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A special class of Jacobi series and some applications

by

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1. A convolution structure for Jacobi series.

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, of order (α,β), defined by

$$(1-x)^{\alpha}\ (1+x)^{\beta}\ P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\ n!}\ \frac{d^n}{dx^n}\ \left[(1-x)^{n+\alpha}\ (1+x)^{n+\beta}\right],\ \alpha,\ \beta \geq -\tfrac{1}{2}.$$

These polynomials are orthogonal on the interval $(-1,1)$ with respect to the weight function $(1-x)^{\alpha}\ (1+x)^{\beta}$ and normalized by

$$(1.1) \qquad P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} = \mathcal{O}(n^{\alpha}).$$

For convenience we often change the variable $x = \cos\Theta$. Then the functions $P_n^{(\alpha,\beta)}(\cos\Theta)$ are orthogonal on $(0,\pi)$ with respect to

$$(1.2) \qquad \rho^{(\alpha,\beta)}(\Theta) = (\sin\frac{\Theta}{2})^{2\alpha+1}(\cos\frac{\Theta}{2})^{2\beta+1}$$

and with

$$(1.3) \qquad \left[\omega_n^{(\alpha,\beta)}\right]^{-1} = \int_0^{\pi} \{P_n^{(\alpha,\beta)}(\cos\Theta)\}^2\ \rho^{(\alpha,\beta)}(\Theta)\ d\Theta$$

$$= \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} = \mathcal{O}(n^{-1}).$$

The functions $P_n^{(\alpha,\beta)}(\cos\Theta)$ are the eigenfunctions of the differential operator

$$P(\frac{d}{d\Theta}) = -\ (\frac{d^2}{d\Theta^2} + \frac{(\alpha-\beta)+(\alpha+\beta+1)\cos\Theta}{\sin\Theta}\ \frac{d}{d\Theta})$$

which can be written in the form

$$(1.4) \qquad P(\frac{d}{d\Theta}) = -\ (\rho^{(\alpha,\beta)}(\Theta))^{-1}\ \frac{d}{d\Theta}\ \{\rho^{(\alpha,\beta)}(\Theta)\ \frac{d}{d\Theta}\}$$

with the boundary conditions

$$(1.5) \qquad \frac{dP_n^{(\alpha,\beta)}(\cos\Theta)}{d\Theta} = 0, \qquad \Theta = 0,\ \Theta = \pi.$$

The eigenvalues are $\lambda_n = n(n+\alpha+\beta+1)$.
The differential operator is selfadjoint with respect to the scalar product with the weight function $\rho^{(\alpha,\beta)}(\Theta)$:

$$(1.6) \qquad \int_0^\pi Pf(\Theta)\,\overline{g(\Theta)}\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta = \int_0^\pi f(\Theta)\,\overline{Pg(\Theta)}\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta$$

as follows easily from (1.4).
By $A_\Theta = A_\Theta^1$ we shall denote the corresponding realization of P in L_1.

Let $f(\cos\Theta)$ be in $L_1(0,\pi)$ with respect to $\rho^{(\alpha,\beta)}(\Theta)$, i.e.
$||f||_1 = \int_0^\pi |f(\cos\Theta)|\rho^{(\alpha,\beta)}(\Theta)\,d\Theta < \infty$. Then we associate with $f(\cos\Theta)$ the formal Fourier-Jacobi series

$$(1.7) \qquad f(\cos\Theta) \sim \sum_{n=0}^{\infty} a_n\,\omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1)\,P_n^{(\alpha,\beta)}(\cos\Theta)$$

where

$$(1.8) \qquad a_n = \int_0^\pi f(\cos\Theta)\,\frac{P_n^{(\alpha,\beta)}(\cos\Theta)}{P_n^{(\alpha,\beta)}(1)}\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta.$$

Following the paper of Askey and Wainger [3], we introduce the kernel

$$(1.9) \qquad K_r(\Theta,\phi,\psi) = \sum_{n=0}^{\infty} r^n\,\omega_n^{(\alpha,\beta)}\,P_n^{(\alpha,\beta)}(\cos\Theta)\,P_n^{(\alpha,\beta)}(\cos\phi)\,P_n^{(\alpha,\beta)}(\cos\psi)\,[P_n^{(\alpha,\beta)}(1)]^{-1}$$

and define

$$f_r(\cos\Theta,\cos\phi) = \int_0^\pi K_r(\Theta,\phi,\psi)\,f(\cos\psi)\,\rho^{(\alpha,\beta)}(\psi)\,d\psi$$

$$= \sum_{n=0}^{\infty} r^n\,a_n\,\omega_n^{(\alpha,\beta)}\,P_n^{(\alpha,\beta)}(\cos\Theta)\,P_n^{(\alpha,\beta)}(\cos\phi).$$

In order to show that $\lim_{r\to 1^-} f_r(\cos\Theta, \cos\phi) = f(\cos\Theta, \cos\phi)$ for almost every Θ and ϕ, a property clearly satisfied for a dense set of functions, it is necessary and sufficient to show that

$$(1.10) \qquad \int_0^\pi |K_r(\Theta,\phi,\psi)|\ \rho^{(\alpha,\beta)}(\psi)\ d\psi < A \qquad 0 \leq r < 1.$$

See Ljusternik and Sobolew [9], theorem 4, p. 103.
Relation (1.10) has been shown by Askey and Wainger by a long and tedious calculation [3].
Furthermore for a dense set of functions

$$\int_0^\pi |f_r(\cos\Theta,\cos\phi) - f(\cos\Theta,\cos\phi)|\ \rho^{(\alpha,\beta)}(\Theta)\ d\Theta \to 0 \quad \text{as} \quad r \to 1^-$$

for almost every ϕ.
Moreover, using (1.10) and the symmetry of the kernel,

$$\int_0^\pi |f_r(\cos\Theta,\cos\phi)|\ \rho^{(\alpha,\beta)}(\Theta)\ d\Theta =$$

$$= \int_0^\pi \left| \int_0^\pi K_r(\Theta,\phi,\psi)\ f(\cos\psi)\ \rho^{(\alpha,\beta)}(\psi)\ d\psi \right| \rho^{(\alpha,\beta)}(\Theta)\ d\Theta$$

$$\leq \int_0^\pi |f(\cos\psi)|\ \rho^{(\alpha,\beta)}(\psi)\ \left\{\int_0^\pi |K_r(\Theta,\phi,\psi)|\ \rho^{(\alpha,\beta)}(\Theta)\ d\Theta\right\} d\psi$$

$$\leq A \int_0^\pi |f(\cos\psi)|\ \rho^{(\alpha,\beta)}(\psi)\ d\psi.$$

Hence it follows that for almost every ϕ

$$f_r(\cos\Theta,\cos\phi) \to f(\cos\Theta,\cos\phi) \quad \text{as} \quad r \to 1^-$$

in L_1 with respect to the measure $\rho^{(\alpha,\beta)}(\Theta)\ d\Theta$ and almost everywhere in Θ.

In the special case that $\psi = 0$, the kernel

$$K_r(\Theta,\phi,0) = \sum_{n=0}^{\infty} r^n \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos \Theta) P_n^{(\alpha,\beta)}(\cos \phi)$$

is positive as is shown by Bailey [5] by explicit calculation. In that case it follows from the orthogonality that

$$\int_0^{\pi} K_r(\Theta,\phi,0) \rho^{(\alpha,\beta)}(\phi) d\phi = 1.$$

Hence $f_r(\cos \Theta) \to f(\cos \Theta)$ as $r \to 1^-$ for almost every Θ, which shows the Abel summability of the series (1.7). In a recent letter to Prof. Askey, G. Gasper announces to have shown the positivity of the kernel (1.9). If we assume $g(\cos \Theta)$ to be in $L_1(0,\pi)$ with respect to $\rho^{(\alpha,\beta)}(\Theta)$ we can define

$$(1.11) \qquad h(\cos \Theta) = \int_0^{\pi} f(\cos \Theta,\cos \phi) g(\cos \phi) \rho^{(\alpha,\beta)}(\phi) d\phi.$$

From Fubini's theorem it follows that $h(\cos \Theta)$ is in $L_1(0,\pi)$ with respect to $\rho^{(\alpha,\beta)}(\Theta)$ and that

$$(1.12) \qquad ||h||_1 \leq A ||f||_1 ||g||_1.$$

Also it is not hard to derive that if g is in $L_\infty(0,\pi)$ then

$$(1.13) \qquad ||h||_\infty \leq A ||f||_1 ||g||_\infty$$

where

$$||g||_\infty = \sup_{0\leq\Theta\leq\pi} |g(\cos \Theta)| < \infty,$$

We shall call $h(\cos \Theta)$, defined by (1.11), the convolution of $f(\cos \Theta)$ and $g(\cos \Theta)$. From the fact that

$$h(\cos \Theta) = \int_0^\pi f(\cos \Theta,\cos \phi)\ g(\cos \phi)\ \rho^{(\alpha,\beta)}(\phi)\ d\phi$$

$$= \int_0^\pi f(\cos \phi)\ g(\cos \Theta,\cos \phi)\ \rho^{(\alpha,\beta)}(\phi)\ d\phi$$

which is easily verified, and from the inequalities (1.12) and (1.13) many important properties follow. See O'Neil [11].

We shall call $f(\cos \Theta,\cos \phi)$ the generalized translation of $f(\cos \Theta)$. It is a generalized translation in the sense used by Löfström and Peetre [10]. In their paper they make the connection between a generalized translation operator and a differential operator of the form (1.4) with boundary conditions (1.5). They show that the remainder term of the Taylor series can be estimated by

$$(1.14) \qquad ||f(\cos \Theta,\cos \phi) - f(\cos \Theta)||_p \leq C\ \phi^2\ ||A_\Theta f||_p,\ 1 \leq p \leq \infty.$$

We shall use this estimate in the last section.
It is clear that if $f(\cos \Theta)$ has a Fourier-Jacobi expansion (1.7), then

$$(1.15) \qquad A_\Theta\ f(\cos \Theta) \sim \sum_{n=0}^{\infty} a_n\ n(n+\alpha+\beta+1)\ \omega_n^{(\alpha,\beta)}\ P_n^{(\alpha,\beta)}(1)\ P_n^{(\alpha,\beta)}(\cos \Theta).$$

Furthermore it follows from (1.10) and the definition of $f(\cos \Theta,\cos \phi)$ that

$$(1.16) \qquad ||f(\cos \Theta,\cos \phi)||_\infty \leq A\ ||f(\cos \Theta)||_\infty\ .$$

In the following we shall study a special class of Jacobi series such as

$$F(\cos \Theta) \sim \sum_{n=1}^{\infty} n^{-\gamma}(\log n)^{\delta}\ \omega_n^{(\alpha,\beta)}\ P_n^{(\alpha,\beta)}(1)\ P_n^{(\alpha,\beta)}(\cos \Theta).$$

We do this in a way similar to [4], where Askey and Wainger treat the same problem for ultraspherical series. At a few places the proofs could be simplified a little.
In the last section this special class of Jacobi series is used to define fractional integration and differentiation by means of the convolution structure and the differential operator A_Θ defined above
We shall prove that all the usual properties of fractional integration and differentiation remain valid.

We shall use $\mathcal{O}$ and o in the usual manner. We write $F(x) \simeq G(x)$ as x tends to a, to mean $F(x)/G(x)$ tends to 1 as x tends to a.

2. Summation by parts and a criterion for Fourier-Jacobi series.

In this section we develop a method of summation by parts, which depends strongly on the Christoffel-Darboux formula. It can be used to do some work normally done by integretion by parts in the theory of the Fourier integral, when one uses the fact that
$\exp(itx) = (ix)^{-1} \frac{d}{dt} \exp(itx)$.
As an application we shall prove a simple sufficient condition for a series

$$\sum_{n=1}^{\infty} a(n)\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

to be a Fourier-Jacobi series of some function.

Lemma 2.1.
Let $a(n)$ be a function defined on the positive integers. Let

$$H(N,\cos\Theta) = \sum_{n=0}^{N} a(n)\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta).$$

Then
(2.1)
$$H(N,\cos\Theta) = \sum_{n=0}^{N} \Delta'\, a(n)\, \frac{(\alpha+1)}{(2n+\alpha+\beta+2)}\, \omega_n^{(\alpha+1,\beta)}\, P_n^{(\alpha+1,\beta)}(1)\, P_n^{(\alpha+1,\beta)}(\cos\Theta)$$

where, if $d(n)$ is a sequence of numbers,

$$\Delta' d(n) = \Delta\, d(n) = d(n) - d(n+1) \quad (n=0,1,\ldots,N-1)$$

$$\Delta' d(N) = d(N).$$

In particular, if $a(n) = \vartheta(-\varepsilon n)$ $(\varepsilon>0)$, we have
(2.2)
$$H(\cos\Theta) = \sum_{n=0}^{\infty} a(n)\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta) =$$

$$= \sum_{n=0}^{\infty} \Delta\, a(n)\, \frac{(\alpha+1)}{2n+\alpha+\beta+2}\, \omega_n^{(\alpha+1,\beta)}\, P_n^{(\alpha+1,\beta)}(1)\, P_n^{(\alpha+1,\beta)}(\cos\Theta).$$

Proof.

The proof is essentially the application of the Christoffel-Darboux formula for Jacobi polynomials. (Szegō [12], (4.5.3)).

$$H(N,\cos\Theta) = \sum_{n=0}^{N} \Delta' a(n) \sum_{k=0}^{n} \omega_k^{(\alpha,\beta)} P_k^{(\alpha,\beta)}(1) P_k^{(\alpha,\beta)}(\cos\Theta)$$

$$= \sum_{n=0}^{N} \Delta' a(n) \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(\cos\Theta)$$

$$= \sum_{n=0}^{N} \Delta' a(n) \frac{(\alpha+1)}{2n+\alpha+\beta+2} \omega_n^{(\alpha+1,\beta)} P_n^{(\alpha+1,\beta)}(1) P_n^{(\alpha+1,\beta)}(\cos\Theta)$$

(2.2) follows by taking the limit as $N \to \infty$, since $P_n^{(\alpha,\beta)}(\cos\Theta)$ does not grow faster than a polynomial in n.

We shall need a lemma, which deals with the repeated application of lemma 2.1. We shall state the results in terms of derivatives rather than finite differences.

Lemma 2.2.

Let ν be any possitive integer and let a(t) be a function of a real variable t, possessing ν continuous derivatives. Assume that

$$\left|\frac{d^j a(t)}{dt^j}\right| = \mathcal{O}(\exp -\varepsilon t) \quad \text{for} \quad j = 0, 1, \ldots, \nu,$$

and any $\varepsilon > 0$. Define

$$H(\cos\Theta) = \sum_{n=1}^{\infty} a(n)\, \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1) P_n^{(\alpha,\beta)}(\cos\Theta).$$

Then we may write

(2.3)

$$H(\cos\Theta) = (-1)^{\nu} \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)} \sum_{n=1}^{\infty} q_{\nu}(n)\, \omega_n^{(\alpha+\nu,\beta)} P_n^{(\alpha+\nu,\beta)}(1) P_n^{(\alpha+\nu,\beta)}(\cos\Theta) +$$

$$+ E_1(\cos\Theta)_1$$

where

$$q_1(t) = \frac{1}{2t+\alpha+\beta+2} \frac{d}{dt} a(t)$$

$$q_k(t) = \frac{1}{2t+\alpha+\beta+k+1} \frac{d}{dt} q_{k-1}(t) \quad (k\geq 2).$$

Also

$$(2.4) \qquad H(\cos\Theta) = \sum_{j=0}^{\nu-1} c(j,\nu) \sum_{n=1}^{\infty} n^{-\nu-j} \left\{\frac{d^{\nu-j}}{dt^{\nu-j}} a(t)\right\}_{t=n} \omega_n^{(\alpha+\nu,\beta)} P_n^{(\alpha+\nu,\beta)}(1)\, P_n^{(\alpha+\nu,\beta)}(\cos\Theta) + E_2(\cos\Theta).$$

The $c(j,\nu)$ are numbers.
For $i = 1, 2$

$$E_i(\cos\Theta) = \sum_{j=0}^{\nu-1} d_i(j,\nu) \sum_{n=1}^{\infty} n^{-\nu-j} \gamma_{n,\nu-j+1}\, \omega_n^{(\alpha+\nu,\beta)} P_n^{(\alpha+\nu,\beta)}(1)\, P_n^{(\alpha+\nu,\beta)}(\cos\Theta)$$

where $d_i(j,\nu)$ are numbers and

$$|\gamma_{n,j}| \leq \max_{n\leq t\leq n+a_\nu} \left|\frac{d^j a(t)}{dt^j}\right|;$$

a_ν is some integer depending only on ν.

Proof.
We start with equation (2.2) and then apply lemma 2.1 again. We repeat the process ν times in all and then we use the mean value theorem to replace differences by derivatives. This finishes the proof.

Theorem 2.1.

Let $\alpha \geq \beta \geq -\frac{1}{2}$ and let ν be an integer $> \alpha + \frac{3}{2}$. Assume a(t) is continuous on $[0,\infty)$ and that a(t) approaches zero as $t \to \infty$. Furthermore assume a(t) has $\nu + 1$ continuous derivatives on $[0,\infty)$ and let

$$\gamma_{n,j} = \max_{n \leq t \leq n+a_\nu} \left|\frac{d^j a(t)}{dt^j}\right|,$$

with a_ν as in lemma 2.2.

Finally suppose

$$\sum_{n=1}^{\infty} n^{j-1} \gamma_{n,j} < \infty, \qquad j = 1, 2, \ldots, \nu.$$

Then there is a function F(cos Θ) such that

$$\int_0^\pi |F(\cos \Theta)| \; \rho^{(\alpha,\beta)}(\Theta) \, d\Theta < \infty$$

and

$$a_n = \int_0^\pi F(\cos \Theta) \frac{P_n^{(\alpha,\beta)}(\cos \Theta)}{P_n^{(\alpha,\beta)}(1)} \; \rho^{(\alpha,\beta)}(\Theta) \, d\Theta.$$

Proof.

Let

$$F_\varepsilon(\cos \Theta) = \sum_{n=1}^{\infty} e^{-\varepsilon n} \, a(n) \, \omega_n^{(\alpha,\beta)} \, P_n^{(\alpha,\beta)}(1) \, P_n^{(\alpha,\beta)}(\cos \Theta).$$

By lemma 2, equation 2.4 (with $e^{-\varepsilon n}$ a(n) instead of a(n))

$$F_\varepsilon(\cos \Theta) = \sum_{k,j,m} \sum_{n=1}^{\infty} c_{k,j,m,n} \, e^{-\varepsilon n} \, n^{-\nu-j+1} \, \varepsilon^m$$

$$\left\{\frac{d^k}{dt^k} a(t)\right\}_{t=n} P_n^{(\alpha+\nu,\beta)}(1) \, P_n^{(\alpha+\nu,\beta)}(\cos \Theta) + E_1.$$

The coefficients $c_{k,j,m,n}$ are bounded for fixed ν and the summation is extended over non-negative integers k,j and m such that $k + j + m = \nu$, and what is very important, at least one of m and k is equal to or greater than one. The remainder term E_1 is of the same form, but here $k + j + m > \nu$. It can be handled in the same way as the main term. Let $S_\varepsilon(\cos\Theta)$ denote those terms with $k \geq 1$. We use the trivial estimate $\varepsilon^m \exp(-\varepsilon n) = \mathcal{O}(n^{-m})$. So, by (1.1),

$$|S_\varepsilon(\cos\Theta)| = \mathcal{O}\{\sum_{k,j,m} \sum_{n=1}^{\infty} n^{\alpha-j-m+1} \gamma_{n,k} |P_n^{(\alpha+\nu,\beta)}(\cos\Theta)|\}$$

where $k + j + m = \nu$ and $k \geq 1$.

We need the following estimates (see Szegö [12], 7.32.6 and 4.1.3)

$$(2.5)\quad 0 \leq \Theta \leq \pi \quad |P_n^{(\alpha,\beta)}(\cos\Theta)| = \mathcal{O}(\Theta^{-\alpha-\frac{1}{2}}(\pi-\Theta)^{-\beta-\frac{1}{2}}n^{-\frac{1}{2}})$$

$$(2.6)\quad 0 \leq \Theta \leq \pi \quad |P_n^{(\alpha,\beta)}(\cos\Theta)| = \mathcal{O}(n^{\alpha}) \quad \alpha \geq \beta.$$

We then find

$$(2.7)\qquad \int_0^\pi |P_n^{(\alpha+\nu,\beta)}(\cos\Theta)\ \rho^{(\alpha,\beta)}(\Theta)\ d\Theta = \mathcal{O}(n^{\nu-\alpha-2}) \quad \nu > \alpha + \frac{3}{2}$$

Thus by hypothesis

$$\int_0^\pi |S_\varepsilon(\cos\Theta)|\rho^{(\alpha,\beta)}(\Theta)d\Theta = \mathcal{O}(\sum_{k,j,m} \sum_{n=1}^{\infty} n^{\alpha-j-m+1}\gamma_{n,k}\ n^{\nu-\alpha-2})$$

$$= \mathcal{O}(\sum_{n=1}^{\infty} n^{k-1}\ \gamma_{n,k}) < M < \infty.$$

M does not depend on ε. Therefore $S_\varepsilon(\cos\Theta)\ \rho^{(\alpha,\beta)}(\Theta)$ converges weakly to a measure (see Ljusternik and Sobolew [9], p. 175). Moreover, using (2.5), we see that $S_\varepsilon(\cos\Theta)\ \rho^{(\alpha,\beta)}(\Theta)$ converges pointwise as $\varepsilon \to 0^+$ for $0 < \Theta \leq \pi$ and uniformly in any compact subinterval of $(0,\pi]$.
The term $(\pi-\Theta)^{-\beta-\frac{1}{2}}$ of (2.5) is compensated by the factor $(\cos\frac{\Theta}{2})^{2\beta+1}$ of $\rho^{(\alpha,\beta)}(\Theta)$. The factor $(\sin\frac{\Theta}{2})^{2\alpha+1}$ cannot compensate the singularity at 0 completely because we have here a factor $\Theta^{-\alpha-\nu-\frac{1}{2}}$ due to the summation by parts.

We put $T_\varepsilon(\cos\Theta) = F_\varepsilon(\cos\Theta) - S_\varepsilon(\cos\Theta)$. In these terms $k = 0$ and $m \geq 1$. Using again the fact that $\varepsilon^1 \exp(-\frac{\varepsilon}{2}n) = \mathcal{O}(n^{-1})$ we see

$$T_\varepsilon(\cos\Theta) = \sum_{j=0}^{\nu} c_j \sum_{n=1}^{\infty} e^{-\varepsilon n}\, n^{-\nu-j+1}\, \varepsilon^{\nu-j}\, a(n)\, P_n^{(\alpha+\nu,\beta)}(1)\, P_n^{(\alpha+\nu,\beta)}(\cos\Theta).$$

Thus

$$|T_\varepsilon(\cos\Theta)| = \mathcal{O}\Big(\sum_{n=1}^{\infty} \varepsilon\, e^{-\frac{\varepsilon}{2}n}\, n^{-\nu-j+1}\, n^{-\nu+j+1}\, |a(n)| n^{\alpha+\nu}\, |P_n^{(\alpha+\nu,\beta)}(\cos\Theta)|\Big)$$

$$= \mathcal{O}\Big(\sum_{n=1}^{\infty} \varepsilon\, e^{-\frac{\varepsilon}{2}n}\, n^{-\nu+2+\alpha}\, |a(n)|\, |P_n^{(\alpha+\nu,\beta)}(\cos\Theta)|\Big).$$

Since $a(n)$ is bounded and (2.7) holds we have

$$\int_0^\pi |T_\varepsilon(\cos\Theta)|\, \rho^{(\alpha,\beta)}(\Theta)d\Theta = \mathcal{O}\Big(\varepsilon \sum_{n=1}^{\infty} \exp(-\frac{\varepsilon}{2}n)\Big) = \mathcal{O}(1).$$

Thus we may conclude that $F_\varepsilon(\cos\Theta)\, \rho^{(\alpha,\beta)}(\Theta)$ converges weakly to a measure μ as $\varepsilon \to 0^+$. Moreover $F_\varepsilon(\cos\Theta)\, \rho^{(\alpha,\beta)}(\Theta)$ converges uniformly on any compact subinterval of $(0,\pi]$. This implies that the singular part of μ is concentrated at 0 and therefore is a δ-function at 0.
We wish to show that μ is actually absolutely continuous, that is μ has no singular part. Let $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous and μ_s a δ-function at 0.
We have

$$\Big|\int_0^\pi P_n^{(\alpha,\beta)}(\cos\Theta)\, d\mu_a\Big| \leq \int_0^{\frac{\varepsilon}{n}} |P_n^{(\alpha,\beta)}(\cos\Theta)|\, |d\mu_a| +$$

$$+ \int_{\frac{\varepsilon}{n}}^{\pi} |P_n^{(\alpha,\beta)}(\cos\Theta)|\, |d\mu_a| = o(n^\alpha).$$

From (1.1) it follows that if μ_s is not zero,

$$\int_0^\pi P_n^{(\alpha,\beta)}(\cos\Theta)\, d\mu_s \quad \text{is not } o(n^\alpha) \text{ as } n \to \infty.$$

On the other hand

$$\int_0^\pi P_n^{(\alpha,\beta)}(\cos\Theta)\,d\mu = \lim_{\varepsilon\to 0^+}\int_0^\pi P_n^{(\alpha,\beta)}(\cos\Theta)\,F_\varepsilon(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)d\Theta$$
$$= a(n)\,P_n^{(\alpha,\beta)}(1) \;=\; o(n^\alpha).$$

This is a contradiction so μ_s is zero.

We let $H(\Theta)$ be the derivative of μ and take $F(\cos\Theta) = H(\Theta)\,\{\rho^{(\alpha,\beta)}(\Theta)\}^{-1}$. Since $H(\Theta)$ is in L_1, $F(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)$ is in L_1. Also $F_\varepsilon(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)$ tends to $F(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)$ weakly. Therefore

$$a(n) = \int_0^\pi \frac{P_n^{(\alpha,\beta)}(\cos\Theta)}{P_n^{(\alpha,\beta)}(1)}\,F(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta.$$

This finishes the proof.

3. Elementary properties of slowly varying functions.

We begin with two definitions.

Definition 3.1. We shall say that a function $b(t)$ is slowly varying if $b(t)$ satisfies the following three conditions:

i) $b(t)$ is in $C^{\infty}(0,\infty)$.

ii) For any $\delta > 0$, there is a $t_0 > 0$ such that $t^{\delta}\ |b(t)|$ is increasing for $t > t_0$.

iii) For any $\delta > 0$, there is a $t_1 > 0$ such that $t^{-\delta}|b(t)|$ is decreasing for $t > t_1$.

We shall need a more restricted class of functions S. We shall use the following notation:

$$h_0(t) = b(t),$$

$$h_n(t) = t\,\frac{d}{dt}\,h_{n-1}(t) \qquad n = 1,\ 2,\ 3,\ \dots\ .$$

We use $h_n(t)$ in the following definition.

Definition 3.2. A slowly varying function $b(t)$ is said to belong to the class S if all its associated $h_n(t)$ are slowly varying.

Common examples of functions of the class S are

$\log^{a}(t+10)$, $\log\log^{a}(t+100)$ and $\log^{a}(t+10)\log\log^{c}(t+100)$

(a and c are arbitrary numbers). A large class of slowly varying functions using Hardy's L-functions is given in the appendix to Wainger [13].

We shall give some simple properties of slowly varying functions. They can also be found in [4], [13] and [14], but they are so easy that we include the proofs.

Lemma 3.1.

Let $b(t)$ be slowly varying. Then

i) $b(t)$ is either non-positive or non-negative for sufficiently large t.

ii) $|b'(t)| = o(t^{-1}|b(t)|)$ as $t \to \infty$.

Proof.

i) There is a t_0 such that $t\,|b(t)|$ is increasing for $t \geq t_0$; therefore $t\,b(t)$ must be either non-positive or non-negative for $t \geq t_0$. Thus $b(t)$ must be either non-positive or non-negative for $t \geq t_0$. This proves i).

ii) Let δ be any positive number. It suffices to exhibit a $t_2(\delta)$ such that

$$(3.1) \qquad |b'(t)| < \delta\, t^{-1}\, |b(t)|$$

for $t \geq t_2(\delta)$. By part i) of this lemma, we may assume with no loss of generality that $b(t) \geq 0$. Then by definition of slowly varying, there is a $t_0(\delta)$ such that $t^\delta\, b(t)$ is increasing for $t \geq t_0(\delta)$. Therefore, $\frac{d}{dt}\{t^\delta\, b(t)\} \geq 0$ for $t \geq t_0(\delta)$. Hence, for $t \geq t_0(\delta)$, $\delta\, t^{\delta-1}\, b(t) + t^\delta\, b'(t) \geq 0$, and we see

$$(3.2) \qquad b'(t) \geq -\,\delta t^{-1}\, b(t) \qquad t \geq t_0(\delta).$$

Similarly, differentiation of $t^{-\delta}\, b(t)$ shows

$$(3.3) \qquad b'(t) \leq \delta t^{-1}\, b(t), \qquad t \geq t_1(\delta).$$

(3.2) and (3.3) imply (3.1) with $t_2 = \max\,(t_0(\delta), t_1(\delta))$. This completes the proof.

<u>Lemma 3.2</u>.

Let $b(t)$ be a slowly varying function. Let ξ_2 and ξ_3 be positive numbers with $\xi_2 < \xi_3$. Then

$$\max_{\xi_2/R \le t \le \xi_3/R} |b(t) - b(1/R)| = o(|b(1/R)|)$$

as $R \to 0$.

<u>Proof</u>.

We may assume without loss of generality that $\xi_2 < 1 < \xi_3$.
Suppose $\xi_2/R \le t \le 1/R$. Let $\delta > 0$. Then by lemma 3.1, there is an R_1, such that $|b'(t)| \le \delta\, t^{-1}\, |b(t)|$ for $t \ge \xi_2/R_1$, and such that $t^{\frac{1}{2}}\, |b(t)|$ is increasing for $t \ge \xi_2/R_1$. Thus for $R < R_1$,

$$\begin{aligned} |b(t) - b(1/R)| &= \left|\int_t^{1/R} b'(t)\, dt\right| \\ &\le \delta \int_{\xi_2/R}^{1/R} t^{-1}\, |b(t)|\, dt \\ &\le \delta \int_{\xi_2/R}^{1/R} t^{-\frac{3}{2}}\, t^{\frac{1}{2}}\, |b(t)|\, dt \\ &\le \delta\, R^{-\frac{1}{2}}\, |b(1/R)| \int_{\xi_2/R}^{1/R} t^{-\frac{3}{2}}\, dt \\ &\le 2\delta\, \xi_2^{-\frac{1}{2}}\, |b(1/R)|. \end{aligned}$$

A similar argument shows that there exists an R_2 such that if $R < R_2$, and $1/R \le t \le \xi_3/R$,

$$|b(t) - b(1/R)| \le 2\delta\, \xi_3^{\frac{1}{2}}\, |b(1/R)|.$$

Since δ is an arbitrary positive number, this completes the proof.

Lemma 3.3.

Let $b(t)$ be in the class S. Then for $n \geq 1$

$$\frac{d^n b(t)}{dt^n} = t^{-n} \sum_j \beta_j h_j(t).$$

The β_j are some numbers. The $h_j(t)$ are the slowly varying functions associated with $b(t)$. The sum is extended over a finite range of summation, and the value $j = 0$ does not occur.

Proof.

The proof is by induction. For $n = 1$, the statement is obvious since $t\, b'(t)$ is $h_1(t)$. Suppose the lemma is true for $n = k$; then

$$\frac{d^{k+1}}{dt^{k+1}} b(t) = \frac{d}{dt} \{t^{-k}\left[t^k \frac{d^k b(t)}{dt^k}\right]\}$$

$$= - k\, t^{-k-1}\left[t^k \frac{d^k b(t)}{dt^k}\right] + t^{-k-1}\left[t\frac{d}{dt} (t^k \frac{d^k}{dt^k} b(t))\right].$$

Now the conclusion for $n = k + 1$ follows easily from the inductive hypothesis and the definition of the h_j's.

Lemma 3.4.

Let $b(t)$ be in the class S and let $b(t) \to 0$ as $t \to \infty$. Then $\sum_{n=1}^{\infty} |b'(n)| < \infty$.

Proof.

By lemma 3.1, there exists a number N, such that $b(t)$, $b'(t)$ and $b''(t)$ are of constant sign if $t \geq N$.

We may assume without loss of the generality that $b(t) \geq 0$ for $t \geq N$. $b'(t)$ is of constant sign it follows that $b(t)$ tends to zero monotonically for $t \geq N$. From the fact that $t\,|b'(t)| = o(b(t))$, see lemma 3.1, it follows $b'(t)$ tends to zero and because $b''(t)$ has a constant sign for $t \geq N$, $b'(t)$ tends monotonically to zero for $t \geq N$.

But then

$$\sum_{n=N}^{M} b'(n) \leq \sum_{n=N}^{M} \int_{n}^{n+1} b'(u)\, du = -b(N) + b(M) \to 0$$

as $M,N \to \infty$.

Lemma 3.5.

Let $b(t)$ be slowly varying.

i) $|b(kt)| \sim |b(t)|$ for every fixed $k > 0$ and uniformly in every interval $\eta \leq k \leq \frac{1}{\eta}$, $0 < \eta < 1$;

ii) If we write $B(t) = \int_{1}^{t} \tau^{-1} |b(\tau)|\, d\tau$, $B^{*}(t) = \sum_{n=1}^{[t]} n^{-1} |b(n)|$, as $t \to \infty$ and $B(t) \neq \mathcal{O}(1)$ then

$$|b(t)| = o(B(t)) \quad \text{and} \quad B(t) \sim B^{*}(t).$$

Proof.

i) If $1 \leq k \leq 1/\eta$, then

$\frac{|b(kt)|}{(kt)^{\delta}} \leq \frac{|b(t)|}{t^{\delta}}$, $|b(kt)| \leq k^{\delta} |b(t)| \leq \eta^{-\delta} |b(t)|$ for sufficiently large t. Similarly $|b(kt)| \geq \eta^{\delta} |b(t)|$. Making δ arbitrarily small, this proves statement i) for $1 \leq k \leq 1/\eta$. The case $\eta \leq k \leq 1$ is proved in an analogous way.

ii) Let $k > 1$. For large t, using i) we obtain

$$B(t) > \int_{t/k}^{t} \tau^{-1} |b(\tau)|\, d\tau \sim |b(t)| \int_{t/k}^{t} \tau^{-1}\, d\tau = |b(t)| \log k.$$

Taking k large we obtain $|b(t)| = o(B(t)$.

Since $t^{-1} |b(t)|$ is ultimately decreasing we have for large k

$$\frac{|b(k)|}{k} \leq B(k) - B(k-1) \leq \frac{|b(k-1)|}{(k-1)} \quad \text{which implies}$$

$$0 \leq B(k) - B(k-1) - \frac{|b(k)|}{k} \leq \frac{|b(k-1)|}{k-1} - \frac{|b(k)|}{k}.$$

Since $\sum \{\frac{|b(k-1)|}{k-1} - \frac{|b(k)|}{k}\}$ converges, so does the series

$\sum \{B(k) - B(k-1) - \frac{|b(k)|}{k}\}$; but the n th partial sum is

$B(n) - B^{*}(n)$ + constant, which proves ii).

4. Behaviour of a special class of Jacobi series.

The main goal in this section is to study the behaviour near $\Theta = 0$ of a Jacobi series of the form

$$\sum_{n=1}^{\infty} b(n)\, n^{-\gamma}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

with $\gamma \geq 0$. $b(n)$ is a slowly varying function. (See section 3). Theorem 4.1 treats the case $0 < \gamma < 2\alpha + 2$. In theorem 4.2 we shall investigate the case $\gamma = 0$. Finally, theorem 4.3. will deal with the case $\gamma \geq 2\alpha + 2$. We need the following lemma

Lemma 4.1.
Let $\gamma > \alpha + \frac{1}{2}$.
For $0 < \Theta < \pi$

$$F(\cos\Theta) = \lim_{N\to\infty} \sum_{n=1}^{N} n^{-\gamma}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

exists. Also $F(\cos\Theta)$ is continuous for $0 < \Theta < \pi$.
If $\gamma > 2\alpha + 2$, $F(\cos\Theta)$ is continuous for $0 \leq \Theta \leq \pi$.
At $\Theta = 0$

$$F(\cos\Theta) = \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)} (\sin\frac{\Theta}{2})^{\gamma-2\alpha-2} + E(\cos\Theta).$$

$$E(\cos\Theta) = \mathcal{O}\{(\sin\frac{\Theta}{2})^{\gamma-2\alpha-1}\} \text{ if } \alpha + \tfrac{1}{2} < \gamma \leq 2\alpha + 1;$$

if $\gamma > 2\alpha + 1$ $E(\cos\Theta)$ is continuous and has the form

$$E(\cos\Theta) = A + \mathcal{O}\{(\sin\frac{\Theta}{2})\}^{\gamma-2\alpha-1}$$

Proof.

As is easily derived from the formula (Erdélyi [6] (10.20.(3)))

$$(4.1)\qquad \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)}\,(\sin\frac{\Theta}{2})^{\gamma-2\alpha-2} =$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1-\frac{\gamma}{2})}{\Gamma(n+\alpha+1)\Gamma(n+\beta+\frac{\gamma}{2}+1)}\,\omega_n^{(\alpha,\beta)}\,P_n^{(\alpha,\beta)}(1)\,P_n^{(\alpha,\beta)}(\cos\Theta).$$

$$(\gamma>\alpha+\tfrac{1}{2})$$

For any positive j there exist λ_j such that

$$\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1-\frac{\gamma}{2})}{\Gamma(n+\alpha+1)\Gamma(n+\beta+\frac{\gamma}{2}+1)} = \frac{1}{n^{\gamma}} +$$

$$+ \sum_{l=1}^{j} \lambda_l \frac{\Gamma(n+\beta+1)\Gamma(n+\alpha-\frac{\gamma+l}{2}+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+\frac{\gamma+l}{2}+1)} + \mathcal{O}(n^{-\gamma-j-1}).$$

If we choose j sufficiently large, lemma 4.1. follows immediately from (4.1) and (2.6).

Lemma 4.2.

Let ω and Ω be fixed.

Then

i) If $\gamma < 2\alpha + 2$

$$(4.2)\qquad \sum_{n=1}^{[\omega\Theta^{-1}]} n^{-\gamma}\,\omega_n^{(\alpha,\beta)}\,P_n^{(\alpha,\beta)}(1)\,|P_n^{(\alpha,\beta)}(\cos\Theta)| = \mathcal{O}\{(\Theta^{-1}\omega)^{2\alpha+2-\gamma}\}$$

ii) If Θ tends to 0^+ and if $\gamma > \alpha + \frac{3}{2}$

$$(4.3)\qquad \sum_{n=[\Omega\Theta^{-1}]}^{\infty} n^{-\gamma}\omega_n^{(\alpha,\beta)}\,P_n^{(\alpha,\beta)}(1)\,|P_n^{(\alpha,\beta)}(\cos\Theta)| =$$

$$= \mathcal{O}\{\Theta^{-(2\alpha+2-\gamma)}\,\Omega^{-\gamma+\alpha+\frac{3}{2}}\}.$$

The $\mathcal{O}$'s do not depend on ω or Ω.

Proof.

(4.2) follows from (2.6) by application of

$$\sum_{n=1}^{N} n^p = \mathcal{O}(N^{p+1}) \qquad p > -1.$$

(4.3) follows from (2.5) by using

$$\sum_{n=N}^{\infty} n^p = \mathcal{O}(N^{p+1}) \text{ if } p < -1.$$

Lemma 4.3.

i) Let $\alpha + \frac{3}{2} < \gamma < 2\alpha + 2$, assume $b(t)$ is slowly varying and let $\omega \leq 1 \leq \Omega$. Choose $\delta < \min(\gamma-\alpha-\frac{3}{2}, 2\alpha+2-\gamma)$.

Then

$$(4.4) \qquad \sum_{n=1}^{[\omega\Theta^{-1}]} n^{-\gamma} |b(n)| \, \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1) \, |P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$= \mathcal{O}(1) + \mathcal{O}\{|b(\Theta^{-1})| \, \Theta^{\gamma-2\alpha-2} \, \omega^{2\alpha+2-\gamma-\delta}\}$$

ii) Let $\gamma > \alpha + \frac{3}{2}$ and let $\delta < \gamma - \alpha - \frac{3}{2}$.

As $\Theta \to 0^+$

$$(4.5) \qquad \sum_{n=[\Omega\Theta^{-1}]}^{\infty} n^{-\gamma} |b(n)| \, \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1) \, |P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$= \mathcal{O}\{|b(\Theta^{-1})| \, \Theta^{\gamma-2\alpha-2} \, \Omega^{-\gamma+\alpha+\frac{3}{2}+\delta}\}.$$

The $\mathcal{O}$'s do not depend on ω and Ω.

Proof.

i) Choose $\delta < \min(\gamma-\alpha-\frac{3}{2}, 2\alpha+2-\gamma)$ and let m be an integer so large that $t^{\delta} |b(t)|$ is increasing for $t \geq m$ and $t^{-\delta} |b(t)|$ is decreasing for $t \geq m$. (Such m exists, see section 3). Then, using (4.2),

$$\sum_{n=1}^{[\omega\Theta^{-1}]} n^{-\gamma} \;|b(n)|\; \omega_n^{(\alpha,\beta)} \; P_n^{(\alpha,\beta)}(1) \; |P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$= \mathcal{O}(1) + \sum_{n=m}^{[\omega\Theta^{-1}]} n^{-\gamma-\delta} \; n^{\delta}|b(n)|\omega_n^{(\alpha,\beta)} \; P_n^{(\alpha,\beta)}(1)|P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$\leq \mathcal{O}(1) + \Theta^{-\delta}|b(\Theta^{-1})| \sum_{n=m}^{\omega\Theta^{-1}} n^{-\gamma-\delta} \; \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1)|P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$= \mathcal{O}(1) + \mathcal{O}\{|b(\Theta^{-1})|\Theta^{\gamma-2\alpha-2} \; \omega^{2\alpha+2-\gamma-\delta}\}$$

ii) Let Θ be so close to 0 that $[\Omega\Theta^{-1}] \geq m$. Then application of (4.3) leads to

$$\sum_{n=[\Omega\Theta^{-1}]}^{\infty} n^{-\gamma}|b(n)| \; \omega_n^{(\alpha,\beta)} \; P_n^{(\alpha,\beta)}(1) \; |P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$= \sum_{n=[\Omega\Theta^{-1}]}^{\infty} n^{-\gamma+\delta} \; n^{-\delta} \; |b(n)| \; P_n^{(\alpha,\beta)}(1) \; |P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$\leq \mathcal{O}\{|b(\Theta^{-1})| \; \Theta^{\gamma-2\alpha-2} \; \Omega^{-\gamma+\alpha+\frac{3}{2}+\delta}\}.$$

<u>Lemma 4.4.</u>

Assume $b(t)$ is slowly varying. Let

$$F(\cos\Theta) = \sum_{n=1}^{\infty} n^{-\gamma} \; b(n) \; \omega_n^{(\alpha,\beta)} \; P_n^{(\alpha,\beta)}(1) \; P_n^{(\alpha,\beta)}(\cos\Theta) \tag{4.6}$$

with $\alpha + \frac{3}{2} < \gamma < 2\alpha + 2$.

The sum of (4.6) converges absolutely and uniformly in any compact subinterval of $(0,\pi)$.

As $\Theta \to 0^+$

$$F(\cos\Theta) = \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)} \; (\sin\frac{\Theta}{2})^{\gamma-2\alpha-2} \; b(\Theta^{-1}) + E(\Theta). \tag{4.7}$$

(4.8) $$E(\Theta) = o\{b(\Theta^{-1})\,\Theta^{\gamma-2\alpha-2}\} + \mathcal{O}(1).$$

Proof.

The fact that the series (4.6) converges uniformly and absolutely in any compact subinterval of $(0,\pi)$ follows from the estimate (2.5).
Now let ω and Ω be fixed (but arbitrary) numbers $\omega < 1 < \Omega$.
Then

$$F(\cos\Theta) = b(\Theta^{-1}) \sum_{n=1}^{\infty} n^{-\gamma}\omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta) + E(\Theta)$$

$$E(\Theta) = E_1(\Theta) + E_2(\Theta) + E_3(\Theta) + E_4(\Theta) + E_5(\Theta).$$

$$E_1(\Theta) = \sum_{n=[\omega\Theta^{-1}]}^{[\Omega\Theta^{-1}]} \{b(n)-b(\Theta^{-1})\} n^{-\gamma}\omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1) P_n^{(\alpha,\beta)}(\cos\Theta)$$

$$E_2(\Theta) = -\, b(\Theta^{-1}) \sum_{n=[\Omega\Theta^{-1}]}^{\infty} n^{-\gamma}\,\omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

$$E_3(\Theta) = -\, b(\Theta^{-1}) \sum_{n=1}^{[\omega\Theta^{-1}]} n^{-\gamma}\,\omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

$$E_4(\Theta) = \sum_{n=[\Omega\Theta^{-1}]}^{\infty} b(n)\, n^{-\gamma}\,\omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

$$E_5(\Theta) = \sum_{n=1}^{[\omega\Theta^{-1}]} b(n)\, n^{-\gamma}\omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

Now by lemma 4.1 it suffices to such that the terms E_1 to E_5 satisfy (4.8).

Consider first E_1. If we choose $\varepsilon > 0$, then by lemma 3.2 and lemma 4.2 it follows that

$$|E_1(\Theta)| \le \max_{\omega\Theta^{-1}\le n\le\Omega\Theta^{-1}} |b(n)-b(\Theta^{-1})|$$

$$\sum_{n=1}^{\Omega\Theta^{-1}} n^{-\gamma}\,\omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, |P_n^{(\alpha,\beta)}(\cos\Theta)|$$

$$\leq \varepsilon \; |b(\Theta^{-1})| \; \mathcal{O}(\Theta^{\gamma-2\alpha-2} \; \Omega^{2\alpha+2-\gamma})$$

$$= o\{|b(\Theta^{-1})| \; \Theta^{\gamma-2\alpha-2}\}.$$

E_2 and E_3 may be estimated by lemma 4.2, E_4 and E_5 by lemma 4.3. We observe that each of these terms is

$$\mathcal{O}\{|b(\Theta^{-1})| \; \Theta^{\gamma-2\alpha-2} \; (\Omega^{-\gamma+\alpha+\frac{3}{2}+\delta} + \omega^{2\alpha+2-\gamma-\delta}) + 1\}$$

where $\mathcal{O}$ is independent of ω, Ω and Θ and $\delta < \min(\gamma-\alpha-\frac{3}{2}, 2\alpha+2-\gamma)$. The desired conclusion now follows by taking ω sufficiently small and Ω sufficiently large.

Theorem 4.1.

Let $b(t)$ be in S and let $0 < \gamma < 2\alpha + 2$. For $\varepsilon > 0$ define

$$F_\varepsilon(\cos\Theta) = \sum_{n=1}^{\infty} b(n)\, n^{-\gamma}\, e^{-\varepsilon n}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta). \tag{4.9}$$

Then, for each closed subinterval of $(0,\pi)$, $F(\cos\Theta)$ converges uniformly in ε. For $\Theta \neq 0$, $F(\cos\Theta) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\cos\Theta)$ exists in the pointwise sense. Also, $F(\cos\Theta)$ is continuous for $0 < \Theta \leq \pi$. At $\Theta = 0$

$$F(\cos\Theta) = \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)}\, b(\Theta^{-1})\, (\sin\tfrac{\Theta}{2})^{\gamma-2\alpha-2} + E(\Theta) \tag{4.10}$$

$$E(\Theta) = o\{b(\Theta^{-1})\, \Theta^{\gamma-2\alpha-2}\} + \mathcal{O}(1).$$

Finally, $\int_0^{\pi} |F(\cos\Theta)|\, \rho^{(\alpha,\beta)}(\Theta)\, d\Theta < \infty$

and

$$F(\cos\Theta) \sim \sum_{n=1}^{\infty} b(n)\, n^{-\gamma}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta). \tag{4.11}$$

Proof.

We are going to apply lemma 2.2 with $a(t) = b(t)\, t^{-\gamma}\, e^{-\varepsilon t}$. We take the integer ν so large that $\gamma + \nu > \alpha + \frac{5}{2}$.

We obtain

$$F_\varepsilon(\cos\Theta) = M_\varepsilon(\Theta) + E_{\varepsilon,1}(\Theta) + E_{\varepsilon,2}(\Theta) + E_{\varepsilon,3}(\Theta).$$

The terms $M_\varepsilon(\Theta)$, $E_{\varepsilon,1}(\Theta)$ and $E_{\varepsilon,2}(\Theta)$ all come from the main term of equation (2.3) which in the present context is

$$\frac{1}{t}\frac{d}{dt}\{\frac{1}{t}\frac{d}{dt}\{\frac{1}{t}\frac{d}{dt}\{\ldots\frac{1}{t}\frac{d}{dt}\{b(t)\, t^{-\gamma}\, e^{-\varepsilon t}\}\}\}\}$$

The main term M_ε arises from taking derivatives only on powers of t. $E_{\varepsilon,1}$ consists of the remaining contribution of terms not involving derivatives of $e^{-\varepsilon t}$. $E_{\varepsilon,2}$ is made up by terms in which at least one derivative is taken on $e^{-\varepsilon t}$. $E_{\varepsilon,3}$ corresponds to the term $E_1(\cos\Theta)$ in equation (2.3).

$$M_\varepsilon(\Theta) = \frac{(-1)^\nu}{2^\nu}\,\frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty} b(n)\, e^{-\varepsilon n}\left[\frac{1}{t}\frac{d}{dt}\{\frac{1}{t}\frac{d}{dt}\{\ldots\frac{1}{t}\frac{d}{dt}(t^{-\gamma})\}\}\right]_{t=n}$$

$$\omega_n^{(\alpha+\nu,\beta)}\, P_n^{(\alpha+\nu,\beta)}(1)\, P_n^{(\alpha+\nu,\beta)}(\cos\Theta)$$

$$= \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)}\,\frac{\Gamma(\frac{\gamma}{2}+\nu)}{\Gamma(\frac{\gamma}{2})}$$

$$\sum_{n=1}^{\infty} b(n)\, e^{-\varepsilon n}\, n^{-\gamma-2\nu}\, \omega_n^{(\alpha+\nu,\beta)}\, P_n^{(\alpha+\nu,\beta)}(1)\, P_n^{(\alpha+\nu,\beta)}(\cos\Theta).$$

As $\gamma + \nu > \alpha + \frac{5}{2}$ it follows immediately from (2.5) that the series $M_\varepsilon(\Theta)$ converges uniformly in ε in any closed subinterval of $(0,\pi)$. $M_\varepsilon(\Theta)$ with $\varepsilon = 0$ is a sum of the type treated in lemma 4.4. Hence applying lemma 4.4 and using the regularity of Abel summability, we find that $M(\Theta) = \lim_{\varepsilon\to 0} M_\varepsilon(\Theta)$ exists and is continuous for $0 < \Theta < \pi$.

Moreover as $\Theta \to 0^+$

$$M(\Theta) = \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\frac{\gamma}{2}+\nu)}{\Gamma(\frac{\gamma}{2})} \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2}+\nu)\Gamma(\alpha+\nu+1)} (\sin\frac{\Theta}{2})^{\gamma-2\alpha-2} b(\Theta^{-1}) + E_4$$

$$= \frac{\Gamma(\alpha+1-\frac{\gamma}{2})}{\Gamma(\frac{\gamma}{2})\Gamma(\alpha+1)} b(\Theta^{-1}) (\sin\frac{\Theta}{2})^{\gamma-2\alpha-2} + E_4$$

$$E_4 = o\{b(\Theta^{-1})\ \Theta^{\gamma-2\alpha-2}\} + \mathcal{O}(1).$$

We now investigate $E_{\varepsilon,1}(\Theta)$.

$$E_{\varepsilon,1}(\Theta) = \sum_{n=1}^{\infty} e^{-\varepsilon n} \left(\sum_{j=1}^{\nu} c_j\, n^{-\gamma-2\nu+j} \left.\frac{d^j b(t)}{dt^j}\right|_{t=n} \right)$$

$$\omega_n^{(\alpha+\nu,\beta)} P_n^{(\alpha+\nu,\beta)}(1)\, P_n^{(\alpha+\nu,\beta)}(\cos\Theta)$$

where c_j are numbers, independent of n.

We now use lemma 3.3 and write $\left.\frac{d^j b(t)}{dt^j}\right|_{t=n} = n^{-j} \sum_{k=1}^{j} \beta_k\, h_k(n)$.

We obtain

$$E_{\varepsilon,1}(\Theta) = \sum_{j=1}^{\nu} \sum_{k=1}^{j} d(j,k) \sum_{n=1}^{\infty} e^{-\varepsilon n}\, n^{-\gamma-2\nu}\, h_k(n)$$

$$\omega_n^{(\alpha+\nu,\beta)} P_n^{(\alpha+\nu,\beta)}(1)\, P_n^{(\alpha+\nu,\beta)}(\cos\Theta)$$

$d_{j,k}$ are numbers.

Thus $E_{\varepsilon,1}(\Theta)$ consists of a finite lineair combination of series, which converge uniformly in ε in any closed subinterval of $(0,\pi)$. Moreover the functions $h_k(t)$ are slowly varying because $b(t)$ belongs to the class S. From lemma 3.1 and lemma 4.4 it is easily seen that

$E_1(\Theta) = \lim_{\varepsilon \to 0^+} E_{\varepsilon,1}(\Theta)$ exists and is easily continuous for $0 < \Theta < \pi$.

Also as $\Theta \to 0^+$

$$E_1(\Theta) = o\{b(\Theta^{-1})\ \Theta^{\gamma-2\alpha-2}\} + \mathcal{O}(1).$$

Next we consider $E_{\varepsilon,2}(\Theta)$.

$$E_{\varepsilon,2}(\Theta) = \sum_{j,l,m} g_{j,l,m} \sum_{n=1}^{\infty} e^{-\varepsilon n}\ \varepsilon^m\ n^{-\gamma-\nu-j}\ b^{(l)}(n)$$

$$\omega_n^{(\alpha+\nu,\beta)}\ P_n^{(\alpha+\nu,\beta)}(1)\ P_n^{(\alpha+\nu,\beta)}(\cos\Theta).$$

Here $g_{j,l,m}$ are numbers. The first summation is over nonnegative values of j, l, m with $m \geq 1$ and $j + l + m = \nu$. Application of (2.5) and the fact that $\varepsilon^{m-1}\ e^{-\varepsilon n} = \mathcal{O}(n^{-m+1})$ gives

$$E_{\varepsilon,2}(\Theta) = \varepsilon\ \Theta^{-\alpha-\nu-\frac{1}{2}}\ (\pi-\Theta)^{-\beta-\frac{1}{2}}\ \mathcal{O}\Big(\sum_{j,l,m}\ \sum_{n=1}^{\infty} n^{-\gamma-\nu-j-m+1+\alpha+\nu+\frac{1}{2}}|b^{(l)}(n)|\Big)$$

$$= \varepsilon\ \Theta^{-\alpha-\nu-\frac{1}{2}}\ (\pi-\Theta)^{-\beta-\frac{1}{2}}\ \mathcal{O}\Big(\sum_{n=1}^{\infty} n^{-\gamma-\nu+\frac{3}{2}+\alpha}\ |b(n)|\Big).$$

Thus $E_{\varepsilon,2}$ converges uniformly in ε in any closed subinterval of $(0,\pi)$. Now we see that for $0 < \Theta < \pi$, $E_{\varepsilon,2} \to 0$ as $\varepsilon \to 0$, since $\gamma + \nu > \alpha + \frac{5}{2}$ and $|b(n)| = \ (n^{\delta})$ for any $\delta > 0$.

Finally we consider $E_{\varepsilon,3}(\Theta)$. $E_{\varepsilon,3}(\Theta)$ contains terms similar to those of M_ε, $E_{\varepsilon,1}$, $E_{\varepsilon,2}$, except that here $m + j + l = \nu + 1$ instead of ν. Hence, if we apply to $E_{\varepsilon,3}(\Theta)$ reasoning similar to that of the previous terms, we find that $E_{\varepsilon,3}$ is a series which converges uniformly in ε in any subinterval of $0 < \Theta < \pi$. Also we find $E_3(\Theta) = \lim_{\varepsilon \to 0^+} E_{\varepsilon,3}(\Theta)$ exists and is continuous for $0 < \Theta < \pi$.

Furthermore as $\Theta \to 0^+$,

$$E_3(\Theta) = o\{b(\Theta^{-1})\ \Theta^{\gamma-2\alpha-2}\} + \mathcal{O}(1).$$

We now examine the behaviour of $F(\cos\Theta)$ near $\Theta = \pi$.
It suffices to show that $F_\varepsilon(\cos\Theta)$ converges uniformly for Θ sufficiently close to π. For $\Theta = \pi$ the convergence follows from the well-known relation

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\, P_n^{(\alpha,\beta)}(-x)$$

and theorem 7 of Wainger [13], with $x = \frac{\pi}{2}$.
We use the Bateman integral, (see Askey and Fitch [2], formula 3.4)

$$(1+x)^\beta\ \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(-1)} = \frac{\Gamma(\beta+1)}{\Gamma(\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \int_{-1}^{x} (1+y)^{-\frac{1}{2}}\ \frac{P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(y)}{P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(-1)}\ (x-y)^{\beta-\frac{1}{2}}\ dy$$

or, writing $x = 2u^2 - 1$, $y = 2z^2 - 1$,

$$\frac{u^{2\beta}}{\Gamma(n+\beta+1)}\ P_n^{(\alpha,\beta)}(2u^2-1) = \frac{2}{\Gamma(\beta+\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_0^u P_n^{(\alpha+\beta+\frac{1}{2},-\frac{1}{2})}(2z^2-1)(u^2-z^2)^{\beta-\frac{1}{2}}dz.$$

Thus, applying Szegő [12], (4.1.5),

$$P_n^{(\alpha,\beta)}(2u^2-1) = \frac{2u^{-2\beta}\ \Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+\frac{3}{2})\Gamma(2n+1)}{\Gamma(\beta+\frac{1}{2})\Gamma(n+\frac{1}{2})\ \Gamma(2n+\alpha+\beta+\frac{3}{2})\Gamma(n+1)}$$

$$\int_0^u P_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(z)\,(u^2-z^2)^{\beta-\frac{1}{2}}\ dz$$

$$= \mathcal{O}\{n^{\beta+\frac{1}{2}}\ u^{-2\beta} \int_0^u P_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(z)\,(u^2-z^2)^{\beta-\frac{1}{2}}\ dz\}.$$

We investigate $F_\varepsilon(\cos\Theta)$ near π. If we put $\cos\Theta = 2u^2 - 1$ we have to study u in the neighbourhood of 0.

$$F_\varepsilon(2u^2-1) = \mathcal{O}\{u^{-2\beta} \int_0^u (u^2-z^2)^{\beta-\frac{1}{2}} \left(\sum_{n=1}^{\infty} b(n)\ n^{-\gamma}\ e^{-\varepsilon n}\ \omega_n^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})} \right.$$

$$\left. P_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(1)\ P_{2n}^{(\alpha+\beta+\frac{1}{2},\alpha+\beta+\frac{1}{2})}(z)\right) dz\}.$$

In the first part of this theorem we have shown that the series in the integrand converges uniformly in ε in any closed subinterval of $(-1,1)$ and that its $\lim_{\varepsilon\to 0^+}$ exists and is continuous. Indeed, if $\sum a_n\ P_n^{(\alpha,\alpha)}(x)$ and $\sum a_n\ P_n^{(\alpha,\alpha)}(-x)$ are continuous functions for x near $x=0$, then so is their sum $\sum a_{2n}\ P_{2n}^{(\alpha,\alpha)}(x)$ which is a series of the kind used in the integrand. By the dominated convergence theorem $F_\varepsilon(2u^2-1)$ converges pointwise to a limit as $\varepsilon \to 0^+$, at least if u is sufficiently small.

Moreover

$$F(2u^2-1) = \mathcal{O}(u^{-2\beta} \int_0^u c(z)\ (u^2-z^2)^{\beta-\frac{1}{2}}\ dz)$$

where $c(z)$ is continuous near $z = 0$. And the convergence is uniform since

$$|u^{-2\beta} \int_0^u (u^2-z^2)^{\beta-\frac{1}{2}}\ dz| = \mathcal{O}(1)$$

is uniformly bounded near $u = 0$.
To finish the proof we need to show that

$$F(\cos\Theta) \sim \sum_{n=1}^{\infty} b(n)\ n^{-\gamma} \omega_n^{(\alpha,\beta)}\ P_n^{(\alpha,\beta)}(1)\ P_n^{(\alpha,\beta)}(\cos\Theta).$$

By theorem 2.1 the sum of the right is a Fourier-Jacobi series of a function $G(\cos\Theta)$. In section 1 we showed that this series is Abel summable to $G(\cos\Theta)$ almost everywhere.
So $G(\cos\Theta) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\cos\Theta) = F(\cos\Theta)$.

Theorem 4.2.

Let

$$F_\varepsilon(\cos\Theta) = \sum_{n=1}^{\infty} b(n)\, e^{-\varepsilon n}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

with $b(t)$ in S. Then $F(\cos\Theta) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\cos\Theta)$ exists in the pointwise sense for $\Theta \neq 0$. Moreover $F(\cos\Theta)$ is continuous for $\Theta \neq 0$.

At $\Theta = 0$

$$F(\cos\Theta) \simeq k\left(\sin\frac{\Theta}{2}\right)^{-2\alpha-3} b'(\Theta^{-1})$$

provided $b'(t)$ is not zero for all large t. $k \neq 0$.

Finally

$$\int_0^{\pi} |F(\cos\Theta)|\, \rho^{(\alpha,\beta)}(\Theta)\, d\Theta < \infty$$

if and only if $b(t)$ tends to 0 as $t \to \infty$. If $b(t)$ tends to zero

$$F(\cos\Theta) \sim \sum_{n=1}^{\infty} b(n)\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta).$$

Proof.

The proof of theorem 4.2. is essentially the same as the proof of theorem 4.1. As in theorem 4.1, the proof of the first part is reduced to lemma 4.4 by lemma 2.2, where we take $a(t) = e^{-\varepsilon t} b(t)$. The fact that $a(t)$ contains no power of t accounts for the different conclusion of theorem 4.1 and 4.2. For the second part of the theorem we apply theorem 2.1 which is possible in view of the lemmas 3.3 and 3.4.

Theorem 4.3.

Let $b(t)$ be in S and let $\gamma \geq 2\alpha + 2$. For $\varepsilon > 0$ define

$$F_\varepsilon(\cos \Theta) = \sum_{n=1}^{\infty} b(n)\, n^{-\gamma}\, e^{-\varepsilon n}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos \Theta).$$

Then

$F(\cos \Theta) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\cos \Theta)$ exists for $0 < \Theta \leq \pi$ and $F(\cos \Theta)$ is continuous in this interval.

Furthermore

$$\int_0^\pi |F(\cos \Theta)|\, \rho^{(\alpha,\beta)}(\Theta)\, d\Theta < \infty$$

and

$$F(\cos \Theta) \sim \sum_{n=1}^{\infty} b(n)\, n^{-\gamma}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos \Theta).$$

Let

$$B(y) = \int_1^y b(t)\, t^{-1}\, dt.$$

As $\Theta \to 0^+$

i) $F(\cos \Theta) = \dfrac{2}{\{\Gamma(\alpha+1)\}^2} B(\Theta^{-1}) + \mathcal{O}(|b(\Theta^{-1})|)$ if $\gamma = 2\alpha + 2$ and if $\int_1^\infty |b(t)|\, t^{-1}\, dt = \infty$.

ii) If $\gamma > 2\alpha + 2$ or if $\gamma = 2\alpha + 2$ and $\int_1^\infty |b(t)| t^{-1}\, dt < \infty$

$\lim_{\Theta\to 0^+} F(\cos \Theta)$ exists and thus $F(\cos \Theta)$ is continuous on $0 \leq \Theta \leq \pi$.

Proof.

Everything except i) and ii) follows as in the proof of theorem 4.1. The proof of ii) is trivial since the hypothesis implies that

$$\sum_{n=1}^{\infty} b(n)\, n^{-\gamma}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

converges uniformly in view of (2.6).

So we only need to prove i). By lemma 4.3, equation (4.5),

$$\sum_{n=[\Theta^{-1}]}^{\infty} n^{-2\alpha-2}\, |b(n)|\, \omega_n^{(\alpha,\beta)}\, |P_n^{(\alpha,\beta)}(\cos\Theta)|\, P_n^{(\alpha,\beta)}(1) =$$

$$= \mathcal{O}\{b(\Theta^{-1})\}.$$

Now

$$\sum_{n=1}^{[\Theta^{-1}]} n^{-2\alpha-2}\, b(n)\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(\cos\Theta)\, P_n^{(\alpha,\beta)}(1) =$$

$$= A_1(\Theta) + A_2(\Theta)$$

where

$$A_1(\Theta) = \sum_{n=1}^{[\Theta^{-1}]} n^{-2\alpha-2}\, b(n)\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(1)$$

and

$$|A_2(\Theta)| = \sum_{n=1}^{[\Theta^{-1}]} n^{-2\alpha-2}\, |b(n)|\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, |P_n^{(\alpha,\beta)}(1) - P_n^{(\alpha,\beta)}(\cos\Theta)|.$$

We consider first $A_2(\Theta)$.

$$|P_n^{(\alpha,\beta)}(1) - P_n^{(\alpha,\beta)}(\cos\Theta)| = \mathcal{O}\Big((1-\cos\Theta) \max_{-1\leq x\leq 1} \Big|\frac{d}{dx} P_n^{(\alpha,\beta)}(x)\Big|\Big).$$

By Szegö [12], 7.32.10 we have

$$\left|\frac{d}{dx} P_n^{(\alpha,\beta)}(x)\right| = \mathcal{O}(n^{\alpha+2}).$$

So

$$|A_2(\Theta)| = \mathcal{O}\{\Theta^2 \sum_{n=1}^{[\Theta^{-1}]} n\, |b(n)|\} = \mathcal{O}\{|b(\Theta^{-1})|\}.$$

We now examine $A_1(\Theta)$.

$$A_1(\Theta) = \frac{2}{\{\Gamma(\alpha+1)\}^2} \sum_{n=1}^{[\Theta^{-1}]} n^{-1}\, b(n) + \mathcal{O}(1)$$

Hence we have

$$F(\cos\Theta) = \frac{2}{\{\Gamma(\alpha+1)\}^2} \sum_{n=1}^{[\Theta^{-1}]} n^{-1}\, b(n) + \mathcal{O}\{|b(\Theta^{-1})|\}.$$

Now according to lemma 3.5, $|b(t)| = o(B(t))$ as $t \to \infty$ and $B(t) \simeq \sum_{n=1}^{[t]} b(n)\, n^{-1}$ which gives us the proof of i).

<u>Remark 4.1.</u>
Theorem 1 and theorem 3 yield more information in the special case $b(t) = 1$. Let

$$F(\cos\Theta) \sim \sum_{n=1}^{\infty} n^{-\gamma}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

with $\gamma > 0$.
Then

$$(4.12) \qquad F(\cos\Theta) = \sum_{j=0}^{k} \beta_j \left(\sin\frac{\Theta}{2}\right)^{\gamma-2\alpha-2+j} + \mu \log|\Theta^{-1}| + E(\Theta).$$

$E(\Theta)$ is at least $\mathcal{O}(1)$ and has at least $\gamma - 2\alpha - 2 + k$ continuous derivatives. The β_j and μ are numbers. μ is zero unless $\gamma + j = 2\alpha + 2$ for some integer j, $0 \le j \le k$.

Remark 4.2.

An important part of theorem 4.1 goes through if $\gamma < 0$ ($\gamma \neq 2k, k=0,1,2,\dots$). When $F_\varepsilon(\cos\Theta)$ is defined by (4.9) with $\gamma < 0$, then for $\Theta \neq 0$ $F(\cos\Theta) = \lim_{\varepsilon\to 0^+} F_\varepsilon(\cos\Theta)$ still exists and is continuous for $0 < \Theta \leq \pi$.
At $\Theta = 0$ (4.10) holds.
In this case the series (4.10) does not satisfy the conditions of theorem 2.1 and therefore we cannot conclude that it is the Fourier-Jacobi series of $F(\cos\Theta)$.
Although $F(\cos\Theta)$ is no longer a function in $L_1(0,\pi)$ with respect to $\rho^{(\alpha,\beta)}(\Theta)$ but a distribution, we are still able to convolve the function $F(\cos\Theta)$ with another function $G(\cos\Theta)$ whenever $G(\cos\Theta)$ is sufficiently smooth near to the origin.

5. Fractional integration.

Let $f(\cos\Theta)$ be a function in $L_1(0,\pi)$ with respect to $\rho^{(\alpha,\beta)}(\Theta)$ defined by

$$(5.1)\qquad f(\cos\Theta) = \sum_{n=0}^{\infty} a_n\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta).$$

In section 1 we introduced the differential operator A_Θ and defined

$$A_\Theta\, f(\cos\Theta) = a_0 + \sum_{n=1}^{\infty} a_n\, n(n+\alpha+\beta+1)\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta).$$

We now introduce the inverse operator I_2 given by

$$I_2\, f(\cos\Theta) = a_0 + \sum_{n=1}^{\infty} a_n [n(n+\alpha+\beta+1)]^{-1}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

such that $A_\Theta\, I_2\, f(\cos\Theta) = f(\cos\Theta)$, or by (1.4),

$$(5.2)\qquad I_2\, f(\cos\Theta) = \int_0^\Theta \frac{d\phi}{\rho^{(\alpha,\beta)}(\phi)} \int_0^\phi f(\cos t)\, \rho^{(\alpha,\beta)}(t)\, dt + c.$$

If we have

$$(5.3)\qquad g_\sigma(\cos\Theta) = 1 + \sum_{n=1}^{\infty} [n(n+\alpha+\beta+1)]^{-\frac{\sigma}{2}}\, \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta)$$

$I_2\, f(\cos\Theta)$ can be generalized to the fractional integral $I_\sigma\, f(\cos\Theta)$ by taking the convolution of $f(\cos\Theta)$ and $g_\sigma(\cos\Theta)$ which is

$$(5.4)\qquad I_\sigma\, f(\cos\Theta) = \int_0^\pi f(\cos\Theta,\cos\phi)\, g_\sigma(\cos\phi)\, \rho^{(\alpha,\beta)}(\phi)\, d\phi.$$

where $f(\cos\Theta,\cos\phi) = \sum_{n=1}^{\infty} a_n\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(\cos\Theta)\, P_n^{(\alpha,\beta)}(\cos\phi)$
(see section 1).
It follows that

$$(5.5)\qquad I_\sigma\, f(\cos\Theta) = a_0 + \sum_{n=1}^{\infty} a_n [n(n+\alpha+\beta+1)]^{-\frac{\sigma}{2}}\, \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta).$$

It is clear that this fractional integration satisfies the semi-group property

$$(5.6) \qquad I_{\sigma_1}(I_{\sigma_2} f(\cos\Theta)) = I_{\sigma_1+\sigma_2} f(\cos\Theta).$$

Many of the classical theorems for fractional integration (see Zygmund [14], ch XII) can be carried over. This will be done in this section. We first introduce Lipschitz classes.

Definition 5.1.

Let f be in $L_\infty(0,\pi)$.

For $0 < \tau \leq 2$ we define f to be in Lip τ if

$$||f(\cos\Theta,\cos\phi) - f(\cos\Theta)||_\infty < A(1-\cos\phi)^{\frac{\tau}{2}} = \mathcal{O}(\phi^\tau).$$

For $\tau > 2$ we can write $\tau = 2k + \tau_1$ (k integer ≥ 1, $0 < \tau_1 \leq 2$) and we say that f is in Lip τ if the k times repeated application on f of the differential operator A_Θ leads to a function in Lip τ_1.

Theorem 5.1.

Let $0 < \sigma < 2$, $0 < \tau < 2$ and suppose $f \in$ Lip τ.

Then $I_\sigma f(\cos\Theta) \in \text{Lip}(\sigma+\tau)$ if $\sigma + \tau < 2$.

Proof.

We need the following inequalities

$$(5.7) \qquad |g_\sigma(\cos\Theta)| = \mathcal{O}(\Theta^{\sigma-2\alpha-2}) \quad 0 < \sigma < 2\alpha + 2$$

where $g_\sigma(\cos\Theta)$ is defined by (5.3). This estimate can be derived from theorem 4.1 noticing that

$$[n(n+\alpha+\beta+1)]^{-\frac{\sigma}{2}} = n^{-\sigma} + \sum_{j=1}^{[2\alpha+2-\sigma]} c_j\, n^{-\sigma-j} + o(n^{-(2\alpha+2)})$$

for certain numbers c_j. (5.7) follows applying (4.12) and (2.6). It is clear that (5.7) can only be used for values of σ less than $2\alpha + 2$. However, we can come beyond this value by breaking up σ in parts $\sigma = \sigma_1 + \sigma_2 + \dots + \sigma_k$ ($\sigma_j < 2\alpha + 2$, $1 \leq j \leq k$) and applying (5.6).

(5.8) $$|g_\sigma(\cos\Theta,\cos\phi) - g_\sigma(\cos\Theta)| \leq C\,\phi^2 |g_{\sigma-2}(\cos\Theta)| = C_1\,\phi^2\,\Theta^{\sigma-2\alpha-4}.$$

This follows from (1.14) and remark 4.2.
We can now go on with the proof and follow Zygmund [14] II, p 136.

Suppose $f \in \mathrm{Lip}\,\tau$ and $0 < \phi \leq \frac{\pi}{2}$.

$$I_\sigma f(\cos t) = \int_0^\pi f(\cos\Theta,\cos t)\, g_\sigma(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta$$

$$= \int_0^\pi \{f(\cos\Theta,\cos t) - f(\cos t)\}\, g_\sigma(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta$$

$$I_\sigma f(\cos t,\cos\phi) = \int_0^\pi f(\cos\Theta,\cos t,\cos\phi)\, g_\sigma(\cos\Theta)\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta$$

$$= \int_0^\pi (f(\cos\Theta,\cos t) - f(\cos t)\; g_\sigma(\cos\Theta,\cos\phi)\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta$$

Thus

(5.9)
$$I_\sigma f(\cos t,\cos\phi) - I_\sigma f(\cos t) = \int_0^\pi \{f(\cos\Theta,\cos t) - f(\cos t)\}\{g_\sigma(\cos\Theta,\cos\phi) - g_\sigma(\cos\Theta)\}\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta$$

$$= \int_0^\phi + \int_\phi^\pi = A + B.$$

$$|A| = \int_0^\phi \mathcal{O}(\Theta^\tau)\,\{|g_\sigma(\cos\Theta,\cos\phi)| + |g_\sigma(\cos\Theta)|\}\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta$$

$$= \int_0^\phi \mathcal{O}(\Theta^\tau)\,|g_\sigma(\cos\Theta)|\,\rho^{(\alpha,\beta)}(\Theta)\,d\Theta \quad \text{(this follows from (1.16))}$$

$$= \int_0^\phi \mathcal{O}(\Theta^\tau)\,\Theta^{\sigma-2\alpha-2}\,\Theta^{2\alpha+1}\,d\Theta = \mathcal{O}(\phi^{\sigma+\tau})$$

$$|B| = \int_{\phi}^{\pi} \Theta^{\tau} \; |g_{\sigma}(\cos\Theta,\cos\phi) - g_{\sigma}(\cos\Theta)| \; \rho^{(\alpha,\beta)}(\Theta)\; d\Theta$$

$$\leq \mathcal{O}(\phi^2) \int_{\phi}^{\infty} \Theta^{\tau+\sigma-2\alpha-4}\; \Theta^{2\alpha+1}\; d\Theta =$$

$$= \mathcal{O}(\phi^2) \int_{\phi}^{\infty} \Theta^{\sigma+\tau-3}\; d\Theta = \mathcal{O}(\phi^{\sigma+\tau})$$

Since $\sigma + \tau < 2$. Hence $A + B = \mathcal{O}(\phi^{\sigma+\tau})$ which proves theorem 5.1.

Remark 5.1.

Theorem 5.1 is valid for all positive values of σ and τ except the case that $\sigma + \tau$ = even integer.

Let $\sigma = 2k + \sigma_1$ and $\tau = 2l + \tau_1$ (k,l integer ≥ 0, $0 < \sigma_1 < 2$, $0 < \tau_1 < 2$). If $\sigma_1 + \tau_1 < 2$ we apply the differential operator A_Θ $k + l$ times in (5.9) and show that the result is $\mathcal{O}(\phi^{\sigma_1+\tau_1})$. If $\sigma_1 + \tau_1 > 2$ we apply the differential operator A_Θ $k + l + 1$ times in (5.9) and show that the result is $\mathcal{O}(\phi^{\sigma_1+\tau_1-2})$. If we had been working with another definition of Lipschitz spaces($\tau > 2$) using higher order differences, defined by

$$\Delta^k f(\cos\Theta) = \sum_{n=0}^{\infty} a_n\, \omega_n\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta) \left[\frac{P_n^{(\alpha,\beta)}(\cos\Theta)}{P_n^{(\alpha,\beta)}(1)} - 1\right]^k ,$$

instead of the differential operator A_Θ, we would not have to make an exception for $\sigma + \tau$ = even integer.

Theorem 5.2.

Suppose $f \in L_q$, $1 < q < \infty$.

If $\frac{2\alpha+2}{q} < \sigma < 2 + \frac{2\alpha+2}{q}$ then $I_\sigma f \in \mathrm{Lip}(\sigma - \frac{2\alpha+2}{q})$

Proof.

By Hölder's inequality

$$(5.10)\quad ||I_\sigma f(\cos\Theta,\cos\phi) - I_\sigma f(\cos\Theta)||_\infty =$$

$$= ||\int_0^{\pi} f(\cos\Theta,\cos t)\, \{g_\sigma(\cos t,\cos\phi) - g_\sigma(\cos t)\}\, \rho^{(\alpha,\beta)}(t)\, dt||_\infty$$

$$\leq \{\int_0^{\pi} |f(\cos t)|^q\, \rho^{(\alpha,\beta)}(t)dt\}^{\frac{1}{q}} \{\int_0^{\pi} |g_\sigma(\cos t,\cos\phi) - g_\sigma(\cos t)|^{q'}\, \rho^{(\alpha,\beta)}(t)dt\}^{\frac{1}{q'}}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.
We have to show that the last factor is $\mathcal{O}(\sigma - \frac{2\alpha+2}{q})$.
Using (5.7) and (5.8) we write

$$\int_0^\pi |g_\sigma(\cos t, \cos \phi) - g_\sigma(\cos t)|^{q'} \rho^{(\alpha,\beta)}(t)\, dt = \int_0^\phi + \int_\phi^\pi = A + B.$$

$$A \leq 2^{q'} \int_0^\phi |g_\sigma(\cos t)|^{q'} \rho^{(\alpha,\beta)}(t)\, dt = \int_0^\phi \mathcal{O}(t^{(\sigma-2\alpha-2)q'}) \rho^{(\alpha,\beta)}(t) dt$$

$$= \int_0^\phi \mathcal{O}(t^{(\sigma-2\alpha-2)q'+2\alpha+1}) dt = \mathcal{O}(\phi^{(\sigma-2\alpha-2)q'+2\alpha+2})$$

$$B \leq C\, \phi^{2q'} \int_\phi^\pi |g_{\sigma-2}(t)|^{q'} \rho^{(\alpha,\beta)}(t) dt =$$

$$= \mathcal{O}(\phi^{2q'}) \int_\phi^\infty t^{(\sigma-2\alpha-4)q'+2\alpha+1}\, dt = \mathcal{O}(\phi^{(\sigma-2\alpha-2)q'+2\alpha+2})$$

So the last factor of (5.10) is $\mathcal{O}(\phi^{\sigma-\frac{2\alpha+2}{q}})$. The inequalities $(\sigma - 2\alpha - 2)q' + 2\alpha + 1 > -1$ and $(\sigma - 2\alpha - 4)q' + 2\alpha + 1 < -1$, which we used in estimating A and B, are equivalent to the hypothesis $\frac{2\alpha+2}{q} < \sigma < 2 + \frac{2\alpha+2}{q}$.

<u>Theorem 5.3</u>.
If $q > 1$, $0 < \sigma < \frac{2\alpha+2}{q}$ and if $f \in L_q$, then $I_\sigma f$ is in L_r where $\frac{1}{r} = \frac{1}{q} - \frac{\sigma}{2\alpha+2}$

<u>Proof</u>.
This is a consequence of our theorem 4.1 and theorem 2.6 of O'Neil [11].
To use this theorem we need to calculate $g_\sigma^{**}(\cos \Theta)$.
We define the set $E_y = \{\Theta : |g_\sigma(\cos \Theta)| > y\}$ and define $g_\sigma^{*}(\cos \Theta)$ as the inverse function of $m(g_\sigma(\cos \Theta), y) = \text{meas}\,(E_y)$.
In view of (5.7) we have essentially $E_y = \{\Theta : \Theta > y^{1/(\sigma-2\alpha-2)}\}$ and

$$\text{meas}(E_y) = \int_{y^{1/(\sigma-2\alpha-2)}}^\pi (\sin \tfrac{\Theta}{2})^{2\alpha+1} (\cos \tfrac{\Theta}{2})^{2\beta+1}\, d\Theta = \mathcal{O}(y^{(2\alpha+2)/(\sigma-2\alpha-2)}).$$

So the inverse function $g_\sigma^*(\cos\Theta) = \mathcal{O}(\Theta^{(\sigma-2\alpha-2)/(2\alpha+2)})$ and

$$g_\sigma^{**}(\cos\Theta) = \frac{1}{\Theta}\int_0^\Theta g_\sigma^*(\cos\Theta)\,d\Theta = \mathcal{O}(\Theta^{(\sigma-2\alpha-2)/(2\alpha+2)}).$$

We use the norm $||g_\sigma(\cos\Theta)||_{p,\infty} = \sup_{x>0} \Theta^{\frac{1}{p}} g^{**}(\cos\Theta)$ and it follows that $g(\cos\Theta) \in L(\frac{2\alpha+2}{2\alpha+2-\sigma},\infty)$.

O'Neil's theorem 2.6 now states that if $f \in L(q.q) = L_q$ and $g_\sigma \in L(\frac{2\alpha+2}{2\alpha+2-\sigma},\infty)$ with the conditions $\frac{1}{q} + \frac{2\alpha+2-\sigma}{2\alpha+2} > 1$, then $I_\sigma\, f \in L(r,s)$ where $\frac{1}{r} = \frac{1}{q} - \frac{\sigma}{2\alpha+2}$ and any number $s \geq q$.

If we choose $s = r$ theorem 5.3 is proved.

We now define the fractional derivative of order σ by
$D_\sigma\, f(\cos\Theta) = A_\Theta \cdot I_{2-\sigma}\, f(\cos\Theta)$.

<u>Theorem 5.4</u>.
Let $0 < \sigma < \tau < 2$. Then $D_\sigma\, f \in \mathrm{Lip}(\tau-\sigma)$ if $f \in \mathrm{Lip}\,\tau$.

<u>Proof</u>.
We have to show that $D_\sigma\, f(\cos\Theta) = A_\Theta\, I_{2-\sigma}\, f(\cos\Theta)$ exists and is in $\mathrm{Lip}(\tau-\sigma)$. We write

$$I_{2-\sigma}\, f(\cos\Theta) = \int_0^\pi A_t\, \{I_2\, f(\cos\Theta,\cos t) - I_2\, f(\cos\Theta)\} g_{2-\sigma}(\cos t)\rho^{(\alpha,\beta)}(t)dt$$

$$= \int_0^\pi \{I_2\, f(\cos\Theta,\cos t) - I_2\, f(\cos\Theta)\}\, A_t\, g_{2-\sigma}(\cos t)\rho^{(\alpha,\beta)}(t)dt$$

because of the selfadjointness of the operator A_t.
Then

$$D_\sigma\, f(\cos\Theta) = A_\Theta\, I_{2-\sigma}\, f(\cos\Theta) =$$

$$= \int_0^\pi \{f(\cos\Theta,\cos t) - f(\cos\Theta)\}\, A_t\, g_{2-\sigma}(\cos t)\, \rho^{(\alpha,\beta)}(t)\, dt$$

exists, since the integral on the right converges absolutely and uniformly. We have

$$D_\sigma f(\cos\Theta,\cos\phi) - D_\sigma f(\cos\Theta) = \int_0^\pi \Delta(\Theta,t,\phi)\, A_t\, g_{2-\sigma}(\cos t)\, \rho^{(\alpha,\beta)}(t)dt$$

where $\Delta(\Theta,t,\phi) = f(\cos\Theta,\cos t,\cos\phi) - f(\cos\Theta,\cos\phi) - f(\cos\Theta,\cos t) + f(\cos\Theta)$. Clearly, by (1.16), $\Delta = \mathcal{O}(t^\tau)$ and regrouping terms we also find that $\Delta = \mathcal{O}(\phi^\tau)$. Applying these estimates and (1.15) combined with remark 4.2, we find

$$||D_\sigma f(\cos\Theta,\cos\phi) - D_\sigma f(\cos\Theta)||_\infty =$$

$$= \int_0^\phi \mathcal{O}(t^\tau)\, t^{-\sigma-2\alpha-2+2\alpha+1}\, dt + \phi^\tau \int_\phi^\pi \mathcal{O}(t^{-\sigma-2\alpha-2+2\alpha+1})\, dt = \mathcal{O}(\phi^{\tau-\sigma}).$$

Remark 5.2.

Theorem 5.4 is valid for all positive values of σ and τ with $0 < \sigma < \tau$ except the case $\tau - \sigma$ = even integer. This can be done by using A_Θ in the same way as mentioned in remark 5.1.

As an application we give sufficient conditions for $f(\cos\Theta)$ to have a uniformly convergent or an absolutely convergent Fourier-Jacobi series. The partial sum $S_N(\cos\Theta)$ of the series (1.7) can be written as the convolution of $D_\sigma f(\cos\Theta)$ for some σ with a kernel $g_\sigma^N(\cos\Theta)$ where

$$g_\sigma^N(\cos\Theta) = 1 + \sum_{n=1}^{N} (n(n+\alpha+\beta+1))^{-\frac{\sigma}{2}}\, \omega_n^{(\alpha,\beta)}\, P_n^{(\alpha,\beta)}(1)\, P_n^{(\alpha,\beta)}(\cos\Theta).$$

If there exists a σ_1 such that $D_{\sigma_1} f(\cos\Theta)$ is continuous and $g_{\sigma_1}^N(\cos\Theta)$ is in $L_1(0,\pi)$ with respect to $\rho^{(\alpha,\beta)}(\Theta)$ it follows from (1.13) that

$$||S_N(\cos\Theta)||_\infty \leq A\, ||g_{\sigma_1}^N(\cos\Theta)||_1\, ||D_{\sigma_1} f(\cos\Theta)||_\infty$$

which implies that $f(\cos\Theta)$ has an uniformly convergent Fourier-Jacobi series. In order to find the behaviour of $|g_{\sigma_1}^N(\cos\Theta)|$, a straightforward calculation similar to those in section 4, using a summation by parts (lemma 2.1) and splitting up the sum in

$\sum_{n=1}^{[1/\Theta]}$ and $\sum_{n=[1/\Theta]+1}^{N}$, leads to the estimate

$$|g^N_{\sigma_1}(\cos\Theta) = \mathcal{O}(\Theta^{\sigma_1-2\alpha-2}(\pi-\Theta)^{\sigma_1-\alpha-\beta-1}) \text{ if } \sigma_1 > \alpha + \tfrac{1}{2},$$

and in this case $||g^N_{\sigma_1}(\cos\Theta)||_1 < \infty$.

From theorem 5.4 it follows that for certain $\sigma_1 > \alpha + \frac{1}{2}$ $D_{\sigma_1} f$ is still continuous if $f \in \mathrm{Lip}(\alpha+\frac{1}{2}+\varepsilon)$. Therefore if $f \in \mathrm{Lip}(\alpha+\frac{1}{2}+\varepsilon)$ $f(\cos\Theta)$ has a uniformly convergent Fourier-Jacobi series.
Let $g_\sigma(\cos\Theta) = \lim_{N\to\infty} g^N_\sigma(\cos\Theta)$ as in (5.3).

If there exists a σ_2 such that $D_{\sigma_2} f(\cos\Theta)$ is continuous and $g_{\sigma_2}(\cos\Theta)$ is in the weighted $L_2(0,\pi)$ it follows from the Canchy-Schwarz inequality, that $f(\cos\Theta)$ has an absolutely convergent Fourier-Jacobi series.
From (5.7)it follows that $g_{\sigma_2}(\cos\Theta)$ is in the weighted $L_2(0,\pi)$ if $\sigma_2 > \alpha + 1$. Therefore $f(\cos\Theta)$ has an absolutely convergent Fourier-Jacobi series if $f \in \mathrm{Lip}(\alpha+1+\varepsilon)$.
We have to mention that these results are not best possible, but almost best possible, whereas the proofs are very simple. Best possible results concerning uniform convergence are given by Agahanov and Natanson [1] (or by the much older results of Gronwall [8] for Legendre polynomials). For slightly better results on absolute convergence we refer to the paper of Ganser [7].

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