \in I_{<j,k}} \mathbb{D}_{l,m}$
  $\mathbb{D}_{=j,k} = \bigcup_{(l,m) \in I_{=j,k}} \mathbb{D}_{l,m}$
  $\mathbb{D}_{\ll j,k} = \bigcup_{(l,m) \in I_{\ll j,k}} \mathbb{D}_{l,m}$
- d: Skorokhod metric on $\mathbb{D}([0, 1], \mathbb{R})$
- $\hat{S}_j = \{(x, u) \in \mathbb{R}_+^{j\downarrow} \times [0, 1]^j : 0, 1, u_1, \ldots, u_j \text{ are all distinct}\}$
  $S_j = \{(x, u) \in \mathbb{R}_+^{\infty\downarrow} \times [0, 1]^\infty : 0, 1, u_1, \ldots, u_j \text{ are all distinct}\}$
  $S_{j,k} = \{(x, y, u, v) \in \mathbb{R}_+^{\infty\downarrow} \times \mathbb{R}_+^{\infty\downarrow} \times [0, 1]^\infty \times [0, 1]^\infty : 0, 1, u_1, \ldots, u_j, v_1, \ldots, v_k \text{ are all distinct}\}$.
  $\hat{T}_j : \hat{S}_j \to \mathbb{D}_j$ defined by $\hat{T}_j(x, u) = \sum_{i=1}^j x_i 1_{[u_i, 1]}$

$T_m : S_m \to \mathbb{D}$ defined by $T_m(x,u) = \sum_{i=1}^m x_i 1_{[u_i,1]}$
$T_{j,k} : S_{j,k} \to \mathbb{D}$ defined by $T_{j,k}(x,y,u,v) = \sum_{i=1}^j x_i 1_{[u_i,1]} - \sum_{i=1}^k y_i 1_{[v_i,1]}$

- $\nu_\alpha(x,\infty) = x^{-\alpha}$
  ν_α^j: restriction (to $\mathbb{R}_+^{j\downarrow}$) of j-fold product measure of ν_α
- U_i, V_i: i.i.d. uniform random variables on $[0,1]$
- $C_j(\cdot) = \mathbf{E}\left[\nu_\alpha^j\{y \in (0,\infty)^j : \sum_{i=1}^j y_i 1_{[U_i,1]} \in \cdot\}\right]$
  $C_{j,k}(\cdot) = \mathbf{E}\Big[\nu_\alpha^j \times \nu_\beta^k\{(x,y) \in (0,\infty)^j \times (0,\infty)^k : \sum_{i=1}^j x_i 1_{[U_i,1]} - \sum_{i=1}^k y_i 1_{[V_i,1]} \in \cdot\}\Big]$
- $\nu_1^+ = \nu[1,\infty)$
  $\mu_1^+ = \frac{1}{\nu_1^+}\int_{[1,\infty)} x\nu(dx)$
  $\nu_1^- = \nu(-\infty,-1]$
  $\mu_1^- = \frac{1}{\nu_1^-}\int_{(-\infty,-1]} x\nu(dx)$
- $\mathcal{D}_+(\xi)$: number of upward jumps of $\xi \in \mathbb{D}$
  $\mathcal{D}_-(\xi)$: number of downward jumps of $\xi \in \mathbb{D}$
- $\mathcal{J}(A) = \inf_{\xi \in \mathbb{D}_s^\uparrow \cap A} \mathcal{D}_+(\xi)$
- $\mathcal{I}(j,k) = (\alpha-1)j + (\beta-1)k$
- $(\mathcal{J}(A), \mathcal{K}(A)) = \arg\min_{\substack{(j,k)\in\mathbb{Z}_+^2 \\ \mathbb{D}_{j,k}\cap A \neq \emptyset}} \mathcal{I}(j,k)$
- $I(\xi) \triangleq \begin{cases} (\alpha-1)\mathcal{D}_+(\xi) + (\beta-1)\mathcal{D}_-(\xi) & \text{if } \xi \text{ is a step function with } \xi(0)=0 \\ \infty & \text{otherwise} \end{cases}$
- $\bar{\gamma} = \gamma \max_{(l,m)\in I_{<j,k}} (4k)^l (4j)^m$
  $\gamma_{l,m} = \bar{\gamma}(4k)^{-l}(4j)^{-m}$
  $\hat{\mathbb{D}}_{l,m} = (\mathbb{D}_{l,m})_{\gamma_{l,m}}$
  $\mathbb{W}_{l,m} = \hat{\mathbb{D}}_{l,m} \setminus (\hat{\mathbb{D}}_{l,m-1} \cup \hat{\mathbb{D}}_{l-1,m})$
- $\delta_{(x,y)}$: Dirac measure concentrated on (x,y)
- $Q_n(x) = n\nu[x,\infty)$
  $Q_n^\leftarrow(y) = \inf\{s > 0 : n\nu[s,\infty) < y\}$
  $\tilde{N}_n = N_n\big([0,1]\times[1,\infty)\big)$
  $N_n = \sum_{l=1}^\infty \delta_{(U_l, Q_n^\leftarrow(\Gamma_l))}$
- $R_n(x) = n\nu(-\infty,-x]$
  $R_n^\leftarrow(y) = \inf\{s > 0 : n\nu(-\infty,-s] < y\}$
  $\tilde{M}_n = M_n\big([0,1]\times[1,\infty)\big)$
  $M_n = \sum_{l=1}^\infty \delta_{(V_l, Q_n^\leftarrow(\Gamma_l))}$
- $J_n(s) = \sum_{l=1}^{\tilde{N}_n} Q_n^\leftarrow(\Gamma_l) 1_{[U_l,1]}(s) \stackrel{\mathcal{D}}{=} \int_{x>1} xN([0,ns]\times dx)$, $\qquad \Gamma_l = E_1 + E_2 + ... + E_l$
  $K_n(s) = \sum_{l=1}^{\tilde{M}_n} R_n^\leftarrow(\Delta_l) 1_{[V_l,1]}(s) \stackrel{\mathcal{D}}{=} \int_{x<-1} -xN([0,ns]\times dx)$, $\qquad \Delta_l = F_1 + F_2 + ... + F_l$
  E_i's and F_i's are i.i.d. standard exponential random variables
  U_l's and V_l's are i.i.d. uniform variables in $[0,1]$

- $\bar{J}_n = \frac{1}{n}\sum_{l=1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]}$
  $\hat{J}_n^{\leqslant j} = \frac{1}{n}\sum_{l=1}^{j} Q_n^{\leftarrow}(\Gamma_l)1_{[U_l,1]}$
  $\check{J}_n^{\leqslant j} = \frac{1}{n}\sum_{l=1}^{j} - \mu_1^+ 1_{[U_l,1]}$,
  $\bar{J}_n^{>j} = \frac{1}{n}\sum_{l=j+1}^{\tilde{N}_n}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]}$,
  $\bar{R}_n^{+} = \frac{1}{n}\mathbb{I}(\tilde{N}_n < j)\sum_{l=\tilde{N}_n+1}^{j}(Q_n^{\leftarrow}(\Gamma_l) - \mu_1^+)1_{[U_l,1]}$,
  $\hat{K}_n^{\leqslant k} = \frac{1}{n}\sum_{l=1}^{k} R_n^{\leftarrow}(\Delta_l)1_{[V_l,1]}$,
  $\bar{K}_n = \frac{1}{n}\sum_{l=1}^{\tilde{M}_n}(R_n^{\leftarrow}(\Delta_l) - \mu_1^-)1_{[V_l,1]}$,
  $\check{K}_n^{\leqslant k} = \frac{1}{n}\sum_{l=1}^{k} - \mu_1^- 1_{[V_l,1]}$,
  $\bar{K}_n^{>k} = \frac{1}{n}\sum_{l=k+1}^{\tilde{M}_n}(R_n^{\leftarrow}(\Delta_l) - \mu_1^-)1_{[V_l,1]}$,
  $\bar{R}_n^{-} = \frac{1}{n}\mathbb{I}(\tilde{M}_n < j)\sum_{l=\tilde{M}_n+1}^{j}(R_n^{\leftarrow}(\Delta_l) - \mu_1^-)1_{[V_l,1]}$

Acknowledgements

We are grateful to Arkadi Nemirovski for insightful comments on the example in Section 5.3.

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