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Solution of the Laplace inversion problem for a special function

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§ 1. Introduction

A problem in the theory of electromagnetic waves studied by B. van der Pol [1] led to the question: Does there exist a function h(t) so that

$$(1) \qquad f(p) = \int_0^\infty \frac{e^{-z\sqrt{x^2+a^2p^2}}\, J_0(\rho x)\, x\, dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}$$

is the Laplace transform of h(t) in the sense that

$$(2) \qquad f(p) = p\int_0^\infty e^{-pt}\, h(t)dt \ ?$$

And if the answer is affirmative, give a manageable expression for this function h(t).

These problems will be solved in this paper by means of the complex inversion theorem for Laplace transforms ([2], Satz 21.2, p. 182). However, this theorem cannot be applied to f(p). Therefore, in § 2, we shall study a function $f_\mu(p)$, to which the inversion theorem applies if $\mu > 0$, and which has the property $f_\mu(p) \to f(p)$ if $\mu \to 0$. We shall find a function $h_\mu(t)$ which is related to $f_\mu(p)$ by (2) if $\mu > 0$. In § 3 we prove that $h_\mu(t)$ has a limit h(t) if $\mu \to 0$, and in § 4 it will be shown that this function h(t) solves our problem.

Finally, in § 5 we shall give the required manageable expressions, namely complete elliptic integrals.

Throughout the paper it will be assumed that ρ, z, a, b, c, d are positive numbers, and $a \neq b$.

The method of this paper applies equally well if in (1) $J_0(\rho x)$ is replaced by $J_\nu(\rho x)$, where ν is a natural number.

§ 2. A generalization.

In this section we consider the function

$$\text{(3)} \qquad f_{\mu}(p) = \int_0^{\infty} \frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} J_0(\rho x)\,x\,dx$$

for positive values of μ. (If $\mu=0$ we have again the function $f(p)$ defined in the previous section.) We shall try to find the original $h_{\mu}(t)$ of $f_{\mu}(p)$ in the sense of (2). As we intend to apply the complex inversion formula for Laplace transforms, we have to investigate the analytic continuation of $f_{\mu}(p)$ into a right half-plane. Therefore it is necessary to define the functions $\sqrt{x^2+a^2p^2}$ and $\sqrt{x^2+b^2p^2}$ for complex values of p. We make two cuts C_a and C_b in the complex w-plane. C_a consists of the two intervals $(\frac{i}{a}, i\infty)$ and $(-\frac{i}{a}, -i\infty)$ on the imaginary axis. $\sqrt{1+a^2w^2}$ is defined in the w-plane with cut C_a so that the root is positive on the real axis. If $p=xw$ then $\sqrt{x^2+a^2p^2}$ is defined as $x\sqrt{1+a^2w^2}$. In an analogous way C_b and $\sqrt{x^2+b^2p^2}$ are defined. It is not difficult to prove that

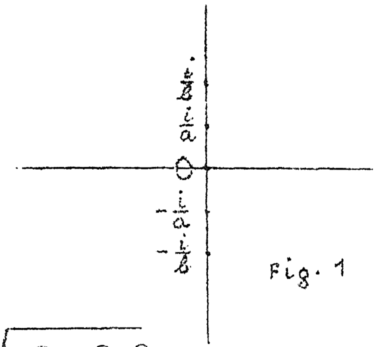

Fig. 1

$$\text{(4)} \qquad \text{Re}\sqrt{1+a^2w^2} \geq \text{Re}\, aw.$$

Applying this in the case Re $p > 0$, we find

$$\text{Re}\sqrt{x^2+a^2p^2} \geq 0, \qquad \text{Re}\sqrt{x^2+b^2p^2} \geq 0.$$

If $p=\sigma+i\tau$ $(\sigma>0,\ \tau>0)$, then we have

$$\text{Im}\sqrt{x^2+a^2p^2} > 0,$$

and

$$\left|\sqrt{x^2+a^2p^2}\right| = \sqrt{\left|x^2+a^2(\sigma^2-\tau^2)+2a^2\sigma\tau i\right|} \geq a\sqrt{2\sigma\tau}.$$

And in a similar way

$$\text{Im}\sqrt{x^2+b^2p^2} > 0, \quad \left|\sqrt{x^2+b^2p^2}\right| \geq b\sqrt{2v\tau}.$$

Therefore we find

$$\left|c\sqrt{x^2+a^2p^2} + d\sqrt{x^2+b^2p^2}\right| \geq \sqrt{2\sigma|\tau|}\sqrt{a^2c^2+b^2d^2}.$$

The same result holds if $\tau < 0$.

If $|\tau|<\sigma$ we also have the estimations

$$\left|\sqrt{x^2+a^2p^2}\right| \geqslant a\sqrt{\sigma^2-\tau^2}, \quad \left|\sqrt{x^2+b^2p^2}\right| \geqslant b\sqrt{\sigma^2-\tau^2},$$

$$\left|c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}\right| \geqslant \sqrt{\sigma^2-\tau^2}\sqrt{a^2c^2+b^2d^2}.$$

Applying these results we find

$$(5)\qquad \left|\frac{e^{-\mu x-z\sqrt{x^2+a^2p^2}}J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}\right| \leqslant \gamma\frac{e^{-\mu x}|J_0(\rho x)|x}{\sqrt{a^2c^2+b^2d^2}},$$

where $\gamma=(2\sigma|\tau|)^{-\frac{1}{2}}$, and if $|\tau|<\sigma$ we may also take $\gamma=(\sigma^2-\tau^2)^{-\frac{1}{2}}$. It follows from this that the integral (3) is absolutely convergent in the half plane Re $p>0$, and that

$$|f_\mu(p)|\to 0 \quad \text{if} \quad |p|\to\infty$$

uniformly in the halfplane Re $p \geqslant \beta$ (β is an arbitrary positive number).

Another consequence of (5) is

$$\int_{\alpha-i\infty}^{\alpha+i\infty}\left|\frac{f_\mu(p)}{p}\right| dp < \infty$$

if $\alpha>0$.

Finally, it can be shown that $f_\mu(p)$ is an analytic function in the half plane Re $p>0$.

$f_\mu(p)$ satisfies the conditions of the complex inversion theorem ([2] , Satz 21.2, p.182). Hence, the function $h_\mu(t)$ defined by

$$(6)\qquad h_\mu(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{e^{pt}}{p}f_\mu(p)dp \qquad (\alpha>0)$$

equals 0 if $t<0$ and $f_\mu(p)$ is the Laplace transform of $h_\mu(t)$.

Next in (6) we substitute the integral expression (3) for $f_\mu(p)$ and interchange the order of integration. This procedure can be justified in the following way. If Re $p=\alpha$, it follows from (5) that

$$g(p)=\int_0^\infty\left|\frac{e^{-\mu x-z\sqrt{x^2+a^2p^2}}J_0(x\rho)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}\right|dx \leqslant \begin{cases}\dfrac{C}{\sqrt{|\tau|}} & \text{if } |\tau|\geqslant\alpha\\[2mm] \dfrac{C}{\sqrt{\alpha^2-\tau^2}} & \text{if } |\tau|<\alpha,\end{cases}$$

where C does not depend on τ. Therefore

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{pt}}{p} \; g(p)\, dp$$

converges absolutely, and we have

$$(7) \qquad h_\mu(t) = \int_0^\infty J_0(\rho x)dx \; \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{pt-\mu x - z\sqrt{x^2+a^2p^2}}\, x \; dp}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}\, p} =$$

$$\int_0^\infty J_0(\rho x)dx \; \frac{1}{2\pi i} \int_{\frac{\alpha}{x}-i\infty}^{\frac{\alpha}{x}+i\infty} \frac{e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}}{c\sqrt{1+a^2w^2}+d\sqrt{1+b^2w^2}} \; \frac{dw}{w} \, .$$

It is easily seen that

$$(8) \qquad \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}}{c\sqrt{1+a^2w^2}+d\sqrt{1+b^2w^2}} \; \frac{dw}{w}$$

is independent of β, as long as $\beta > 0$. For, if $0 < \beta_1 \leq \mathrm{Re}\; w \leq \beta_2$, then

$$e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}$$

is a bounded function of w. This follows from the estimate

$$\mathrm{Re}(\mu+z\sqrt{1+a^2w^2}-wt)x \geq \left\{ \mu + (az-t)\mathrm{Re}\; w \right\} x .$$

Another consequence of this inequality is that $e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}$ is a bounded function of w in the half plane Re $w > 0$ if $t \leq az$. Therefore, the integral (8) and hence $h_\mu(t)$ equals zero in this case. From now on we assume $t > az$. We have deduced that, if $\beta > 0$,

$$(9) \qquad h_\mu(t) = \int_0^\infty J_0(\rho x)dx \; \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}}{c\sqrt{1+a^2w^2}+d\sqrt{1+b^2w^2}} \; \frac{dw}{w} \, .$$

We again want to change the order of the integrations. This can easily be justified, if

$$0 < \beta < \frac{\mu}{t-az} \, .$$

In that case we have

$$\left| \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{e^{-(\mu + z\sqrt{1+a^2w^2} - wt)x}}{c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}} \frac{dw}{w} \right| \leq e^{-(\mu + (az-t)\beta)x} C ,$$

where

$$C = \frac{1}{2\pi} \int_{\beta - i\infty}^{\beta + i\infty} \frac{|dw|}{|c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}||w|}$$

is independent of x. As $\mu + (az-t)\beta > 0$, the integral

$$\int_0^\infty J_o(\rho x) e^{-(\mu + (az-t)\beta)x} \, dx$$

converges absolutely. Hence, we have proved

$$h_\mu(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{dw}{w\{c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}\}}.$$

(10)

$$\int_0^\infty J_o(x\rho) e^{-(\mu + z\sqrt{1+a^2w^2} - wt)x} \, dx =$$

$$= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{dw}{w\{c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}\}\sqrt{\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2}}$$

Here $\sqrt{\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2}$ must be taken positive if $w = \beta$ ([3] ,p.47).

$\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2$ can be factorized into

$$(z\sqrt{1+a^2w^2} - wt + \lambda)(z\sqrt{1+a^2w^2} - wt + \bar{\lambda}) \qquad (\lambda = \mu + i\rho).$$

Each factor has only one zero in the w-plane with cut C_a. These zeros w_1 en $\overline{w_1}$ have real parts $\geq \frac{\mu}{t-az} > \beta$. Hence, we can replace the integration contour Re $w = \beta$ by the contour W, which is shown in fig. 2. In A we have to take $\sqrt{\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2}$ positive.

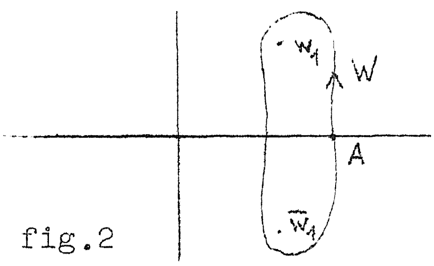

fig.2

Another integral representation of the function $h_\mu(t)$ is obtained by applying the conformal mapping

(11)
$$u = \frac{\sqrt{1+a^2w^2}}{w} .$$

The cut C_a is mapped onto the interval $(-a,a)$. As to the cut C_b we have to distinguish the two cases: I. $a<b$ and II. $a>b$. In fig. 3 and fig.4 the cuts for the integrand and the integration contours V_1 and V_2 are sketched. u_1 and $\overline{u_1}$ are the images of w_1 and $\overline{w_1}$. A' is the image of A.

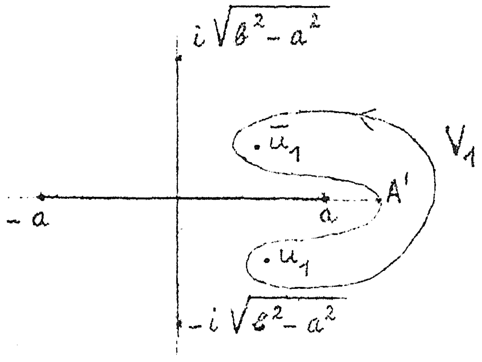

fig.3

I. $a<b$

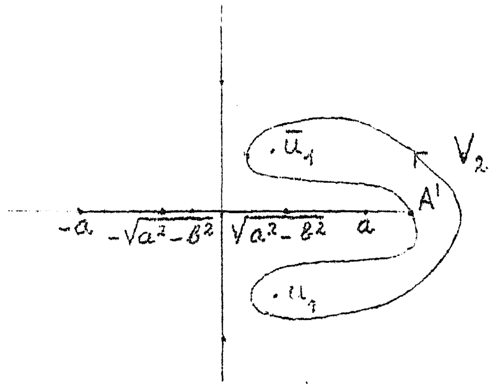

fig.4

II. $a>b$

In this way we find, if $j=1,2$,

$$(12)\quad h_\mu(t)= -\frac{1}{2\pi i}\int_{V_j}\frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+uz-t)^2}},$$

where in A', $\sqrt{u^2-a^2}$, $\sqrt{u^2+b^2-a^2}$ and $\sqrt{\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+uz-t)^2}$ are positive.

Finally, if $\mu\to 0$ we can derive in the case $t>Ra$, where

$$R=\sqrt{\rho^2+z^2}$$

that u_1 and $\overline{u_1}$ tend to

$$\frac{zt\pm i\rho\sqrt{t^2-a^2R^2}}{R^2},$$

whereas in the case $t<Ra$ u_1 and $\overline{u_1}$ tend to the same point

$$\frac{zt+\rho\sqrt{a^2R^2-t^2}}{R^2}.$$

§ 3. The limit case $\mu \to 0$.

In accordance with the method explained in § 1, we shall try to extend the results of the last section to the limit case $\mu \to 0$. It will be proved here, that $f_\mu(p)$ and $h_\mu(t)$ have limits if $\mu \to 0$ $(\mu > 0)$.

First of all, if $p > 0$ then $f_\mu(p) \to f(p)$ $(\mu \to 0)$. This follows from Lebesgue's theorem on majorized convergence, for we have

$$\frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} \to \frac{e^{-z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}$$

if $\mu \to 0$, and

$$(15) \qquad \left| \frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} \right| \leq \frac{e^{-zx}|J_0(\rho x)|\,x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} .$$

The function on the right of (15) is integrable over $(0,\infty)$. So the conditions of Lebesgue's theorem are satisfied and we have

$$f_\mu(p) = \int_0^\infty \frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} \to \int_0^\infty \frac{e^{-z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x\,dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} = f(p).$$

In the following we consider $\lim h_\mu(t)$ in the two cases I. $a < b$ and II. $a > b$.

I. We take the integration contour V_1 of § 2 fig.3. u_1 and $\overline{u_1}$ are complex continuous functions of μ $(\mu \geq 0)$, which assume real values only in the case $t < Ra$, $\mu = 0$, and take never purely imaginary values.

It is easily seen that $h_\mu(t)$ depends continuously on μ $(\mu \geq 0)$ in those points μ_0 where u_1 and $\overline{u_1}$ are not real. For we can take $\delta > 0$ so small that the sets

$$S = \{u_1(\mu)\,\big|\,|\mu - \mu_0| \leq \delta\} \quad \text{and} \quad T = \{\overline{u_1}(\mu)\,\big|\,|\mu - \mu_0| \leq \delta\},$$

do not contain for any μ with $|\mu - \mu_0| \leq \delta$ other singularities of the integrand $k_\mu(u,t)$ of (12) than $u_1(\mu)$, $\overline{u_1}(\mu)$ and we may take V_1 such that S and T are entirely inside V_1. Further, the integrand $k_\mu(u,t)$ tends uniformly to the limit

(16)
$$\frac{u}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}}$$

if $u \in V_1$. So we have proved

(17)
$$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_1} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $t > Ra$.

Next we suppose $t < Ra$. The forégoing considerations can be extended to the Riemann surface of $k_\mu(u,t)$. If $\mu \to 0$, then $u_1(\mu)$ and $\overline{u_1}(\mu)$ tend to points over the same point u_1 of the u-plane. The sets S and T on this Riemann surface are defined as in the case $t > Ra$. V_1^* (fig.5) is a simple contour on the Riemann surface which encircles S and T in the positive direction such that for all μ with $|\mu - \mu_0| \leq \delta$ the only singularities of $k_\mu(u,t)$ in the domain with boundary V_1^* are those in the sets S and T. If μ satisfies $0 < \mu \leq \delta$ we can deform V_1^* into a contour V_1 of the type described above without changing the value of the integral. It is also true that $k_\mu(u,t)$ tends to (16) uniformly on V_1^* if $\mu \downarrow 0$. Hence we may conclude

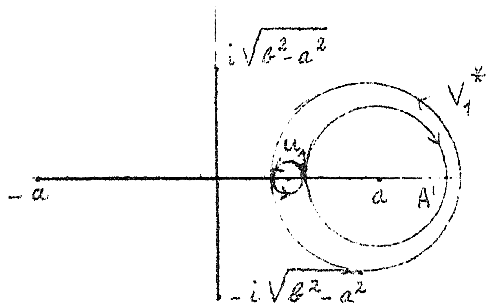

fig. 5

(18)
$$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_1^*} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}}$$

if $\mu \downarrow 0$.

II. In an analogous way we can prove the existence of $\lim_{\mu \to 0} h_\mu(t)$ if $a > b$. We shall confine ourselves to a description of the limit function. Using now the integration contour V_2 of § 2 fig.4, we can deduce

(19)
$$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_2} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $t > Ra$.

If $t < Ra$ we have to take care of the singularity $+\sqrt{a^2-b^2}$ of $k_\mu(u,t)$.

$u_1 = \frac{zt+\rho\sqrt{a^2R^2-t^2}}{R^2}$ is the greater root of

(19a) $$\rho^2(u^2-a^2) + (uz-t)^2 = 0.$$

Fig.6 is a u,t-diagram of this equation. It is an ellips with center in the origin. For the further discussion it is of interest to know whether $u_1 < \sqrt{a^2-b^2}$ or $u_1 > \sqrt{a^2-b^2}$. From the picture it is easily seen that $u_1 > \sqrt{a^2-b^2}$ if $\frac{za}{R} > \sqrt{a^2-b^2}$, that is if $Rb > \rho a$. However, if $Rb < \rho a$, it is also possible that $u_1 > \sqrt{a^2-b^2}$. Solving t from (19a) we find the condition

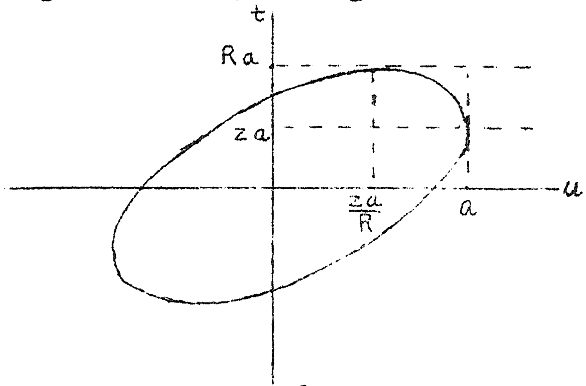

fig. 6

$t < z\sqrt{a^2-b^2}+\rho b$. Finally, $u_1 < \sqrt{a^2-b^2}$ only if $Rb < \rho a$ and $t > z\sqrt{a^2-b^2}+\rho b$. As in the case I we take a closed contour V_2^* if $u_1 > \sqrt{a^2-b^2}$ and a closed contour V_2^{**} if $u_1 < \sqrt{a^2-b^2}$ (fig.7a and 7b), and we find

(20) $$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_2^*} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $Rb > \rho a$ or $Rb < \rho a$ and $t < z\sqrt{a^2-b^2}+\rho b$.

(21) $$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_2^{**}} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $Rb < \rho a$ and $t > z\sqrt{a^2-b^2}+\rho b$.

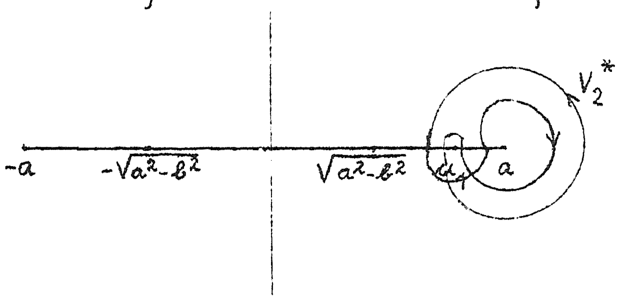

fig. 7a

$Rb > \rho a$ or $Rb < \rho a$ and $t < z\sqrt{a^2-b^2}+\rho b$

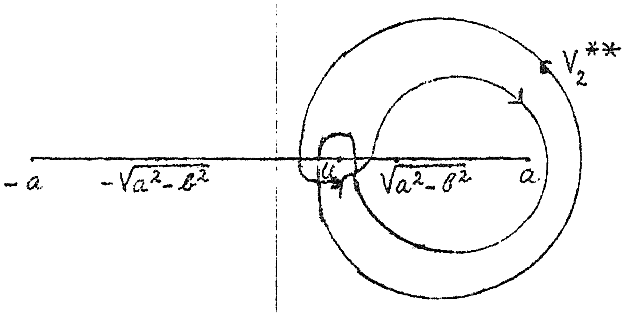

fig. 7b

$Rb < \rho a$ and $t > z\sqrt{a^2-b^2}+\rho b$

§ 4. Justification of the passage to the limit $\mu \to 0$

In the last section it was shown that

$$\lim_{\mu \downarrow 0} h_{\mu}(t) = h(t)$$

exists for all but a finite number of values of t. From the same section we know that

$$\lim_{\mu \downarrow 0} f_{\mu}(p) = f(p)$$

exists if $p > 0$. As we have

$$p \int_0^{\infty} h_{\mu}(t) e^{-pt} dt = f_{\mu}(p)$$

if $\mu > 0$, it is natural to expect that this equality holds even in the limit $\mu \to 0$. This can be proved by applying Lebesgue's theorem on majorized convergence. The conditions of this theorem are satisfied if a function g(t) exists such that

$$|h_{\mu}(t)| \leq g(t),$$

and

$$\int_0^{\infty} g(t) e^{-pt} dt < \infty \qquad (p > 0).$$

Such a function g(t) will be given here for the two cases I. $b > a$ and II. $b < a$.

I. If $b > a$ we can deform the integration contour V_1 of § 2, fig.3 into V_1' as is shown in fig.8. Taking into account the residu at $u = \infty$ we find

$$(22) \quad h_{\mu}(t) = \frac{1}{(c+d)\sqrt{\rho^2 + (\mu + z)^2}} -$$

$$\frac{1}{2\pi} \int_{V_1'} \frac{u\, du}{(cu + d\sqrt{u^2 + b^2 - a^2})\sqrt{\rho^2 (u^2 - a^2) + (\mu \sqrt{u^2 - a^2} + uz - t)^2}}$$

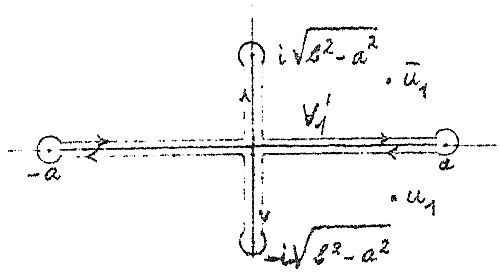

fig.8

As $c > 0$ and $d > 0$ we can show that by our definition of $\sqrt{u^2+b^2-a^2}$ $\psi(u)=cu+d\sqrt{u^2+b^2-a^2}$ has no zeros in the u-plane. Furthermore, $|\psi(u)|$ tends to infinity as $|u|\to\infty$. So we have $|\psi(u)|\geq k$ for some $k>0$.

Now the following inequalities hold

$$|h_\mu(t)| \leq \frac{1}{(c+d)R} + \frac{1}{2\pi k}\int_{V_1'} \frac{|u|\ |du|}{\sqrt{|\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+\mu z-t)^2|}}$$

(23)
$$\frac{1}{(c+d)R} + \frac{1}{\pi k}\int_{-a}^{a} \frac{|x|\ dx}{\sqrt{|\rho^2(x^2-a^2)+(\pm i\mu\sqrt{a^2-x^2}+xz-t)^2|}} + \frac{1}{\pi k}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} \frac{|x|\, dx}{\sqrt{|-\rho^2(x^2+a^2)+(\pm i\mu\sqrt{x^2+a^2}+ixz-t)^2|}}\ .$$

If α, β, γ are real numbers, then

$$|-\alpha^2+(i\beta+\gamma)^2| \geq |-\alpha^2+\gamma^2|\ .$$

Using this inequality we find

$$|h_\mu(t)| \leq \frac{1}{(c+d)R} + \frac{1}{\pi k}\int_{-a}^{a} \frac{|x|\, dx}{\sqrt{|\rho^2(x^2-a^2)+(xz-t)^2|}} + \frac{1}{\pi k}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} \frac{|x|dx}{\sqrt{|-\rho^2(x^2+a^2)+t^2|}}\ .$$

The function on the right of (24) can be taken as a majorizing $g(t)$. It depends continuously on t, except at the points $t=Ra$ and $t=\rho a$, where the quadratic expressions in x in the first and the second integral respectively have coinciding zeros. But in this points we can give the estimations $O(\log|t^2-R^2a^2|)$ and $O(\log|t^2-\rho^2a^2|)$ respectively. It is easily seen that $g(t)$ is bounded if $t\to\infty$.

II. If $b < a$ we deform the integration contour V_2 of § 2, fig.4 into V_2' (see fig.9).

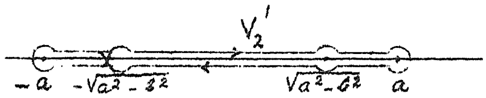

fig.9

Again we have a residu in $u=\infty$ and we find

$$h_\mu(t) = \frac{1}{(c+d)\sqrt{\rho^2+(\mu+z)^2}}$$

$$- \frac{1}{2\pi i} \int_{V_2} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+uz-t)^2}} .$$

Proceeding as in the case I we obtain

$$(25) \quad |h_\mu(t)| \leq \frac{1}{(c+d)R} + \frac{1}{\pi k} \int_{-a}^{a} \frac{dx}{\sqrt{|(-\rho^2(a^2-x^2)+(xz-t)|^2}} .$$

The function at the right of (25) is continuous except at the point $t=Ra$, where the estimation $O(\log|R^2a^2-t^2|)$ holds. The function is also bounded if $t \to \infty$.

§ 5. Summary of the results.

It is possible to put the solution h(t) of our problem in the form of complete elliptic integrals over intervals of the real axis. This can easily be done if we start from the formulae deduced in § 3. We again distinguish the two cases I and II.

I. If $a < b$ and $t > Ra$ we use (17). We deform the integration contour V_1 into the contour shown in fig.10, taking into account the residue at $u=\infty$. After some calculations we find

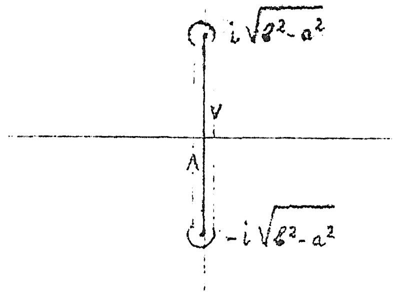

fig.10

$$(26)\quad h(t) = +\frac{1}{(c+d)R} - \frac{1}{\pi i}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} \frac{dx\sqrt{b^2-a^2-x^2}\,dx}{\{(c^2-d^2)x^2+d^2(b^2-a^2)\}\sqrt{-R^2x^2+2iztx+t^2-\rho^2a^2}},$$

where the roots are positive if $x=0$. It is not difficult to see that h(t) assumes only real values.

If $t < Ra$ it is seen from (18) and fig.5 that V_1^* can be shrunk to the point u_1 and so $h(t)=0$ in this case.

II. When $a > b$, we consider first the case $t > Ra$. We deform the integration contour V_2 in (19) into the contour shown in fig.11. Proceeding as in the case $a < b$ we find

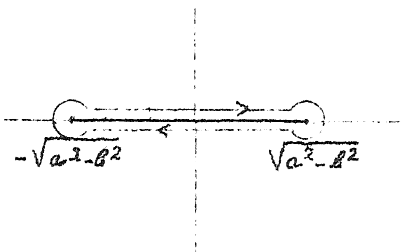

fig.11

$$(27)\quad h(t) = +\frac{1}{(c+d)R} + \frac{1}{\pi}\int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} \frac{x\,d\sqrt{a^2-b^2-x^2}\,dx}{\{(c^2-d^2)x^2+d^2(a^2-b^2)\}\sqrt{R^2x^2-2ztx+t^2-\rho^2a^2}},$$

where the roots are non-negative.

Next we consider $Rb < \rho a$ and $z\sqrt{a^2-b^2}+\rho b < t < Ra$. It can be seen that V_2^{**} in fig.7b may be replaced by the contour of fig.12.

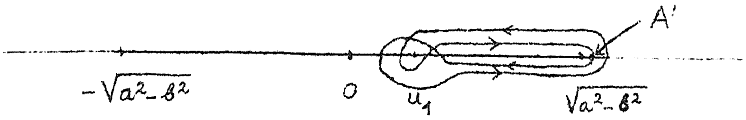

fig. 12

Therefore we find in this case

$$(28)\quad h(t) = +\frac{2}{\pi}\int_{u_1}^{\sqrt{a^2-b^2}} \frac{x\, d\sqrt{a^2-b^2-x^2}\,dx}{\{(c^2-d^2)x^2+d^2(a^2-b^2)\}\sqrt{R^2x^2-2ztx+t^2-\rho^2a^2}},$$

where $u_1 = \dfrac{zt+\rho\sqrt{R^2a^2-t^2}}{R^2}$ and the roots are non-negative.

Finally, if $Rb > \rho a$ or $Rb < \rho a$ and $t < z\sqrt{a^2-b^2}+\rho b$ we use (20). The contour V_2^* of fig.7a can again be shrunk to u_1. Therefore we find $h(t)=0$.

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