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ON R. VON MISES' CONDITION FOR THE DOMAIN OF ATTRACTION OF EXP($-e^{-x}$)

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Abstract

On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$

There exist well-known necessary and sufficient conditions for a distribution function to belong to the domain of attraction of the double exponential distribution Λ. For practical purposes a simple sufficient condition due to Von Mises is very useful. It is shown that each distribution function F in the domain of attraction of Λ is close to some distribution function satisfying Von Mises' condition.

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On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$.*)

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Suppose X_1, X_2, X_3, ... are independent real-valued random variables with common distribution function F. We say that F is in the domain of attraction of the double exponential distribution (notation $F \in D(\Lambda)$; $\Lambda(x) = \exp(-e^{-x})$) if there exist two sequences of real constants $\{b_n\}$ and $\{a_n\}$ (with $a_n > 0$ for n = 1, 2, ...) such that for all real x

$$(1) \qquad \lim_{n\to\infty} P\{\frac{\max(X_1,X_2,\ldots,X_n)-b_n}{a_n} \leq x\} = \exp(-e^{-x}).$$

Necessary and sufficient conditions for $F \in D(\Lambda)$ are well-known ([1] and [2]) but rather intricate. The following relatively simple criterion is due to R. von Mises ([4] p. 285). It is convenient for the formulation of the theorem to use the symbol x_0 for the upper bound of X_1 defined by

$$x_0(F) = \sup\{x \mid F(x) < 1\}.$$

<u>Theorem 1</u>. Suppose F(x) is a distribution function with a density f(x) which is positive and differentiable on a left neighbourhood of x_0. If

$$(2) \qquad \lim_{x\uparrow x_0} \frac{d}{dx}\left(\frac{1-F(x)}{f(x)}\right) = 0,$$

then $F \in D(\Lambda)$.

A distribution function F satisfying (2) will be called a <u>Von Mises function</u>.

*) Report SW 10/71, Afdeling Mathematische Statistiek, Mathematisch Centrum, Amsterdam.

We shall prove

Theorem 2 A distribution function F lies in the domain of attraction of Λ if and only if there exists a Von Mises function F_* such that

$$(3) \qquad \lim_{x \uparrow x_0} \frac{1-F(x)}{1-F_*(x)} = 1.$$

For the proof we need three lemma's.

Lemma 1 Let F and G be distribution functions and let $a_n > 0$ and b_n be real constants such that

$$(4) \qquad \lim_{n\to\infty} G^n(a_n x+b_n) = \Lambda(x).$$

If

$$(5) \qquad \lim_{x \uparrow x_0(G)} \frac{1-F(x)}{1-G(x)} = 1$$

then

$$(6) \qquad x_0(F) = x_0(G)$$

and

$$(7) \qquad \lim_{n\to\infty} F^n(a_n x+b_n) = \Lambda(x).$$

Proof of lemma Since $0 \le \Lambda(x) < 1$ for all real x relation (4) implies

$$(8) \qquad \lim_{n\to\infty} 1 - G(a_n x+b_n) = 0$$

and hence

$$(9) \qquad a_n x + b_n \text{ converges to } x_0 \text{ from the left for } n \to \infty,$$

which together with (5) implies (6).

The relation (8) implies that

$$n\{1-G(a_n x+b_n)\} \sim \log G^n(a_n x+b_n) \sim \log \Lambda(x) \quad \text{for } n \to \infty.$$

These asymptotic relations also hold for F instead of G because of (9) and (5). This proves (7). Q.E.D.

In order to increase the differentiability of F we define the sequence of distribution functions F_0, F_1, F_2, ... by

$$F_0(x) = F(x)$$

$$1 - F_{n+1}(x) = \min\{1, \int_x^{x_0} (1-F_n(t))dt\} \qquad \text{for } x < x_0.$$

<u>Lemma 2</u> If $F \in D(\Lambda)$, then $F_1 \in D(\Lambda)$. In particular the sequence F_n is well defined.

<u>Proof</u> See De Haan [2] lemma 2.5.1 or [3] lemma 6.

<u>Lemma 3</u> If $F \in D(\Lambda)$, then

$$\text{(10)} \qquad \lim_{x \uparrow x_0} \frac{\{1-F(x)\}.\{1-F_2(x)\}}{\{1-F_1(x)\}^2} = 1.$$

<u>Proof</u> See De Haan [2] th. 2.5.2 or [3] th. 10.

Note that (10) is the integral form of (2).

<u>Corollary</u> If $F \in D(\Lambda)$, then for n = 1, 2, ...

$$\text{(11)} \qquad \lim_{x \uparrow x_0} \frac{\{1-F_{n-1}(x)\}\ \{1-F_{n+1}(x)\}}{\{1-F_n(x)\}^2} = 1.$$

<u>Proof of theorem 2</u> The if statement is a trivial result of lemma 1. Now suppose $F \in D(\Lambda)$.

Define the function U_* by

$$U_*(x) = U_3^4(x)\ U_4^{-3}(x)$$

where $U_n(x) = 1-F_n(x)$. Then $U(x)$ is twice differentiable in a left neighbourhood of x_0 and

$$(12)\qquad \frac{d}{dx}\log U_* = -4\,\frac{U_2}{U_3} + 3\,\frac{U_3}{U_4} = \frac{3-4\,U_2U_3^{-2}U_4}{U_4U_3^{-1}}\,.$$

Consider

$$\frac{U_4U_3^{-1}}{3-4\,U_2U_3^{-2}U_4} = \frac{U_*}{\frac{d}{dx}U_*}\,.$$

By (11) the denominator is asymptotic to -1 for $x \uparrow x_0$ and both $\frac{d}{dx}U_4U_3^{-1}$ and $U_4U_3^{-1}\frac{d}{dx}(3-4\,U_2U_3^{-2}U_4)$ vanish for $x \uparrow x_0$. Hence

$$(13)\qquad \lim_{x\uparrow x_0} \frac{d}{dx}\left(\frac{U_*(x)}{U_*'(x)}\right) = 0.$$

Observe that

$$U_0 = \frac{U_0U_2}{U_1^2}\,.\,\left(\frac{U_1U_3}{U_2^2}\right)^2\,.\,\left(\frac{U_2U_4}{U_3^2}\right)^3\,.\,U_*.$$

Hence by (11) we obtain

$$(14)\qquad \lim_{x\uparrow x_0} \frac{U_0(x)}{U_*(x)} = 1.$$

Then $\lim_{x\uparrow x_0} U_*(x) = 0$, and since by (12) U_* is decreasing on a left neighbourhood of x_0, there exists a twice differentiable distribution function $F_*(x)$ which coincides with $1 - U_*(x)$ on a left neighbourhood of x_0. F_* is a Von Mises function by (13) and satisfies (2) by (14). Q.E.D.

Remarks Lemma 1 is a particular case of a theorem due to Resnick [5] (th. 2.3). Lemma 1 implies that for the convergence of the distribution functions F^n and F_*^n the same norming constants $a_n > 0$ and b_n may be used.

Corollary If $F \in D(\Lambda)$, there exist a positive function c satisfying $\lim_{x\uparrow x_0} c(x) = 1$ and a positive differentiable function ϕ satisfying

$\phi(x_0) = 0$ if x_0 is finite and $\lim_{x \uparrow x_0} \phi'(x) = 0$ such that

$$1 - F(x) = c(x).\ \exp\{-\int_{-\infty}^{x} \frac{dt}{\phi(t)}\} \qquad \text{for } x < x_0.$$

This improves the representation theorem 2.5.3 in De Haan [2].

<u>Proof</u> Set $\phi(x) = \dfrac{1-F_*(x)}{F_*'(x)}$ in a left neighbourhood of x_0.

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