

# MOMENT ANALYSIS OF THE DELAUNAY TESSELLATION FIELD ESTIMATOR

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## Abstract

The Campbell–Mecke theorem is used to derive explicit expressions for the mean and variance of Schaap and Van de Weygaert’s Delaunay tessellation field estimator. Special attention is paid to Poisson processes.

*Keywords & Phrases:* Campbell–Mecke formula, Delaunay tessellation field estimator, generalised weight function estimator, intensity function, mass preservation, Poisson point process, second order factorial moment measure, second order product density.

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## 1 Preliminaries and notation

Let  $\varphi$  be a locally finite point pattern in  $\mathbb{R}^d$  arising as a realisation of simple point processes  $\Phi$  on  $\mathbb{R}^d$  [4, 9]. In practice,  $d \in \{1, 2, 3\}$ . We shall assume that the points are in general quadratic position [11], that is, (a) no  $d+2$  points are located on the boundary of a sphere, and (b) in the plane no three points are co-linear; in higher dimensions, no  $k+1$  points lie in a  $k-1$  dimensional affine subspace for  $k = 2, \dots, d$ . These assumptions are satisfied almost surely for realisations of a Poisson process with locally finite intensity function  $\lambda : \mathbb{R}^d \rightarrow [0, \infty)$  or, more generally, for Gibbs point processes defined by their probability density with respect to such a Poisson process.

Any point pattern  $\varphi$  gives rise to two interesting tessellations. First consider the set

$$C(x_i | \varphi) := \{y \in \mathbb{R}^d : \|x_i - y\| \leq \|x_j - y\| \quad \forall x_j \in \varphi\}$$

that consists of all points in  $\mathbb{R}^d$  that are at least as close to  $x_i \in \varphi$  as to any other point of  $\varphi$ , which is called the *Voronoi cell* of  $x_i$ . The ensemble of all Voronoi cells is the *Voronoi tessellation* of  $\varphi$  [20]. An equivalent definition is

$$C(x_i | \varphi) = \bigcap_{x_j \neq x_i \in \varphi} H(x_i, x_j),$$

where  $H(x_i, x_j)$  is the closed halfspace  $\{y \in \mathbb{R}^d : \langle y - (x_i + x_j)/2, x_i - x_j \rangle \geq 0\}$  consisting of points that are at least as close to  $x_i$  as to  $x_j$ . In  $\mathbb{R}^1$ , for  $x_i < x_j$ ,  $H(x_i, x_j) = (-\infty, (x_i + x_j)/2]$ . In the plane,  $H(x_i, x_j)$  is the closed halfplane bounded by the bisecting line  $L(x_i, x_j)$  of the

segment connecting  $x_i$  and  $x_j$  that contains  $x_i$ . Note that the Voronoi cells are closed and convex, but not necessarily bounded.

Under our assumptions, intersections between  $k = 2, \dots, d + 1$  different Voronoi cells are either empty or of dimension  $d - k + 1$ . In particular,

$$\bigcap_{i=1}^{d+1} C(x_i | \varphi) \neq \emptyset \Leftrightarrow b(x_1, \dots, x_{d+1}) \cap \varphi = \emptyset$$

where  $b(x_1, \dots, x_{d+1})$  is the open ball spanned by  $x_1, \dots, x_{d+1}$  on its boundary, and in that case is a single point, usually referred to as a *vertex* of the Voronoi diagram.

Vertices can be used to define the second tessellation of interest to us in this paper, the *Delaunay tessellation*. Indeed, suppose that  $\varphi$  contains at least  $d + 1$  points. Each Voronoi vertex arising as the intersection of  $d + 1$  cells  $C(x_i | \varphi)$  defines a closed simplex, the convex hull of  $\{x_1, \dots, x_{d+1}\}$ , which is called a *Delaunay cell* [5] and denoted by  $D(x_1, \dots, x_{d+1})$ . Note that for  $d = 1$ , Delaunay cells are intervals, whilst in the plane they form triangles. An alternative, equivalent, edge based construction is to join points  $x_1, x_2 \in \varphi$  that share a common Voronoi border  $C(x_1 | \varphi) \cap C(x_2 | \varphi) \neq \emptyset$  into a Delaunay edge. In this case,  $x_1$  and  $x_2$  are called *Voronoi neighbours*. The set of neighbours of  $x_1$  in  $\varphi$  is denoted by  $\mathcal{N}(x_1 | \varphi)$ . Either way, the partition of space formed by the Delaunay cells is referred to as the *Delaunay tessellation*. The union of Delaunay cells containing  $x_i \in \varphi$  is known as the *contiguous Voronoi cell*  $W(x_i | \varphi)$  of  $x_i$  in  $\varphi$ .

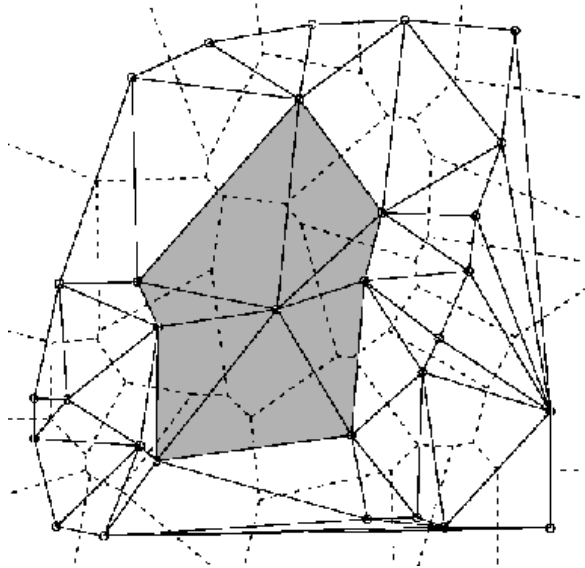


Figure 1: A set of thirty points with their Voronoi (dashed lines) and Delaunay (solid lines) tessellations. A contiguous Voronoi cell is indicated by shading.

For more details, including an historical account, the reader is referred to the comprehensive textbooks [11, 12]. An illustration is given in Figure 1, which was obtained using the DELDIR package [19].

## 2 Delaunay tessellation field estimator

Recently, Schaap and Van de Weygaert [14, 15] proposed to estimate the intensity function of a spatial point process by the so-called *Delaunay tessellation field estimator* (DTFE). The method estimates the intensity at points in a realisation reciprocal to the volume of their contiguous Voronoi cell, and distributes these estimated field values over Delaunay cells by linear (or other) interpolation. They also consider interpolation of fields  $x \mapsto f(x) \in \mathbb{R}^+$  observed at sampling points. Earlier suggestions to use Voronoi tessellations for field interpolation include those by Ord [13] and Sibson [16].

Based on extensive simulations, Schaap and Van de Weygaert claim that, in contrast to kernel estimators [1], the DTFE preserves the total mass of the field and fine structural details, appears to result in smooth interpolation, adapts itself to the local scale and geometry, and is relatively robust. The limitations of the method lie in its sensitivity to measurement error, boundary effects, and triangular artefacts [15]. Our aim in this paper is a rigorous analysis of this estimator.

Throughout this paper, let  $\Phi$  be a simple point process on  $\mathbb{R}^d$  having realisations in general quadratic position for which the expected number of points placed in bounded Borel sets is finite so that its (first order) moment measure exists as a  $\sigma$ -finite Borel measure. Furthermore, assume that the moment measure is absolutely continuous with respect to Lebesgue measure with Radon–Nikodym derivative  $\lambda : \mathbb{R}^d \rightarrow [0, \infty)$ , its *intensity function*.

**Definition 1.** Consider a point process  $\Phi$  observed in a convex bounded Borel subset  $A$  of  $\mathbb{R}^d$ . For  $x \in \Phi \cap A$ , define

$$\widehat{\lambda}(x) := \frac{d+1}{|W(x | \Phi \cap A)|}, \quad (1)$$

where  $|\cdot|$  denotes  $d$ -volume. For any  $x_0 \in A$  in the interior of some Delaunay cell, define

$$\widehat{\lambda}(x_0) := \frac{1}{d+1} \sum_{x \in \Phi \cap D(x_0 | \Phi \cap A)} \widehat{\lambda}(x) \quad (2)$$

as the average of the estimated intensity function values at the  $d+1$  vertices  $x$  of the Delaunay cell  $D(x_0 | \Phi \cap A)$  containing  $x_0$ .

A few remarks are in order. Should a particular realisation  $\varphi$  of  $\Phi$  happen to contain less than  $d+1$  points in  $A$ , the intensity function estimate may be set to zero, or (cf. the Lemma below) to the number of points divided by  $|A|$ . On the sides of the Delaunay cells, any averaging may be used – it is a null set. Finally,  $\widehat{\lambda}(x_0)$  is set to zero for points that do not fall in any Delaunay cell.

Edge effects arise due to the fact that  $\Phi$  is not observed, only  $\Phi \cap A$ , the Delaunay tessellation of which partitions the convex hull of  $\Phi \cap A \subseteq A$ . Such effects may be dealt with in many ways. For example, one might use torus corrections, add arbitrary points on the boundary of  $A$  (the corners for example in the generic case of a cube), or draw lines orthogonal to the edges emanating from points on the boundary of the convex hull, etc. Further examples can be found in chapter 6 of [12].

**Lemma 1.** (Schaap and Van de Weygaert [14, 15])

Let  $\varphi$  be a realisation of the simple point process  $\Phi$  containing at least  $d + 1$  points in  $A$ . Then the estimator of Definition 1 preserves total mass, that is,

$$\int_A \widehat{\lambda}(x_0) dx_0 = n(\varphi \cap A),$$

the number of points of  $\varphi$  in  $A$ .

**Proof:** Write  $\mathcal{D}(\varphi \cap A)$  for the family of Delaunay cells defined by  $\varphi \cap A$ , and note that

$$\begin{aligned} \int_A \widehat{\lambda}(x_0) dx_0 &= \sum_{D_j \in \mathcal{D}(\varphi \cap A)} |D_j| \left[ \sum_{x \in \varphi \cap D_j} \frac{1}{|W(x | \varphi \cap A)|} \right] \\ &= \sum_{x \in \varphi \cap A} \frac{1}{|W(x | \varphi \cap A)|} \left[ \sum_{D_j \in \mathcal{D}(\varphi \cap A)} 1_{\{x \in D_j\}} |D_j| \right] = n(\varphi \cap A), \end{aligned}$$

cf. [15, p. 62 ff.]. □

### 3 Mean and variance of the Delaunay tessellation field estimator

In this section, we derive the first two moments of the Delaunay tessellation field estimator. Our first result concerns the expectation.

**Theorem 1.** Let  $\Phi$  be observed in a convex bounded Borel subset  $A$ , and, for a point pattern  $\varphi$  with  $n(\varphi \cap A) \geq d + 1$  in general quadratic position, set

$$g(x_0 | x, \varphi) := \frac{\sum_{D_j \in \mathcal{D}(\varphi \cap A)} 1_{\{x_0 \in D_j^\circ; x \in D_j\}}}{|W(x | \varphi \cap A)|}, \quad (3)$$

for  $x_0 \in A \setminus \varphi$ ,  $x \in \varphi$ , and let  $g(x | x, \varphi) := (d + 1)/|W(x | \varphi \cap A)|$  if  $x \in \varphi \cap A$ . Then the Delaunay tessellation field estimator defined by (2) and (1) has expectation

$$\mathbb{E} \left[ \widehat{\lambda}(x_0) \right] = \int_A \mathbb{E}_x [g(x_0 | x, \Phi)] \lambda(x) dx,$$

where  $\mathbb{E}_x$  denotes the expectation with respect to the Palm distribution of  $\Phi$  at  $x$ .

For patterns  $\varphi$  with less than  $d + 1$  points falling in  $A$ , it is also possible to write  $\widehat{\lambda}(x_0) = \sum_{x \in \varphi \cap A} g(x_0 | x, \varphi)$  with the function  $g$  chosen to suit the particular type of edge correction adopted, see Section 2.

**Proof:** Note that

$$\widehat{\lambda}(x_0) = \sum_{x \in \Phi \cap A} g(x_0 | x, \Phi).$$

Hence, by the Campbell–Mecke theorem [18],

$$\mathbb{E}\widehat{\lambda(x_0)} = \int_A \mathbb{E}_x [g(x_0 | x, \Phi)] \lambda(x) dx.$$

□

Recall that the second order *factorial moment measure*  $\mu^{(2)}$  is defined in integral terms by

$$\mathbb{E} \left[ \sum_{x_1, x_2 \in \Phi}^{\neq} f(x_1, x_2) \right] = \int \int f(x_1, x_2) d\mu^{(2)}(x_1, x_2) \quad (4)$$

for any non-negative measurable function  $f$ . The sum is over all pairs of different points. We shall say that the second order factorial moment measure exists, if it is locally finite. If furthermore  $\mu^{(2)}$  is absolutely continuous with respect to the 2-fold product measure of Lebesgue measure with itself, a Radon–Nikodym derivative exists known as second order *product density* and denoted by  $\rho^{(2)}$ . In this case, (4) reduces to

$$\int \int f(x_1, x_2) \rho^{(2)}(x_1, x_2) dx_1 dx_2.$$

**Theorem 2.** *Let  $\Phi$  be observed in a convex bounded Borel subset  $A$  and define the function  $g$  by (3). Assume that second order product densities exist. Then the Delaunay tessellation field estimator defined by (2) and (1) has variance*

$$\begin{aligned} \text{Var}(\widehat{\lambda(x_0)}) &= \int_A \int_A \mathbb{E}_{x,y}^{(2)} [g(x_0 | x, \Phi) g(x_0 | y, \Phi)] \rho^{(2)}(x, y) dx dy \\ &+ \int_A \mathbb{E}_x [g^2(x_0 | x, \Phi)] \lambda(x) dx - \left( \int_A \mathbb{E}_x [g(x_0 | x, \Phi)] \lambda(x) dx \right)^2, \end{aligned}$$

where  $E_{x,y}^{(2)}$  denotes the two-fold Palm distribution of  $\Phi$ .

**Proof:** Remark that

$$\mathbb{E} \left[ \widehat{\lambda(x_0)}^2 \right] = \mathbb{E} \left[ \sum_{x,y \in \Phi \cap A}^{\neq} g(x_0 | x, \Phi) g(x_0 | y, \Phi) \right] + \mathbb{E} \left[ \sum_{x \in \Phi \cap A} g^2(x_0 | x, \Phi) \right].$$

The cross term on the right hand side is equal to

$$\int_A \int_A \mathbb{E}_{x,y}^{(2)} [g(x_0 | x, \Phi) g(x_0 | y, \Phi)] \rho^{(2)}(x, y) dx dy,$$

see e.g. [4], where  $\mathbb{E}_{x,y}^{(2)}$  denotes the two-fold Palm distribution of  $\Phi$  [8]. Another appeal to the Campbell–Mecke theorem yields

$$\mathbb{E} \left[ \sum_{x \in \Phi \cap A} g^2(x_0 | x, \Phi) \right] = \int_A \mathbb{E}_x [g^2(x_0 | x, \Phi)] \lambda(x) dx.$$

Finally, the variance is obtained using Theorem 1.  $\square$

In general, the integrals involved in Theorems 1–2 must be evaluated by numerical or simulation methods.

## 4 Comparison to a classic estimator

The classic estimator of intensity is the kernel estimator

$$\widehat{\lambda_{BD}}(x_0) := \frac{n(\Phi \cap b(x_0, h) \cap A)}{|b(x_0, h) \cap A|}, \quad x_0 \in A. \quad (5)$$

proposed by Berman and Diggle [1]. The estimator can be regarded as a kernel estimator [17] with  $k_h(x_0 | x) = 1\{|x - x_0| < h\}/|b(x_0, h) \cap A|$ , where  $b(x_0, h)$  denotes the open ball around  $x_0$  with radius  $h > 0$ . The choice of bandwidth  $h$  determines the amount of smoothing.

Note that when the bounded observation window  $A \neq \emptyset$  is open, one never divides by zero. In fact, a stronger statement can be made. The function  $x \mapsto |b(x, h) \cap A|$  is continuous and attains its minimum on the closure  $\bar{A}$ . Since any point on the boundary  $\partial A$  always has a neighbour within distance  $h$  in  $A$ ,  $\inf_{x \in A} |b(x, h) \cap A| > 0$ . Further details may be found e.g. in [3, 6, 18].

Although (5) is a natural estimator, it does not necessarily preserve the total mass in  $A$  [15], nor is it based on a generalised weight function [17]. It is not hard to modify the edge correction in (5) to define an estimator [10] that does preserve total mass and is based on a weight function.

**Definition 2.** Consider a point process  $\Phi$  observed in an open bounded Borel subset  $A$  of  $\mathbb{R}^d$ . For  $x_0 \in A$ , define

$$\widehat{\lambda_K}(x_0) := \sum_{x \in \Phi \cap A} \frac{1\{|x - x_0| < h\}}{|b(x, h) \cap A|}. \quad (6)$$

**Lemma 2.** The estimator of Definition 2 is a generalised weight function estimator with kernel  $k_h(x_0 | x) = 1\{|x - x_0| < h\}/|b(x, h) \cap A|$  that preserves total mass, that is,

$$\int_A \widehat{\lambda_K}(x_0) dx_0 = n(\Phi \cap A),$$

the number of points of  $\Phi$  in  $A$ .

**Proof:** Note that

$$\int_A k_h(x_0 | x) dx_0 = \int_A \frac{1\{|x - x_0| < h\}}{|b(x, h) \cap A|} dx_0 \equiv 1$$

for all  $x \in A$ , that is,  $\widehat{\lambda_K}(\cdot)$  is a generalised weight function estimator. Furthermore, for any realised point pattern  $\varphi$ , the restriction  $\varphi \cap A$  in  $A$  is finite and

$$\int_A \left[ \sum_{x \in \varphi \cap A} \frac{1\{|x - x_0| < h\}}{|b(x, h) \cap A|} \right] dx_0 = \sum_{x \in \varphi \cap A} \int_A \frac{1\{|x - x_0| < h\}}{|b(x, h) \cap A|} dx_0 = n(\varphi \cap A).$$

□

Note that the Delaunay tessellation field estimator is based on an *adaptive kernel* (3) as it depends on the underlying point pattern. Indeed, for every  $x \in A$ ,

$$\int_A g(x_0 | x, \phi) dx_0 = \int_A \frac{\sum_{D_j \in \mathcal{D}(\phi \cap A)} 1\{x_0 \in D_j^c; x \in D_j\}}{|W(x | \varphi \cap A)|} dx_0 = 1.$$

A clear advantage is that the problem of choosing the bandwidth is avoided.

In order to assess the quality of the estimator, we proceed to compute its mean and variance.

**Theorem 3.** *Let  $\Phi$  be observed in a bounded open Borel subset  $A$ . Then, the estimator of Definition 2 has expectation*

$$\mathbb{E} \left[ \widehat{\lambda_K(x_0)} \right] = \int_A \frac{1\{x \in b(x_0, h)\}}{|b(x, h) \cap A|} \lambda(x) dx.$$

**Proof:** By the Campbell–Mecke theorem

$$\mathbb{E} \left[ \widehat{\lambda_K(x_0)} \right] = \mathbb{E} \left[ \sum_{x \in \Phi \cap A} \frac{1\{\|x - x_0\| < h\}}{|b(x, h) \cap A|} \right] = \int_A \frac{1\{\|x - x_0\| < h\}}{|b(x, h) \cap A|} \lambda(x) dx.$$

□

If we compare Theorem 3 to Theorem 1, the Palm expectation  $\mathbb{E}_x [g(x_0 | x, \Phi)]$  is replaced by  $k_h(x_0 | x)$ , as the latter does not depend on the point process  $\Phi$ .

**Theorem 4.** *let  $\Phi$  be observed in a bounded open Borel subset  $A$  and assume that second order product densities exist. Then*

$$\text{Var}(\widehat{\lambda_K(x_0)}) = \int \int_{(b(x_0, h) \cap A)^2} \frac{\rho^{(2)}(x, y) - \lambda(x)\lambda(y)}{|b(x, h) \cap A| |b(y, h) \cap A|} dx dy + \int_{b(x_0, h) \cap A} \frac{\lambda(x)}{|b(x, h) \cap A|^2} dx.$$

**Proof:** Regarding the second moment, note that

$$\begin{aligned} \mathbb{E} \left[ \widehat{\lambda_K(x_0)}^2 \right] &= \mathbb{E} \left\{ \sum_{x, y \in \Phi \cap A}^{\neq} \left[ \frac{1\{\|x - x_0\| < h\}}{|b(x, h) \cap A|} \frac{1\{\|y - x_0\| < h\}}{|b(y, h) \cap A|} \right] \right\} \\ &+ \mathbb{E} \left\{ \sum_{x \in \Phi \cap A} \left[ \frac{1\{\|x - x_0\| < h\}}{|b(x, h) \cap A|^2} \right] \right\}. \end{aligned}$$

Then rewrite the expectations as integrals with respect to  $\rho^{(2)}$  and  $\lambda$  respectively to obtain that the variance of  $\widehat{\lambda_K(x_0)}$  is equal to

$$\int_{b(x_0, h) \cap A} \int_{b(x_0, h) \cap A} \frac{1}{|b(x, h) \cap A| |b(y, h) \cap A|} \rho^{(2)}(x, y) dx dy + \int_{b(x_0, h) \cap A} \frac{\lambda(x)}{|b(x, h) \cap A|^2} dx.$$

An appeal to Theorem 3 completes the proof.  $\square$

The result should be compared to that of Theorem 2.

Similar arguments as those in the proofs of Theorems 3 and 4 applied to the classic Berman–Diggle estimator (5) give mean

$$\frac{1}{|b(x_0, h) \cap A|} \int_{b(x_0, h) \cap A} \lambda(x) dx$$

and variance

$$\frac{1}{|b(x_0, h) \cap A|^2} \left\{ \int_{b(x_0, h) \cap A} \lambda(x) dx + \int_{(b(x_0, h) \cap A)^2} [\rho^{(2)}(x, y) - \lambda(x) \lambda(y)] dx dy \right\}.$$

Note that for  $x_0 \in A \ominus b(0, 2h)$  separated by  $2h$  from the boundary of  $A$ , no edge correction is necessary, and both kernel estimators are identical.

The disadvantage of kernel estimators is that they involve a bandwidth parameter  $h$ ; the larger  $h$ , the smoother the estimated intensity function. For specific models,  $h$  may be chosen by optimisation of the (integrated) mean squared error [6]. In practice, in a planar setting, Diggle [6] recommends to choose  $h$  proportional to  $n^{-1/2}$ , where  $n$  is the observed number of points. For a fixed bandwidth, neither the Berman–Diggle estimator nor the modification of Definition 2 is universally better. For examples, the reader is referred to [10].

## 5 Intensity estimation for Poisson point processes

In general, the integrals involved in Theorems 1–4 have to be evaluated numerically. An exception is the case where  $\Phi$  is a Poisson point process with a locally finite intensity function.

**Corollary 1.** *Let  $\Phi$  be a Poisson point process observed in a convex bounded Borel subset  $A$ . Then,*

$$\mathbb{E} \left[ \widehat{\lambda(x_0)} \right] = \int_A \mathbb{E} [g(x_0 | x, \Phi \cup \{x\})] \lambda(x) dx$$

and

$$\begin{aligned} \text{Var}(\widehat{\lambda(x_0)}) &= \int_A \int_A \mathbb{E} [g(x_0 | x, \Phi \cup \{x, y\}) g(x_0 | y, \Phi \cup \{x, y\})] \lambda(x) \lambda(y) dx dy \\ &+ \int_A \mathbb{E} [g^2(x_0 | x, \Phi \cup \{x\})] \lambda(x) dx - \left( \int_A \mathbb{E} [g(x_0 | x, \Phi \cup \{x\})] \lambda(x) dx \right)^2. \end{aligned}$$

**Proof:** For a Poisson process, the Palm distribution at  $x$  is equal to the superposition of its distribution  $\mathbb{P}$  with an extra point at  $x$ , the two-fold Palm distribution  $\mathbb{P}_{x,y}^{(2)}$  is the superposition of  $\mathbb{P}$  with  $x$  and  $y$ . Furthermore,  $\rho^{(2)}(x, y) = \lambda(x) \lambda(y)$  is a product density. Plugging these results into the expressions of Theorems 1–2 completes the proof.  $\square$



**Corollary 2.** *let  $\Phi$  be a Poisson point process observed in a bounded open Borel subset  $A$  and assume that second order product densities exist. Then,*

$$\text{Var}(\widehat{\lambda_K}(x_0)) = \int_{b(x_0, h) \cap A} \frac{\lambda(x)}{|b(x, h) \cap A|^2} dx.$$

**Proof:** Use that  $\rho^{(2)}(x, y) = \lambda(x)\lambda(y)$  and apply Theorem 4.  $\square$

The variance of the Berman–Diggle estimator is  $\int_{b(x_0, h) \cap A} \lambda(x) dx / |b(x_0, h) \cap A|^2$ .

For stationary Poisson processes, even more can be said. In the remainder of this section, define  $g$  as in (3) with  $A = \mathbb{R}^d$ .

**Theorem 5.** *Let  $\Phi$  be a stationary Poisson point process in  $\mathbb{R}^d$  with intensity  $\lambda > 0$ . Then, the Delaunay tessellation field estimator  $\widehat{\lambda}(0)$  is asymptotically unbiased.*

**Proof:** Let  $b(x, y_1, \dots, y_d)$  be the open ball spanned by the points  $x, y_1, \dots, y_d$  on its topological boundary, and let  $D^\circ(x, y_1, \dots, y_d)$  be the *open* simplex that is the interior of the convex hull of  $\{x, y_1, \dots, y_d\}$ . Recall that the points  $x, y_1, \dots, y_d$  define a Voronoi vertex, or, equivalently, a Delaunay cell if and only if there are no points in  $b(x, y_1, \dots, y_d)$ .

By Corollary 1, asymptotically

$$\begin{aligned} \mathbb{E}[\widehat{\lambda}(0)] &= \lambda \int_{\mathbb{R}^d} \mathbb{E}[g(0 \mid x, \Phi \cup \{x\})] dx \\ &= \lambda \int \mathbb{E} \left[ \sum_{\{y_1, \dots, y_d\} \subset \Phi}^{\neq} \frac{1\{0 \in D^\circ(x, y_1, \dots, y_d); b(x, y_1, \dots, y_d) \cap (\Phi \cup \{x\}) = \emptyset\}}{|W(x \mid \Phi \cup \{x\})|} \right] dx \\ &= \lambda \int \mathbb{E} \left[ \sum_{\{y_1, \dots, y_d\} \subset \Phi}^{\neq} \frac{1\{0 \in D^\circ(x, y_1, \dots, y_d); b(x, y_1, \dots, y_d) \cap \Phi = \emptyset\}}{|W(x \mid \Phi \cup \{x\})|} \right] dx \\ &= \lambda \int \mathbb{E} \left[ \sum_{\{z_1, \dots, z_d\} \subset \Phi_{-x}}^{\neq} \frac{1\{-x \in D^\circ(0, z_1, \dots, z_d); b(0, z_1, \dots, z_d) \cap \Phi_{-x} = \emptyset\}}{|W(0 \mid \Phi_{-x} \cup \{0\})|} \right] dx \\ &= \lambda \int \mathbb{E} \left[ \sum_{\{z_1, \dots, z_d\} \subset \Phi}^{\neq} \frac{1\{-x \in D^\circ(0, z_1, \dots, z_d); b(0, z_1, \dots, z_d) \cap \Phi = \emptyset\}}{|W(0 \mid \Phi \cup \{0\})|} \right] dx \end{aligned}$$

by stationarity. Hence, by Fubini's theorem,

$$\begin{aligned} \mathbb{E}[\widehat{\lambda}(0)] &= \lambda \mathbb{E} \left[ \frac{\sum_{\{z_1, \dots, z_d\} \subset \Phi}^{\neq} |D^\circ(0, z_1, \dots, z_d)| 1\{b(0, z_1, \dots, z_d) \cap \Phi = \emptyset\}}{|W(0 \mid \Phi \cup \{0\})|} \right] \\ &= \lambda \mathbb{E} \left[ \frac{|W(0 \mid \Phi \cup \{0\})|}{|W(0 \mid \Phi \cup \{0\})|} \right] = \lambda. \end{aligned}$$

$\square$

The asymptotic variance of the Delaunay tessellation field estimator increases quadratically with  $\lambda$  with a constant multiplier that depends on the dimension. The proof rests on the following two lemmata.

**Lemma 3.** *Let  $\Phi$  be a stationary Poisson point process in  $\mathbb{R}^d$  with intensity  $\lambda > 0$ . Then,*

$$\begin{aligned} C(\lambda, d) &:= \int \int \mathbb{E} [g(0 \mid x, \Phi \cup \{x, y\}) g(0 \mid y, \Phi \cup \{x, y\})] \lambda(x) \lambda(y) dx dy \\ &= \lambda^2 \int \mathbb{E}_1 \left[ \frac{|W(0 \mid \Phi \cup \{0, x\}) \cap W(x \mid \Phi \cup \{0, x\})|}{|W(0 \mid \Phi \cup \{0, x\})| |W(x \mid \Phi \cup \{0, x\})|} \right] dx, \end{aligned}$$

where  $\mathbb{E}_1$  denotes expectation with respect to a unit intensity Poisson point process.

By the Nguyen–Zessin formula [9], alternatively

$$\begin{aligned} C(\lambda, d) &= \lambda \mathbb{E} \left[ \frac{1}{|W(0 \mid \Phi \cup \{0\})|} \sum_{y \in \mathcal{N}(0 \mid \Phi \cup \{0\})} \frac{|W(0 \mid \Phi \cup \{0\}) \cap W(y \mid \Phi \cup \{0\})|}{|W(y \mid \Phi \cup \{0\})|} \right] \\ &= \lambda^2 \mathbb{E}_1 \left[ \frac{1}{|W(0 \mid \Phi \cup \{0\})|} \sum_{y \in \mathcal{N}(0 \mid \Phi \cup \{0\})} \frac{|W(0 \mid \Phi \cup \{0\}) \cap W(y \mid \Phi \cup \{0\})|}{|W(y \mid \Phi \cup \{0\})|} \right]. \end{aligned}$$

**Proof:** Write  $\Phi_{d-1}$  for sets of  $d-1$  distinct points in  $\Phi$ . Then, as  $\lambda(x) \equiv \lambda$  is constant, and  $g(0 \mid x, \Phi \cup \{x, y\}) g(0 \mid y, \Phi \cup \{x, y\})$  vanishes when  $x$  and  $y$  do not belong to the same Delaunay cell containing  $0$  in its interior,

$$\begin{aligned} C(\lambda, d) &= \lambda^2 \int \int \mathbb{E} \left[ \sum_{z \in \Phi_{d-1}} \frac{1\{0 \in D^\circ(x, y, z); b(x, y, z) \cap \Phi = \emptyset\}}{|W(x \mid \Phi \cup \{x, y\})| |W(y \mid \Phi \cup \{x, y\})|} \right] dx dy \\ &= \lambda^2 \int \int \mathbb{E} \left[ \sum_{z \in \Phi_{-x; d-1}} \frac{1\{-x \in D^\circ(0, y-x, z); b(0, y-x, z) \cap \Phi_{-x} = \emptyset\}}{|W(0 \mid \Phi_{-x} \cup \{0, y-x\})| |W(y-x \mid \Phi_{-x} \cup \{0, y-x\})|} \right] dx dy. \end{aligned}$$

Because of stationarity,

$$\begin{aligned} C(\lambda, d) &= \lambda^2 \int \int \mathbb{E} \left[ \sum_{z \in \Phi_{d-1}} \frac{1\{-x \in D^\circ(0, y-x, z); b(0, y-x, z) \cap \Phi = \emptyset\}}{|W(0 \mid \Phi \cup \{0, y-x\})| |W(y-x \mid \Phi \cup \{0, y-x\})|} \right] dx dy \\ &= \lambda^2 \int \int \mathbb{E} \left[ \frac{\sum_{z \in \Phi_{d-1}} 1\{-x \in D^\circ(0, y, z); b(0, y, z) \cap \Phi = \emptyset\}}{|W(0 \mid \Phi \cup \{0, y\})| |W(y \mid \Phi \cup \{0, y\})|} \right] dx dy. \end{aligned}$$

Scaling by  $\lambda^{1/d}$  yields that  $\lambda^{-2} C(\lambda, d)$  is equal to

$$\int \int \mathbb{E} \left[ \frac{\sum_{z \in \Phi_{d-1}} 1\{-\lambda^{1/d}x \in D^\circ(0, \lambda^{1/d}y, \lambda^{1/d}z); b(0, \lambda^{1/d}y, \lambda^{1/d}z) \cap \lambda^{1/d}\Phi = \emptyset\}}{\lambda^{-1} |W(0 \mid \lambda^{1/d}\Phi \cup \{0, \lambda^{1/d}y\})| \lambda^{-1} |W(\lambda^{1/d}y \mid \lambda^{1/d}\Phi \cup \{0, \lambda^{1/d}y\})|} \right] dx dy.$$

Since  $\lambda^{1/d}\Phi$  is a unit intensity Poisson point process, we obtain

$$\lambda^{-2} C(\lambda, d) = \int \int \mathbb{E}_1 \left[ \frac{\sum_{z \in \Phi_{d-1}} \mathbf{1}\{-x \in D^\circ(0, y, z); b(0, y, z) \cap \Phi = \emptyset\}}{|W(0 | \Phi \cup \{0, y\})| |W(y | \Phi \cup \{0, y\})|} \right] dx dy.$$

An appeal to Fubini's theorem to integrate out over  $x$  completes the proof.  $\square$

**Lemma 4.** *Let  $\Phi$  be a stationary Poisson point process in  $\mathbb{R}^d$  with intensity  $\lambda > 0$ . Then,*

$$C'(\lambda, d) := \int \mathbb{E} [g^2(0 | x, \Phi \cup \{x\})] \lambda(x) dx = \lambda^2 \mathbb{E}_1 \left[ \frac{1}{|W(0 | \Phi \cup \{0\})|} \right].$$

where  $\mathbb{E}_1$  denotes expectation with respect to a unit intensity Poisson point process.

**Proof:** Using  $\lambda(x) = \lambda$  and arguing as in the proof of Theorem 5, we get

$$\begin{aligned} C'(\lambda, d) &= \lambda \int \mathbb{E} \left[ \left( \sum_{\{z_1, \dots, z_d\} \subset \Phi}^{\neq} \frac{\mathbf{1}\{-x \in D^\circ(0, z_1, \dots, z_d); b(0, z_1, \dots, z_d) \cap \Phi = \emptyset\}}{|W(0 | \Phi \cup \{0\})|} \right)^2 \right] dx \\ &= \lambda \int \mathbb{E} \left[ \sum_{\{z_1, \dots, z_d\} \subset \Phi}^{\neq} \frac{\mathbf{1}\{-x \in D^\circ(0, z_1, \dots, z_d); b(0, z_1, \dots, z_d) \cap \Phi = \emptyset\}}{|W(0 | \Phi \cup \{0\})|^2} \right] dx \quad (7) \end{aligned}$$

as  $-x$  belongs to a single Delaunay interior. Write  $\Phi_d$  for sets of  $d$  distinct points in  $\Phi$  and scale each point in (7) by  $\lambda^{1/d}$  to obtain that  $C'(\lambda, d)$  is equal to

$$\lambda \int \mathbb{E} \left[ \sum_{z \in \Phi_d} \frac{\mathbf{1}\{-\lambda^{1/d}x \in D^\circ(0, \lambda^{1/d}z); b(0, \lambda^{1/d}z) \cap \lambda^{1/d}\Phi = \emptyset\}}{\lambda^{-2}|W(0 | \lambda^{1/d}\Phi \cup \{0\})|^2} \right] dx$$

which, since  $\lambda^{1/d}\Phi$  is a unit rate Poisson process reduces to

$$= \lambda^2 \int \mathbb{E}_1 \left[ \frac{\sum_{\{z_1, \dots, z_d\} \subset \Phi} \mathbf{1}\{-x \in D^\circ(0, z_1, \dots, z_d); b(0, z_1, \dots, z_d) \cap \Phi = \emptyset\}}{|W(0 | \Phi \cup \{0\})|^2} \right] dx.$$

The sum of  $d$ -volumes of Delaunay cells involving 0 is that of its contiguous Voronoi cell, and we conclude that

$$C'(\lambda, d) = \lambda^2 \mathbb{E}_1 \left[ \frac{1}{|W(0 | \Phi \cup \{0\})|} \right].$$

$\square$

The above results can be summarised as follows.

**Theorem 6.** *Let  $\Phi$  be a stationary Poisson point process in  $\mathbb{R}^d$  with intensity  $\lambda > 0$ . Then, the Delaunay tessellation field estimator  $\widehat{\lambda}(0)$  has asymptotic variance  $c_d \lambda^2$  with*

$$c_d = \mathbb{E}_1 \left[ \frac{1}{|W(0 | \Phi \cup \{0\})|} \left\{ 1 + \sum_{y \in \mathcal{N}(0 | \Phi \cup \{0\})} \frac{|W(0 | \Phi \cup \{0\}) \cap W(y | \Phi \cup \{0\})|}{|W(y | \Phi \cup \{0\})|} \right\} \right] - 1.$$

Note that the classic Berman–Diggle estimator (5) is asymptotically unbiased with variance  $\lambda \omega_d^{-1} h^{-d}$ , where  $\omega_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . In words, the Berman–Diggle estimator is more efficient whenever the average number of points per test ball exceeds  $1/c_d$ .

## 6 Poisson processes on the line

For one-dimensional Poisson processes, the distribution of the contiguous Voronoi cell can be calculated explicitly for arbitrary intensity functions. For simplicity, assume that  $A = [-w, w]$  is an interval of radius  $w > 0$  either side of the origin.

The following lemma is well-known.

**Lemma 5.** *Let  $\Phi$  be a Poisson point process on  $[-w, w]$  with finite intensity function  $\lambda$  and write  $\Lambda(a, b) = \int_a^b \lambda(x) dx$  for the moment measure of  $(a, b)$  for any  $-w \leq a \leq b \leq w$ . For  $x \in (-w, w)$ , define the random variables*

$$\begin{aligned}\Phi^-(x) &:= \max\{y \in \{-w\} \cup (\Phi \cap [-w, x])\}; \\ \Phi^+(x) &:= \min\{y \in \{w\} \cup (\Phi \cap (x, w])\}.\end{aligned}$$

Then, their distribution functions are given by

$$F^-(t) = \exp[-\Lambda(t, x)]$$

for  $t \in (-w, x)$ , with an atom of mass  $P(\Phi^-(x) = -w) = \exp[-\Lambda(-w, x)]$  at  $-w$ , respectively

$$F^+(s) = 1 - \exp[-\Lambda(x, s)]$$

for  $s \in (x, w)$  with an atom at  $w$  of mass  $P(\Phi^+(x) = w) = \exp[-\Lambda(x, w)]$ . Moreover, for fixed  $x$ ,  $\Phi^+(x)$  and  $\Phi^-(x)$  are independent random variables.

### 6.1 Expectation of the DTFE for Poisson processes on the line

Note that on the real line, the contiguous Voronoi cell  $W(x | (\Phi \cup \{x\}) \cap [-w, w])$  is the interval  $[\Phi^-(x), \Phi^+(x)]$ . Thus, Lemma 5 can be used to calculate the moments of the Delaunay tessellation field estimator. In this section, we shall deal with edge effects by placing two ghost points at the borders  $-w$  and  $w$ .

**Theorem 7.** *Let  $\Phi$  be a Poisson point process observed in  $A = [-w, w]$  for some  $w > 0$  with locally finite intensity function  $\lambda : \mathbb{R} \rightarrow [0, \infty)$ . Then, for  $x_0 \in A$ ,*

$$\begin{aligned}\mathbb{E} \left[ \widehat{\lambda(x_0)} \right] &= \int_{-w}^{x_0} \int_{x_0}^w \frac{\Lambda(t, s) \lambda(s) \lambda(t)}{s - t} e^{-\Lambda(t, s)} dt ds + \frac{\Lambda(-w, w) e^{-\Lambda(-w, w)}}{2w} \\ &+ \int_{x_0}^w \frac{\Lambda(-w, s) \lambda(s)}{w + s} e^{-\Lambda(-w, s)} ds + \int_{-w}^{x_0} \frac{\Lambda(t, w) \lambda(t)}{w - t} e^{-\Lambda(t, w)} dt.\end{aligned}\quad (8)$$

**Proof:** Fix  $x_0 \neq x \in (-w, w)$ , and let  $\varphi$  be a realisation of  $\Phi$ , which we augment by  $-w$  and  $w$  in order to obtain bounded Delaunay cells. Since almost surely,  $x \notin \Phi$  and  $x_0 \notin \Phi$ , assume  $x_0, x \notin \varphi$ , and consider  $g(x_0 | x, \varphi \cup \{x\})$  as defined in (3). Note that  $x_0$  belongs to a single Delaunay cell interior. If  $x$  is no endpoint of this cell,  $g(x_0 | x, \varphi \cup \{x\}) = 0$ . Otherwise,  $g(x_0 | x, \varphi \cup \{x\}) = 1/(\varphi^+(x_0) - \varphi^-(x_0))$ , cf. Lemma 5.

First, assume  $x < x_0$ . By Lemma 5 applied to the point  $x$ ,

$$\begin{aligned} \mathbb{E}[g(x_0 | x, \Phi \cup \{x\})] &= \int_{-w}^x \int_{x_0}^w \frac{dF^-(t) dF^+(s)}{s-t} = \frac{e^{-\Lambda(-w,w)}}{2w} + \\ &+ \int_{x_0}^w \frac{\lambda(s)}{w+s} e^{-\Lambda(-w,s)} ds + \int_{-w}^x \frac{\lambda(t)}{w-t} e^{-\Lambda(t,w)} dt + \int_{-w}^x \int_{x_0}^w \frac{\lambda(s)\lambda(t)}{s-t} e^{-\Lambda(t,s)} dt ds. \end{aligned}$$

Similarly, for  $x_0 < x$ ,

$$\begin{aligned} \mathbb{E}[g(x_0 | x, \{x\} \cup \Phi)] &= \int_{-w}^{x_0} \int_x^w \frac{dF^-(t) dF^+(s)}{s-t} = \frac{e^{-\Lambda(-w,w)}}{2w} + \\ &\int_x^w \frac{\lambda(s)}{w+s} e^{-\Lambda(-w,s)} ds + \int_{-w}^{x_0} \frac{\lambda(t)}{w-t} e^{-\Lambda(t,w)} dt + \int_{-w}^{x_0} \int_x^w \frac{\lambda(s)\lambda(t)}{s-t} e^{-\Lambda(t,s)} dt ds. \end{aligned}$$

By Theorem 1, the expectation of the Delaunay tessellation field estimator is as stated for  $x_0 \in (-w, w)$ .

It remains to consider  $x_0 = -w$  or  $w$ . In the first case,  $\varphi^-(x_0)$  is replaced by  $-w$ ; for  $x_0 = w$ ,  $\varphi^+(x_0)$  becomes  $w$  in the evaluation of  $g(x_0 | x, \varphi \cup \{x\})$ . Thus, for example,

$$\mathbb{E}[g(-w | x, \Phi \cup \{x\})] = e^{-\Lambda(-w,x)} \int_x^w \frac{dF^+(s)}{w+s} = \frac{e^{-\Lambda(-w,w)}}{2w} + \int_x^w \frac{\lambda(s)}{w+s} e^{-\Lambda(-w,s)} ds,$$

with a similar expression for  $x_0 = w$ . Upon integration, (8) is obtained, under the convention that integrals over intervals of zero length vanish.  $\square$

In general, (8) must be evaluated numerically. For the homogeneous Poisson process, analytic evaluation is possible. In fact, it can be shown that the estimator is unbiased even near the borders of the observation interval.

**Corollary 3.** *Let  $\Phi$  be a stationary Poisson point process observed in  $A = [-w, w]$  for some  $w > 0$  with intensity  $\lambda > 0$ . Then, the Delaunay tessellation field estimator  $\widehat{\lambda}(x_0)$  is unbiased for all  $x_0 \in A$ .*

**Proof:** For a stationary Poisson point process, the double integral in (8) reduces to

$$\lambda \left( e^{\lambda x_0} - e^{-\lambda w} \right) \times \left( e^{-\lambda x_0} - e^{-\lambda w} \right)$$

and in particular vanishes for  $x_0 = -w$  or  $w$ . The three border correction terms are equal to  $\lambda e^{-2\lambda w}$ , to  $\lambda e^{-\lambda w}(e^{-\lambda x_0} - e^{-\lambda w})$ , and to  $\lambda e^{-\lambda w}(e^{\lambda x_0} - e^{-\lambda w})$ , respectively. The sum of all four terms is  $\lambda$ , so the estimator is unbiased.  $\square$

Note that the Berman–Diggle estimator is unbiased as well, but that this may not be true for (6) due to edge correction near the border.

## 6.2 Variance of the DTFE for Poisson processes on the line

In this section, we derive the asymptotic variance of the Delaunay tessellation field estimator for a stationary Poisson process on the line. The result can be used to approximate the variance when the underlying intensity function is smoothly varying.

**Theorem 8.** *Let  $\Phi$  be a stationary Poisson point process observed in  $A = [-w, w]$  for some  $w > 0$  with intensity  $\lambda > 0$ . Then, as  $w \rightarrow \infty$ , the Delaunay tessellation field estimator  $\widehat{\lambda(0)}$  has asymptotic variance  $2\lambda^2(2 - \pi^2/6) \approx 0.7\lambda^2$ .*

The result should be compared to  $\lambda/(2h)$  for the Berman–Diggle kernel estimator [1], see also [10]. If  $2\lambda h > 1.4$ , that is the average number of points per bin at least 1.4, kernel estimation is the better choice. Naturally, in order to compute  $\widehat{\lambda(x_0)}$ , two points of the underlying process are used.

In order to give the proof, some special function theory is needed. Let  $x > 0$ . Recall that the exponential integral is defined as

$$E_1(x) = \int_1^\infty \frac{e^{-tx}}{t} dt = \int_x^\infty \frac{e^{-u}}{u} du.$$

Its integral satisfies

$$E_2(x) = \int_x^\infty E_1(s) ds = e^{-x} - xE_1(x).$$

In the limit,  $E_1(0) = \infty$  and  $E_2(0) = 1$ . Furthermore,

$$\int_0^\infty u e^u E_1(u)^2 du = 2 - \frac{\pi^2}{6}.$$

See for example [7] for further details. We shall also need the equation

$$\int_0^c e^{ax} E_1(ax) dx = \frac{\gamma + \log(ac) + e^{ac} E_1(ac)}{a}$$

where  $a$  and  $c$  are strictly positive constants, and  $\gamma \approx 0.577$  is the Euler-Mascheroni constant.

**Proof:** By Theorem 3, asymptotically  $\mathbb{E}[\widehat{\lambda(0)}] = \lambda$ . For the variance, by Theorem 2, we need to evaluate two further integrals. Now, arguing as in the proof of Theorem 7,

$$\begin{aligned} \int_A \mathbb{E}[g^2(x_0 | x, \Phi \cup \{x\})] \lambda(x) dx &= \int_{-w}^{x_0} \int_{x_0}^w \frac{\Lambda(t, s) \lambda(s) \lambda(t)}{(s-t)^2} e^{-\Lambda(t, s)} dt ds + \\ &\frac{\Lambda(-w, w) e^{-\Lambda(-w, w)}}{4w^2} + \int_{x_0}^w \frac{\Lambda(-w, s) \lambda(s)}{(w+s)^2} e^{-\Lambda(-w, s)} ds + \int_{-w}^{x_0} \frac{\Lambda(t, w) \lambda(t)}{(w-t)^2} e^{-\Lambda(t, w)} dt. \end{aligned} \quad (9)$$

Since the intensity function is constant and we took  $x_0 = 0$ , (9) reduces to

$$\frac{\lambda e^{-2\lambda w}}{2w} + \lambda e^{-\lambda w} \int_0^w \frac{\lambda e^{-\lambda s}}{w+s} ds + \lambda e^{-\lambda w} \int_{-w}^0 \frac{\lambda e^{\lambda t}}{w-t} dt + \int_{-w}^0 \int_0^w \frac{\lambda^3 e^{\lambda t} e^{-\lambda s}}{s-t} dt ds.$$

Clearly, the first term above converges to 0 as  $w \rightarrow \infty$ . Due to symmetry, the two middle terms are equal. Note that

$$2\lambda \int_0^w \frac{\lambda e^{-\lambda(s+w)}}{s+w} ds = 2\lambda^2 \int_{\lambda w}^{2\lambda w} \frac{e^{-u}}{u} du = 2\lambda^2 [E_1(\lambda w) - E_1(2\lambda w)],$$

which converges to zero as  $w \rightarrow \infty$ . Moreover,

$$\lambda^3 \int_{-\infty}^0 \int_0^{\infty} \frac{e^{\lambda t} e^{-\lambda s}}{s-t} dt ds = \lambda^3 \int_{-\infty}^0 E_1(-\lambda t) dt = \lambda^2 E_2(0) = \lambda^2.$$

To calculate the double integral in Theorem 2, let  $x \neq y$  be points of  $(-w, w)$ , fix  $x_0 \notin \{x, y, -w, w\}$ , and let  $\varphi$  be a realisation of  $\Phi$ , which we augment by  $-w$  and  $w$  in order to obtain bounded Delaunay cells. Since almost surely none of  $x, y$  or  $x_0$  lie in  $\Phi$ , assume  $x_0, x, y \notin \varphi$ , and consider  $g(x_0 | x, \varphi \cup \{x, y\})$  as defined in (3). Note that  $x_0$  belongs to a single Delaunay cell interior. If  $x$  and  $y$  are not both endpoints of this cell,  $g(x_0 | x, \varphi \cup \{x, y\}) g(x_0 | y, \varphi \cup \{x, y\}) = 0$ . Otherwise, without loss of generality,  $x < x_0 < y$ , and  $g(x_0 | x, \varphi \cup \{x, y\}) = 1/(y - \varphi^-(x_0))$  and  $g(x_0 | y, \varphi \cup \{x, y\}) = 1/(\varphi^+(x_0) - x)$ .

Thus, for  $x < x_0$  and  $y > x_0$ , let  $F^-$  and  $F^+$  be the cumulative distribution functions of  $\Phi^-(x_0)$  and  $\Phi^+(x_0)$ . By Lemma 5,

$$\begin{aligned} \mathbb{E} [g(x_0 | x, \Phi \cup \{x, y\}) g(x_0 | y, \Phi \cup \{x, y\})] &= \int_{-w}^x \int_y^w \frac{dF^-(t) dF^+(s)}{(y-t)(s-x)} = \\ &= \int_y^w \frac{\lambda(s)}{(w+y)(s-x)} e^{-\Lambda(-w,s)} ds + \int_{-w}^x \frac{\lambda(t)}{(y-t)(w-x)} e^{-\Lambda(t,w)} dt \\ &+ \frac{e^{-\Lambda(-w,w)}}{(w+y)(w-x)} + \int_{-w}^x \int_y^w \frac{\lambda(s)\lambda(t)}{(y-t)(s-x)} e^{-\Lambda(t,s)} dt ds. \end{aligned}$$

By symmetry,

$$\begin{aligned} \int_A \int_A \mathbb{E} [g(x_0 | x, \Phi \cup \{x, y\}) g(x_0 | y, \Phi \cup \{x, y\})] \lambda(x) \lambda(y) dx dy &= \\ &= 2e^{-\Lambda(-w,w)} \int_{-w}^{x_0} \frac{\lambda(x)}{w-x} dx \int_{x_0}^w \frac{\lambda(y)}{w+y} dy \\ &+ 2 \int_{x_0}^w \lambda(s) e^{-\Lambda(-w,s)} \left[ \int_{-w}^{x_0} \frac{\lambda(x)}{s-x} dx \int_{x_0}^s \frac{\lambda(y)}{w+y} dy \right] ds \\ &+ 2 \int_{-w}^{x_0} \lambda(t) e^{-\Lambda(t,w)} \left[ \int_t^{x_0} \frac{\lambda(x)}{w-x} dx \int_{x_0}^w \frac{\lambda(y)}{y-t} dy \right] dt \\ &+ 2 \int_{-w}^{x_0} \int_{x_0}^w \lambda(t) \lambda(s) e^{-\Lambda(t,s)} \left[ \int_t^{x_0} \frac{\lambda(x)}{s-x} dx \int_{x_0}^s \frac{\lambda(y)}{y-t} dy \right] dt ds. \end{aligned} \quad (10)$$

For  $x_0 \in \{-w, w\}$ , formula (10) holds true under the convention that integrals over intervals of zero length vanish, as in this case  $x_0$  cannot belong to any Delaunay cell with endpoints  $x < x_0 < y$ .

Next, we plug in  $x_0 = 0$  and  $\lambda(\cdot) \equiv \lambda$ , and consider each integral in (10) in turn. The main term is the four fold integral

$$\int_{-w}^0 \int_0^w \int_{-w}^x \int_y^w \frac{2\lambda^4 e^{\lambda t} e^{-\lambda s}}{(y-t)(s-x)} dx dy dt ds.$$

Its limit as  $w \rightarrow \infty$  is

$$\begin{aligned} & 2\lambda^4 \int_{-\infty}^0 \int_0^{\infty} e^{\lambda(y-x)} \left[ \int_{-\infty}^x \frac{e^{-\lambda(y-t)}}{y-t} dt \int_y^{\infty} \frac{e^{-\lambda(s-x)}}{s-x} ds \right] dx dy = \\ & 2\lambda^4 \int_{-\infty}^0 \int_0^{\infty} e^{\lambda(y-x)} E_1(\lambda(y-x))^2 dx dy = 2\lambda^3 \int_{-\infty}^0 \int_{-\lambda x}^{\infty} e^u E_1(u)^2 dx du = \\ & 2\lambda^2 \int_0^{\infty} \int_y^{\infty} e^u E_1(u)^2 dy du = 2\lambda^2 \int_0^{\infty} u e^u E_1(u)^2 du = 2\lambda^2(2 - \pi^2/6), \end{aligned}$$

upon a change of integration order.

The first term in (10) reduces to  $2e^{-2\lambda w}(\lambda \log 2)^2$  for a homogeneous Poisson process, which tends to zero as  $w \rightarrow \infty$ .

It remains to consider the sum of the two three fold integrals in (10)

$$\int_{-w}^0 \int_0^w \int_y^w \frac{4\lambda^3 e^{-\lambda(s+w)}}{(s-x)(y+w)} dx dy ds$$

which can be written as

$$\begin{aligned} & 4\lambda^3 \int_0^w \int_0^w \left( \int_0^s \frac{dy}{y+w} \right) \frac{e^{-\lambda(s+w)}}{s+x} dx ds \leq 4\lambda^3 \log 2 \int_0^w e^{-\lambda w + \lambda x} \left( \int_0^w \frac{e^{-\lambda(s+x)}}{s+x} ds \right) dx = \\ & 4\lambda^3 \log 2 \int_0^w e^{-\lambda w + \lambda x} [E_1(\lambda x) - E_1(\lambda x + \lambda w)] dx = 4\lambda^2 h(\lambda, w) \log 2, \end{aligned}$$

where

$$\begin{aligned} h(\lambda, w) &= e^{-\lambda w} \int_0^{\lambda w} e^u [E_1(u) - E_1(u + \lambda w)] du \\ &= \left( e^{-\lambda w} + e^{-2\lambda w} \right) \int_0^{\lambda w} e^u E_1(u) du - e^{-2\lambda w} \int_0^{2\lambda w} e^u E_1(u) du \\ &= e^{-\lambda w} \gamma + (e^{-\lambda w} + e^{-2\lambda w}) \log(\lambda w) - e^{-2\lambda w} \log(2\lambda w) \\ &+ E_1(\lambda w)(1 + e^{-\lambda w}) - E_1(2\lambda w) \end{aligned}$$

tends to zero as  $w \rightarrow \infty$ . The proof is finished upon collection of all terms.  $\square$

As a corollary, the proof gives an expression for the second moment of the Delaunay tessellation field estimator of the intensity function for Poisson processes with not necessarily constant locally finite intensity function on intervals of the form  $[-w, w]$  by combining (9)–(10). A slightly simpler proof can be obtained by an appeal to Theorem 6, but such a proof cannot be generalised to non-homogeneous Poisson processes.



## 7 Discussion

In this paper, we analysed Schaap and Van de Weygaert's Delaunay tessellation field estimator [14, 15] for the intensity function of a point process. We expressed its mean and variance in terms of the first and second order factorial moment measures of the underlying point process, and placed the estimator in the context of adaptive kernel estimation. We then focussed on Poisson point processes, and showed that for stationary Poisson processes, the DTFE is asymptotically unbiased with a variance that is proportional to the squared intensity. The proportionality constant depends on the dimension. For  $d = 1$ , explicit calculation is possible. For  $d = 2$ , we used the DELDIR package [19] to obtain  $C(\lambda, 2) \approx 0.8\lambda^2$  and  $C'(\lambda, 2) \approx 0.6\lambda^2$ , see Lemma 3 and 4. Note that in the plane it is possible to write mean and variance as repeated integrals in the spirit of Calka [2], but explicit evaluation seems difficult. Simulations for the case  $d = 3$  of most interest to cosmologists can be found in Schaap's Ph.D. thesis [15].

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## References

- [1] M. Berman and P.J. Diggle (1989). Estimating weighted integrals of the second-order intensity of a spatial point process. *Journal of the Royal Statistical Society Series B*, 51:81–92.
- [2] P. Calka (2003). An explicit expression for the distribution of the number of sides of the typical Poisson–Voronoi cell. *Advances in Applied Probability (SGSA)*, 35:863–870.
- [3] N.A.C. Cressie (1991, revised edition 1993). *Statistics for spatial data*. John Wiley and Sons, New York.
- [4] D.J. Daley and D. Vere–Jones (1988, revised edition 2003, 2008). *An introduction to the theory of point processes*. Springer, New York.
- [5] B.N. Delaunay (1934). Sur la sphère vide. *Iszvestia Akademii Nauk SSSR, Otdelenie Matematicheskikh i Estestvennykh Nauk*, 7:793–800.
- [6] P.J. Diggle (1983, revised edition 2003). *Statistical analysis of spatial point patterns*. Academic Press, London.
- [7] M. Geller and E.W. Ng (1969). A table of integrals of the exponential integral. *Journal of Research of the National Bureau of Standards*, 73B:191–210.
- [8] K.-H. Hanisch (1982). On inversion formulae for  $n$ -fold Palm distributions of point processes in LCS-spaces. *Mathematische Nachrichten*, 106:171–179.

- [9] M.N.M. van Lieshout (2000). *Markov point processes and their applications*. Imperial College Press, London.
- [10] M.N.M. van Lieshout (2007). Edge corrected non-parametric intensity function estimators for heterogeneous Poisson point processes. EURANDOM Report 2007-042.
- [11] J. Møller (1994). *Lectures on random Voronoi tessellations*. Lecture Notes in Statistics 87. Springer-Verlag, New York.
- [12] A. Okabe, B. Boots, and K. Sugihara (1992). *Spatial tessellations. Concepts and applications of Voronoi diagrams. With a foreword by D.G. Kendall*. Wiley Series in Probability and Mathematical Statistics. John Wiley and Sons, Chichester.
- [13] J.K. Ord (1978). How many trees in a forest? *Mathematical Scientist*, 3:23–33.
- [14] W.E. Schaap and R. van de Weygaert (2000). Letter to the Editor. Continuous fields and discrete samples: reconstruction through Delaunay tessellations. *Astronomy and Astrophysics*, 363:L29–L32.
- [15] W.E. Schaap (2007). *DTFE. The Delaunay Tessellation Field Estimator*. Ph.D. Thesis, University of Groningen.
- [16] R. Sibson (1981). A brief description of natural neighbour interpolation. In: V. Barnett (Ed.) *Interpreting multivariate data*. John Wiley and Sons, New York, 21–36.
- [17] B.W. Silverman (1986). *Density estimation for statistics and data analysis*. Chapman and Hall, London.
- [18] D. Stoyan, W.S. Kendall, and J. Mecke (1987, revised edition 1995). *Stochastic geometry and its applications*. John Wiley and Sons, Chichester.
- [19] R. Turner. DELDIR, An R package to construct the Delaunay triangulation and the Dirichlet (Voronoi) tessellation of a planar point set. <http://www.math.unb.ca/~rolf>.
- [20] G. Voronoi (1908). Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *Reine und Angewandte mathematik* 134:198–287.