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VOLTERRA INTEGRAL EQUATIONS AND SEMIGROUPS OF OPERATORS

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Volterra integral equations and semigroups of operators *)

by

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ABSTRACT

In this paper we develop a semigroup approach for the study of Volterra integral equations of convolution type (renewal equations). Among other things we discuss the variation-of-constants formula, the adjoint semigroup and the decomposition of the state space according to the spectrum of the infinitesimal generator.

KEY WORDS & PHRASES: *renewal equations, variation-of-constants formula, adjoint semigroup, decomposition according to the spectrum of the infinitesimal generator*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In the qualitative theory of ordinary differential equations, the variation-of-constants formula

$$(1.1) \qquad x(t) = e^{B(t-\sigma)}x^{\sigma} + \int_{\sigma}^{t} e^{B(t-\sigma)}h(\tau)d\tau$$

takes in a key-position. It gives an explicit representation of the solution of the forced linear system

$$(1.2) \qquad \begin{cases} x'(t) = Bx(t) + h(t) \\ x(\sigma) = x^{\sigma} \end{cases}$$

as a sum of two contributions, one of which describes the influence of the initial state x^{σ} whereas the other describes the influence of the forcing function h. Starting from this formula one can discuss such things as linearized stability, the saddle point property (invariant manifolds) and bifurcation phenomena. In other words, it enables one to describe the (local) behaviour of solutions of certain nonlinear problems in terms of the eigenvalues of the matrix B (for instance, see HALE [4]).

The success of the variation-of-constants formula is not restricted to ordinary differential equations. Indeed, infinite-dimensional analogues of formula (1.1) provide the basis for the qualitative theory of functional differential equations (HALE [5]) and of semilinear parabolic differential equations (HENRY [6]). The objective of our research is to find a way of looking at Volterra integral equations of convolution type (renewal equations) such that this general approach can be applied to this class of equations as well. In this paper we shall describe the first (and most important) step in this direction, viz., we shall put on the stage the appropriate analogue of (1.1). It is our intention to apply in future work these ideas to nonlinear problems.

So in this paper we consider the linear system of renewal equations

$$\text{RE} \qquad x(t) = \int_{0}^{t} B(t-\tau)x(\tau)d\tau + f(t),$$

which we shall frequently write in the form

$$x = B*x + f.$$

First of all, we observe that it is very easy to solve RE, i.e., to give an explicit representation for the unknown x in terms of the given kernel B and the given forcing function f (see Section 3). Remarkably enough this has had, in our opinion, a confusing influence on attempts to find the analogue of (1.1). In order to explain the conceptual difficulty involved, let us take $\sigma = 0$ in (1.2) and integrate the equation from 0 to t. We arrive at

$$x(t) = \int_0^t Bx(\tau)d\tau + x^0 + \int_0^t h(\tau)d\tau,$$

which is a special case of RE with B constant and

$$f(t) = x^0 + \int_0^t h(\tau)d\tau.$$

Thus we see that f incorporates both the influence of the "initial state" x^0 and the influence from the "outside world", the function h. As we noticed before, the essential feature in (1.1) was that both contributions were clearly separated! This motivates our plan to unravel the contributions to f and it leads to such questions as:

- what does an *autonomous* problem look like in this context?
- what is the "state"-space for RE?
- how can we associate with RE a semigroup of operators?

At first we shall address these questions heuristically guided by an interpretation of RE in terms of the biological model of age-dependent population growth. But in the end we shall define a precise mathematical framework.

Our results depend crucially on the restrictive assumption that the kernel B has compact support. By this assumption we are able to avoid some difficult mathematical problems which are traditionally thought to be inherent in Volterra integral equations. It leads to a fairly easy theory which is applicable to many concrete situations (especially in mathematical biology). Generalizations seem possible but laborious.

From the very beginning of our work we were striving for a theory which is as much alike the theory of retarded functional differential equations as two peas in a pod. After the conceptual difficulty of defining the right setting had been overcome we could imitate quite easily many results and proofs of that theory. In order to bring this out clearly we adopt as much as possible the notation from Hale's stimulating and inspiring book [5].

The organization of the paper is as follows. In Section 2 we collect a number of definitions etc.. In Section 3 we introduce the resolvent and we show how it yields explicitly the solution of RE. In Section 4 we describe in some detail the intuitive ideas which underly the choice of the state space and in Section 5 we associate with RE a semigroup of operators working on that space. In Section 6 we pay attention to forced linear systems and we derive the variation-of-constants formula. In Section 7 we calculate the adjoint semigroup and its infinitesimal generator. Moreover, we establish the connection between these and the adjoint equation. In Section 8 we show how one can decompose the state space according to the spectrum of the infinitesimal generator (and in the appendix we make some of the objects involved more concrete). Finally, in Section 9 we demonstrate that the theory set forth in the preceding sections, can be a handy tool for solving qualitative problems by proving a Fredholm alternative for periodic solutions.

2. ASSUMPTIONS, DEFINITIONS AND NOTATION

In this section we gather together some information (which might be consulted while reading subsequent sections).

Throughout this paper B denotes an $n\times n$-matrix valued function defined on $\mathbb{R}_+ = [0,\infty)$, which is integrable and has compact support. So

$$(2.1) \qquad b := \inf\{\beta \mid \operatorname{supp} B \subset [0,\beta]\} < \infty.$$

The symbol $L_1^{loc}(\mathbb{R}_+)$ will denote the space of locally integrable functions defined on $\mathbb{R}_+$ with values in, depending on the situation at hand, the space of n-column vectors or n-row vectors or $n\times n$-matrices. In general, it should be clear from the context to which of these spaces the range of a given function belongs!

The Laplace transform and the convolution product are defined as usual

$$(2.2)\qquad \bar{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\,dt,$$

$$(2.3)\qquad B*f(t) = \int_0^t B(t-\tau) f(\tau)\,d\tau.$$

For a given function $x = x(t)$ we denote by $x_s = x_s(t)$ its translate to the left over a distance s:

$$(2.4)\qquad x_s(t) := x(t+s).$$

We shall write $f \in AC$ to denote that f is absolutely continuous and $f \in Lip$ to denote that f is Lipschitz continuous (recall that in both cases $f(t) = \int_0^t f'(\tau)d\tau + f(0)$ with $f' \in L_1^{loc}$ in the former and $f' \in L_\infty$ in the latter).

For a given operator A on a Banach space X the spectrum of A will be denoted by $\sigma(A)$ and the point-spectrum by $P\sigma(A)$.

If P and Q are linear subspaces of X and if each $f \in X$ can be written uniquely as $f = p + q$ with $p \in P$ and $q \in Q$, one writes $X = P \oplus Q$ and one says that X is the direct sum of P and Q.

3. PRELIMINARIES: THE RESOLVENT

To begin with, let us do some formal calculations. Applying the Laplace transform to both sides of RE we obtain

$$\bar{x} = \bar{B}\bar{x} + \bar{f}$$

and subsequently

$$\bar{x} = (I - \bar{B})^{-1}\bar{f}.$$

To this identity we may apply the inversion formula to obtain an explicit expression for x. Unfortunately $(I - \bar{B})^{-1}$ is not the Laplace transform of a matrix-valued function (it tends to I as $\mathrm{Re}\,\lambda \to +\infty$). But if we split off the limit then probably the remaining part is! So we write

$$\bar{x} = \bar{f} - \bar{R}\bar{f},$$

where by definition

$$\bar{R} = -(I - \bar{B})^{-1}\bar{B}, \tag{3.1}$$

(the minus sign is a matter of convention), and subsequently

$$x = f - R*f. \tag{3.2}$$

Moreover, multiplying (3.1) with $I - \bar{B}$ we obtain

$$\bar{R} = -\bar{B} + \bar{B}\bar{R}$$

and

$$R = -B + B*R. \tag{3.3}$$

As a first step towards the justification of these manipulations, we observe that $(I - \bar{B})^{-1}$ is nonsingular in a right-half plane. More precisely, there exists a real number λ_0 such that $\det(I - \bar{B}(\lambda)) \neq 0$ for $\mathrm{Re}\,\lambda \geq \lambda_0$. This fact enables one to arrive at the following result.

THEOREM 3.1. *Equation* (3.3) *has a unique (matrix-valued) solution* $R \in L_1^{loc}(\mathbb{R}_+)$. *The function* R, *which is called the resolvent, has the following properties:*

(i) *the function* $t \mapsto R(t)e^{-\lambda_0 t}$ *belongs to* $L_1(\mathbb{R}_+)$ *(so* $\bar{R}$ *is defined for* $\mathrm{Re}\,\lambda \geq \lambda_0$ *and satisfies* (3.1) *there);*

(ii) *for any* $f \in L_1^{loc}(\mathbb{R}_+)$ *the unique solution of* RE *is given by* (3.2);

(iii) R *also satisfies the equation*

$$R = -B + R*B$$

(in other words, $B*R = R*B$*).*

For the proof, which is based on the theorem of Wiener & Lévy, we refer to PALEY & WIENER [15, Section 18], MILLER [12, Section IV.5 and Appendix I.4] or CORDUNEANU [3, Section I.3].

4. WHEN IS THE PROBLEM AUTONOMOUS?

Let $x(t)$ denote the frequency of newly born individuals in a closed population (of one sex) at time t. In a classical model from population dynamics one assumes that x satisfies the dynamical equation

$$x(t) = \int_0^b B(\tau)x(t-\tau)d\tau. \tag{4.1}$$

Here the function B is given as the product of two factors, one describing the fertility of an individual of age τ and the other the chance that an arbitrary neonate will reach that age. B is supposed to be known and to vanish for $\tau \geq b$. The model goes back to historical papers of A.J. Lotka (see KEYFITZ [9] or HOPPENSTEADT [8] for more details and references).

The equation (4.1) is autonomous in the sense that it is translation invariant. This reflects the fact that the model does not take into account any time inhomogeneous effect from or interaction with the outside world.

In order to obtain a well-defined initial value problem we suppose that equation (4.1) does only hold from some time on, say for $t \geq 0$, and that we know at time $t = 0$ the relevant facts from the past:

$$x(t) = \phi(t), \qquad -b \leq t \leq 0, \tag{4.2}$$

where ϕ is a given function.

The problem (4.1) - (4.2) can be rewritten as the usual renewal equation RE (note that the model elucidates this name) with f given by

$$f = L_B\phi, \tag{4.3}$$

where $L_B\colon L_1(\mathbb{R}_-) \to L_1(\mathbb{R}_+)$ is defined by

$$(4.4)\qquad (L_B\phi)(t) := \int_t^b B(\tau)\phi(t-\tau)d\tau = \int_{t-b}^0 B(t-\tau)\phi(\tau)d\tau.$$

So it seems reasonable to call RE autonomous if $f = L_B\phi$ for some ϕ. Note that in this case f vanishes for $t \geq b$.

However, working in the incomplete space $\mathcal{R}(L_B)$ is troublesome from a mathematical point of view. The remedy is obvious, we simply take the closure. The next result gives concrete form to the outcome of this abstract operation, in this special case of $n = 1$ (one equation).

THEOREM 4.1. *Let* $n = 1$ *then*

$$\overline{\mathcal{R}(L_B)} = \{f \in L_1(\mathbb{R}_+) \mid f(t) = 0 \text{ a.e. on } (b,\infty)\}$$

where

$$b = \inf\{\beta \mid \operatorname{supp} B \subset [0,\beta]\}.$$

PROOF. From functional analysis we know that

$$\overline{\mathcal{R}(L_B)} = {}^{\perp}\mathcal{N}(L_B^*)$$

and this motivates us to calculate the adjoint

$$L_B^*\colon L_\infty(\mathbb{R}_+) \to L_\infty(\mathbb{R}_-).$$

$$\langle\psi, L_B\phi\rangle = \int_0^\infty \psi(t) \int_{-\infty}^0 B(t-\tau)\phi(\tau)d\tau dt$$

$$= \int_{-\infty}^0 \int_0^\infty \psi(t)B(t-\tau)dt\phi(\tau)d\tau$$

$$= \langle L_B^*\psi, \phi\rangle.$$

So

$$(L_B^*\psi)(t) = \int_0^\infty \psi(\tau)B(\tau-t)d\tau$$

$$= \int_{-t}^{b} \psi(\tau+t)B(\tau)d\tau,$$

and in particular, $L_B^*\psi$ vanishes for $t \le -b$. Putting $\xi = t+b$ and $\tilde{B}(\tau) = B(b-\tau)$ we can write

$$(L_B^*\psi)(t) = \int_0^\xi \psi(\xi-\tau)\tilde{B}(\tau)d\tau = \psi*\tilde{B}(\xi).$$

Now suppose $L_B^*\psi = 0$, then it follows from the theorem of Titchmarsh (see TITCHMARSH [18, p. 327]) that $\psi(\xi) = 0$ a.e. on $(0,\alpha_1)$ and $\tilde{B}(\xi) = 0$ a.e. on $(0,\alpha_2)$ with $\alpha_1+\alpha_2 \ge b$. The definitions of b and $\tilde{B}$ imply that $\alpha_2 = 0$ and consequently $\alpha_1 \ge b$. On the other hand, the condition $\alpha_1 \ge b$ is also sufficient to have $L_B^*\psi = 0$. Hence

$$N(L_B^*) = \{\psi \in L_\infty(\mathbb{R}_+) \mid \psi(t) = 0 \text{ a.e. on } (0,b)\}$$

and the result follows. □

On the basis of these heuristic considerations we now choose as the underlying state space for the study of RE in the general case of arbitrary n, the Banach space

$$X = \{f \in L_1(\mathbb{R}) \mid f(t) = 0 \text{ a.e. on } (b,\infty)\} \tag{4.5}$$

with the norm given by

$$\|f\| = \int_0^b |f(\tau)|d\tau.$$

We shall call RE autonomous iff $f \in X$.

<u>REMARK</u>. In a study of Volterra integrodifferential equations [13] Miller

uses a state space resembling $\overline{\mathcal{R}(L_B)}$. However, the space remains less concrete and the interpretation less explicit.

In the definition of X, the fact that the functions have compact support is more important than the specific topology chosen. For instance, one can develop the theory in a space of continuous functions, or of L_2-functions (if $B \in L_2$). Then all operators involved are formally the same as those to be discussed in the following sections, but the domains of definition require an appropriate modification.

5. THE SEMIGROUP T(s) AND ITS INFINITESIMAL GENERATOR A

In this section we address ourselves to the problem of associating with RE a semigroup of bounded linear operators on the Banach space X. Several years ago, R.K. MILLER and G.R. SELL published a memoir [14] in which they constructed a topological dynamics framework for a much more general class of Volterra integral equations. The main idea of their approach is simple to describe. The equation corresponds to an initial value problem. If time has gone on for a while we can, in thoughts, do as if we start again with new data. The mapping from old to new data defines, similar to the case of ordinary differential equations, the dynamical system. In the present situation this idea amounts to the following.

If we take in RE the argument equal to t+s then some straightforward manipulations yield the identity

$$x_s(t) = \int_0^t B(t-\tau)x_s(\tau)d\tau + f(t+s) + \int_0^s B(t+s-\tau)x(\tau)d\tau,$$

which can be written as

$$(5.1) \qquad x_s = B*x_s + T(s)f,$$

if we define

$$(5.2) \qquad T(s) = U(s) + V(s),$$

where

(5.3) $$(U(s)f)(t) = f(t+s),$$

(5.4) $$(V(s)f)(t) = \int_0^s B(t+s-\tau)x(\tau)d\tau = (B_t - B_t * R) * f(s)$$

(the second expression for $V(s)f$ is obtained by inserting the explicit formula (3.2) for x into the first). First of all we observe that, for fixed $s \geq 0$, $U(s)$ and $V(s)$ are well-defined as (linear) mappings from $L_1^{loc}(\mathbb{R}_+)$ into itself (for $V(s)$ the proof follows the same lines as the proof of "$g,h \in L_1 \Rightarrow g*h \in L_1$"). Clearly $U(s)$ leaves X invariant and for $V(s)$ we have the even stronger property that it maps all of $L_1^{loc}(\mathbb{R}_+)$ into X (note that B_t vanishes identically for $t \geq b$). Standard estimates show that $T(s)$ is bounded as a mapping from X into itself. We are now ready to formulate the first of our main results.

THEOREM 5.1. *The mapping* $s \mapsto T(s)$ *from* $\mathbb{R}_+$ *into* $L(X)$ *defines a strongly continuous semigroup, i.e.,*

(i) $T(s)T(\sigma) = T(t+\sigma), \qquad s,\sigma \geq 0,$

(ii) $T(0) = I,$

(iii) $\lim_{s \downarrow 0} \|T(s)f - f\| = 0, \qquad \forall f \in X.$

PROOF. From (5.1) it follows that

$$(x_\sigma)_s = B*(x_\sigma)_s + T(s)T(\sigma)f$$

and

$$x_{\sigma+s} = B*x_{\sigma+s} + T(s+\sigma)f.$$

Since $(x_\sigma)_s = x_{\sigma+s}$ this implies the identity (i) (note that we use implicitly the uniqueness of the solution of RE). The identity (ii) is a direct consequence of the definitions and (iii) follows from standard estimates (recall that translation is continuous in the L_1-topology). $\square$

The infinitesimal generator A is defined by

$$Af = \lim_{s\downarrow 0} \frac{T(s)f-f}{s},$$

the domain of A being precisely the set of functions for which this limit exists. From the general theory we know that, for Re λ sufficiently large,

$$\mathcal{D}(A) = \mathcal{R}((\lambda I-A)^{-1})$$

and

$$(\lambda I-A)^{-1}f = \int_0^\infty e^{-\lambda s}T(s)f\,ds$$

(see, for instance, YOSHIDA [19, Section IX.4] or BUTZER & BERENS [2, Section I.3]). These formulas enable us to characterize $\mathcal{D}(A)$ and to give concrete form to A.

THEOREM 5.2.

$$\mathcal{D}(A) = \{f \mid f \in AC \text{ and } f' \in X\}.$$

$$(Af)(t) = f'(t) + B(t)f(0).$$

PROOF. Take any $f \in X$ and let $g = (\lambda I-A)^{-1}f$ then

$$g(t) = \int_0^\infty e^{-\lambda s}(T(s)f)(t)ds = \int_0^\infty e^{-\lambda s}\{f(t+s) + (B_t-B_t*R)*f(s)\}ds$$

$$= e^{\lambda t}\int_t^\infty e^{-\lambda s}f(s)ds + e^{\lambda t}\int_t^\infty e^{-\lambda s}B(s)ds(I-\bar{R}(\lambda))\bar{f}(\lambda).$$

Hence $g \in AC$,

$$g(0) = \bar{f}(\lambda) + \bar{B}(\lambda)(I-\bar{R}(\lambda))\bar{f}(\lambda) = (I-\bar{R}(\lambda))\bar{f}(\lambda)$$

and

$$g'(t) = \lambda g(t) - f(t) - B(t)(I-\bar{R}(\lambda))\bar{f}(\lambda) = \lambda g(t) - f(t) - B(t)g(0),$$

showing that $g' \in X$. From $(\lambda I-A)^{-1}f = g$ we infer that $\lambda g - f = Ag$ which, by comparison with the identity above, shows that

$$(Ag)(t) = g'(t) + B(t)g(0).$$

On the other hand, let now g be any function such that $g \in AC$ and $g' \in X$. Define $f \in X$ by

$$f(t) = \lambda g(t) - g'(t) - B(t)g(0)$$

and define $h \in X$ by $h = (\lambda I-A)^{-1}f$. We intend to show that $h = g$ since this implies that $g \in \mathcal{D}(A)$. It follows, as above, that

$$f(t) = \lambda h(t) - h'(t) - B(t)h(0).$$

So if we put $z = g - h$ then

$$\lambda z(t) - z'(t) - B(t)z(0) = 0$$

and consequently

$$z(t) = e^{\lambda t}(I - \int_0^t e^{-\lambda\tau}B(\tau)d\tau)z(0).$$

Finally, the fact that $\det(I-\bar{B}(\lambda)) \neq 0$ implies that $z \in X$ iff $z(0) = 0$ (and then $z(t) = 0$ for all t). $\square$

REMARK. Let $x(t,f)$ denote the solution of RE and let $f \in \mathcal{D}(A)$. Then differentiation of RE shows that $x(t,Af) = \frac{dx}{dt}(t,f)$.

THEOREM 5.3. *The operator* A *is closed, has compact resolvent and*

$$\sigma(A) = P\sigma(A) = \{\lambda \mid \det \Delta(\lambda) = 0\}$$

where

$$\Delta(\lambda) := I - \bar{B}(\lambda).$$

PROOF. Let us first concentrate on the eigenvalues of A. The eigenvalue problem $Af = \lambda f$ corresponds to the differential equation

$$f'(t) = \lambda f(t) - B(t)f(0),$$

which has the solution

$$f(t) = e^{\lambda t}(I - \int_0^t e^{-\lambda s}B(s)ds)f(0).$$

So if $\det \Delta(\lambda) = 0$ we can achieve that $f \in X$ by choosing $f(0) \in N(\Delta(\lambda))$. Hence λ is an eigenvalue of A in that case.

On the other hand, suppose now that $\det \Delta(\lambda) \neq 0$. We shall show that $(A-\lambda I)^{-1}$ exists and is bounded and compact. With the abstract problem $(A-\lambda I)f = g$ there corresponds the differential equation

$$f'(t) - \lambda f(t) + B(t)f(0) = g(t),$$

which has the solution

$$f(t) = e^{\lambda t}\{(I - \int_0^t e^{-\lambda s}B(s)ds)f(0) + \int_0^t e^{-\lambda s}g(s)ds\}.$$

We can achieve that $f \in X$ by choosing

$$f(0) = -(\Delta(\lambda))^{-1}\bar{g}(\lambda).$$

So $f = (A-\lambda I)^{-1}g$ is given by

$$f(t) = e^{\lambda t}\{\int_0^t e^{-\lambda s}g(s)ds - (I - \int_0^t e^{-\lambda s}B(s)ds)(\Delta(\lambda))^{-1}\bar{g}(\lambda)\} \tag{5.5}$$

$$= e^{\lambda t}\{-\int_t^b e^{-\lambda s}g(s)ds - \int_t^b e^{-\lambda s}B(s)ds(\Delta(\lambda))^{-1}\bar{g}(\lambda)\},$$

and from this explicit expression the correctness of the theorem follows (see, for instance, KUFNER et al. [10, Th. 2.13.1] for the appropriate compactness criterium). $\square$

At this point some remarks seem to be in order. The family of operators $\{T(s)\}$ forms a strongly continuous semigroup on $L_1(\mathbb{R}_+)$ as well. Then the infinitesimal generator is given by the same formal operator, the domain being the appropriate extension of $\mathcal{D}(A)$. However, in this setting the open left-half plane $\{\lambda \mid \mathrm{Re}\,\lambda < 0\}$ belongs to the (point) spectrum of the infinitesimal generator. This is the price one has to pay if one abandons the autonomous point of view!

6. FORCED LINEAR SYSTEMS AND THE VARIATION-OF-CONSTANTS FORMULA

At first, let us resume the heuristic exposition of Section 4. Suppose we perform an experiment, viz., we constantly add to the population neonates emanating from somewhere else. The equation then takes the form

$$x(t) = \int_0^b B(\tau)x(t-\tau)d\tau + h(t), \tag{6.1}$$

where h describes how many neonates are added. Since now the equation is nonautonomous (not translation-invariant), the exact time from which the equation is supposed to hold, matters. We denote this time by σ and we supplement (6.1) with the initial condition

$$x_\sigma(t) = \phi(t), \qquad -b \le t \le 0, \tag{6.2}$$

i.e., we prescribe x on the interval $-b+\sigma \le t \le \sigma$, but we choose our notation such that the data are made up of a function on the (σ-independent) interval $[-b,0]$. Let $f = L_B\phi$ then (6.1) - (6.2) yields

$$x(t) = \int_\sigma^t B(t-\tau)x(\tau)d\tau + f(t-\sigma) + h(t), \qquad t \ge \sigma,$$

or,

$$x_\sigma(t) = \int_0^t B(t-\tau)x_\sigma(\tau)d\tau + f(t) + h_\sigma(t), \qquad t \geq 0.$$

The forcing function contains two terms, the state f and the "true" forcing h_σ. Actually, f is the state at time σ. We embody this in the notation by writing $f = f^\sigma$. This leads to

$$(6.3) \qquad x_\sigma = B*x_\sigma + f^\sigma + h_\sigma.$$

Subsequently, we define, for $s \geq \sigma$, f^s by the formula

$$(6.4) \qquad x_s = B*x_s + f^s + h_s,$$

the interpretation being that f^s is the state at time s, and we investigate the relation between f^s and f^σ. It follows from (6.3), (6.4) and the definition of $T(s-\sigma)$ (as an operator on L_1^{loc}, at first) that

$$f^s + h_s = T(s-\sigma)(f^\sigma + h_\sigma).$$

Hence

$$(6.5) \qquad f^s = T(s-\sigma)f^\sigma + \dot{V}(s-\sigma)h_\sigma ,$$

which shows that $f^s \in X$ indeed. Already this formula can be termed, with some right, the variation-of-constants formula since it has the properties mentioned in the beginning of Section 1. However, we can gain a lot by elaborating it a little bit more.

From the definition of T(s) (cf. (5.2) - (5.4)) and the resolvent equation (3.3) it follows that

$$(6.6) \qquad (T(s)B)(t) = B_t(s) + (B_t - B_t*R)*B(s) = B_t(s) - B_t*R(s).$$

(The fact that we apply T(s) to a matrix-valued function should not lead to confusion; it works on each column separately.) Inserting this identity into the formula (5.4), which defines V(s), we obtain

$$(6.7) \qquad (V(s)f)(t) = \int_0^s (T(s-\tau)B)(t)f(\tau)d\tau,$$

which upon substitution into (6.5) yields

$$(6.8) \qquad f^s = T(s-\sigma)f^\sigma + \int_\sigma^s T(s-\tau)Bh(\tau)d\tau.$$

This is the formula which we shall call *the variation-of-constants formula*. Note that, once the argument has been filled in, the integrand is an $\mathbb{R}^n$-valued function, so that the integral has a well-defined meaning. We summarize the results of this section into the following theorem.

THEOREM 6.1. *Let* $h \in L_1^{loc}(\mathbb{R})$ *be given. Also, let* $\sigma \in \mathbb{R}$ *and* $f^\sigma \in X$ *be given. Define, for* $t \geq \sigma$, $x(t)$ *as the unique solution of the renewal equation*

$$x_\sigma = B{*}x_\sigma + f^\sigma + h_\sigma,$$

and define, for $s \geq \sigma$, $f^s \in X$ *by the relation*

$$x_s = B{*}x_s + f^s + h_s.$$

Then

$$f^s = T(s-\sigma)f^\sigma + \int_\sigma^s T(s-\tau)Bh(\tau)d\tau.$$

REMARK. By formal differentiation of the variation-of-constants formula we obtain an inhomogeneous ordinary differential equation in the Banach space X:

$$\frac{d}{ds} f^s = Af^s + Bh(s)$$

Hence the general theory of such equations can be made to bear on renewal equations.

REMARK. On the basis of the considerations above one can, if one wishes to do so, define a *process* on X, i.e., a two-parameter family of operators which satisfies a product relation analogous to the one for the solution of a nonautonomous ordinary differential equation.

REMARK. The definition of $T(s)$ as given by (5.1) and the resolvent equation (3.3) imply that

$$T(s)B = -R_s + B*R_s.$$

Consequently the following identity holds:

$$-R_s(t) + B*R_s(t) = B_t(s) - B_t*R(s).$$

7. THE ADJOINT SEMIGROUP AND ITS RELATION WITH THE ADJOINT EQUATION

In order to obtain explicit representations of projection operators associated with a splitting of X according to the spectrum of A, it is useful to calculate adjoints. Moreover, this has some interest in itself.

As a realization of the dual space we take

$$X^* = L_\infty(-b,0),$$

the pairing being given by

$$\langle\psi,f\rangle = \int_0^b \psi(-\tau)f(\tau)d\tau.$$

Then some straightforward manipulations yield

$$(7.1)\qquad (T(s)^*\psi)(t) = \begin{cases} \psi(t+s) & \text{for } -b \le t \le \max\{-s,-b\} \\ \int_0^b \psi(-\tau)Q(\tau,s+t)d\tau & \text{for } \max\{-s,-b\} \le t \le 0, \end{cases}$$

where by definition

$$(7.2)\qquad Q(t,s) = B_t(s) - B_t*R(s).$$

It is known that the adjoint operators form a semigroup again which in general, however, need not be strongly continuous (see BUTZER & BERENS [2, Section I.4] or YOSIDA [19, Section IX.13]). Let X_0^* denote the set on which it is, i.e.,

$$X_0^* = \{\psi \in X^* \mid \lim_{s \downarrow 0} \|T(s)^*\psi - \psi\|_{X^*} = 0\}.$$

From (7.1) we deduce that in this case

$$X_0^* = \{\psi \mid \psi \in C[-b,0] \text{ and } K(\psi) = 0\} \tag{7.3}$$

where by definition

$$K(\psi) = \psi(0) - \int_0^b \psi(-\tau)B(\tau)d\tau. \tag{7.4}$$

Next we bend our thoughts towards the determination of A^*. We shall utilize the abstract results

$$\mathcal{D}(A^*) = \mathcal{R}((\lambda I - A^*)^{-1})$$

and

$$(\lambda I - A^*)^{-1} = ((\lambda I - A)^{-1})^*$$

for $\mathrm{Re}\,\lambda$ sufficiently large (YOSTIDA [19, Section IX.13]).

<u>THEOREM 7.1</u>.

$$\mathcal{D}(A^*) = \{\psi \mid \psi \in \mathrm{Lip}[-b,0] \text{ and } K(\psi) = 0\}.$$

$$(A^*\psi)(t) = \psi'(t).$$

<u>PROOF</u>. Let $f,g \in X$ be related by $f = (A-\lambda I)^{-1}g$ and suppose $\langle\phi,f\rangle = \langle\psi,g\rangle$, then it follows from the explicit formula (5.5) that

$$\psi(t) = e^{\lambda t}\{-\int_0^{-t} \phi(-\sigma)e^{\lambda\sigma}d\sigma + \int_0^b \phi(-\sigma)\alpha(\sigma)e^{\lambda\sigma}d\sigma\}, \tag{7.5}$$

where by definition

$$(7.6) \qquad \alpha(t) = -\int_t^b e^{-\lambda\sigma} B(\sigma)\,d\sigma\,(\Delta(\lambda))^{-1}.$$

We can rewrite this as

$$\psi(t) = e^{\lambda t}\{\psi(0) - \int_0^{-t} \phi(-\sigma)e^{\lambda\sigma}d\sigma\},$$

$$\psi(0) = \int_0^b \phi(-\sigma)\alpha(\sigma)e^{\lambda\sigma}d\sigma.$$

Since $\phi \in L_\infty$, clearly $\psi \in$ Lip. Moreover,

$$\int_0^b \psi(-\sigma)B(\sigma)\,d\sigma = \psi(0)\int_0^b e^{-\lambda\sigma}B(\sigma)\,d\sigma - \int_0^b\int_\tau^b \phi(-\tau)e^{\lambda\tau}e^{-\lambda\sigma}B(\sigma)\,d\sigma d\tau$$

$$= \int_0^b \phi(-\tau)e^{\lambda\tau}\{\alpha(\tau)\int_0^b e^{-\lambda\sigma}B(\sigma)\,d\sigma - \int_\tau^b e^{-\lambda\sigma}B(\sigma)\,d\sigma\}d\tau$$

$$= \int_0^b \phi(-\tau)e^{\lambda\tau}\alpha(\tau)\,d\tau = \psi(0),$$

or, in other words, $K(\psi) = 0$.

On the other hand, let now $\psi \in$ Lip be given and such that $K(\psi) = 0$. Since $\psi \in$ Lip it has a well-defined derivative $\psi' \in L_\infty$. Define ϕ by $\phi = \psi' - \lambda\psi$ then

$$\psi(t) = e^{\lambda t}\{\psi(0) - \int_0^{-t} \phi(-\sigma)e^{\lambda\sigma}d\sigma\}.$$

Hence

$$\psi(0) = \int_0^b \psi(-\sigma)B(\sigma)\,d\sigma$$

$$= \psi(0)\int_0^b e^{-\lambda\sigma}B(\sigma)\,d\sigma - \int_0^b \phi(-\sigma)e^{\lambda\sigma}\int_\sigma^b e^{-\lambda\tau}B(\tau)\,d\tau d\sigma,$$

or,

$$\psi(0) = \int_0^b \phi(-\sigma)e^{\lambda\sigma}\alpha(\sigma)d\sigma.$$

Inserting this into the expression for ψ we obtain

$$\psi(t) = e^{\lambda t}\{-\int_0^{-t} \phi(-\sigma)e^{\lambda\sigma}d\sigma + \int_0^b \phi(-\sigma)\alpha(\sigma)e^{\lambda\sigma}d\sigma\},$$

which upon comparison with (7.5) shows that

$$\psi = ((A-\lambda I)^{-1})^*\phi,$$

from which we conclude that $\psi \in \mathcal{D}(A^*)$. This completes the characterization of $\mathcal{D}(A^*)$.

Finally, let $f \in \mathcal{D}(A)$ and $\psi \in \mathcal{D}(A^*)$ then

$$\begin{aligned}<\psi,Af> &= \int_0^b \psi(-\tau)f'(\tau)d\tau + \int_0^b \psi(-\tau)B(\tau)d\tau f(0)\\ &= \psi(-\tau)f(\tau)\Big|_0^b + \int_0^b \psi'(-\tau)f(\tau)d\tau + \psi(0)f(0)\\ &= \int_0^b \psi'(-\tau)f(\tau)d\tau = <A^*\psi,f>.\end{aligned}$$

Hence

$$(A^*\psi)(t) = \psi'(t). \qquad \square$$

We observe that, in accordance with the abstract theory, $\overline{\mathcal{D}(A^*)} = X_0^*$. Let $T_0(s)^*$ denote the restriction of $T(s)^*$ to X_0^*. Then $\{T_0^*(s)\}$ forms a strongly continuous semigroup. Let A_0^* denote its infinitesimal generator with domain $\mathcal{D}(A_0^*)$. It is known (Phillips' Theorem) that A_0^* is the largest restriction of A^* with both domain and range in X_0^*. In this case

$$\mathcal{D}(A_0^*) = \{\psi \mid \psi \in C^1[-b,0] \text{ and } K(\psi) = K(\psi') = 0\},$$

as one can conclude from direct considerations as well.

REMARK.

$$\sigma(A^*) = P\sigma(A^*) = \{\lambda \mid \det \Delta(\lambda) = 0\}.$$

Next we turn our attention to the adjoint equation, by which we mean

$$(7.7) \qquad y(t) = \int_0^b y(t-\tau)B(\tau)\,d\tau.$$

If we supplement (7.7) with the initial condition

$$(7.8) \qquad y(t) = \psi(t), \qquad -b \le t \le 0,$$

where $\psi \in X^*$ is given, it has a well-defined unique solution $y \in L_\infty^{loc}(\mathbb{R}_+)$. This can be concluded most easily by rewriting (7.7) - (7.8) as the renewal equation

$$y(t) = \int_0^t y(t-\tau)B(\tau)\,d\tau + g(t),$$

where

$$g(t) = \int_t^b \psi(t-\tau)B(\tau)\,d\tau = \int_0^b \psi(-\tau)B(t+\tau)\,d\tau,$$

and by applying the results of Section 3. It follows that

$$y = g - g*R,$$

from which we deduce, for $t \ge 0$,

$$\begin{aligned} y(t) &= \int_0^b \psi(-\tau)B(t+\tau)\,d\tau - \int_0^t\int_0^b \psi(-\tau)B(\sigma+\tau)\,d\tau R(t-\sigma)\,d\sigma \\ &= \int_0^b \psi(-\tau)\{B(t+\tau) - \int_0^t B(t-\sigma+\tau)R(\sigma)\,d\sigma\}d\tau = \int_0^b \psi(-\tau)Q(\tau,t)\,d\tau. \end{aligned}$$

Comparison with (7.1) leads to the identity

$$T(s)^{*}\psi = y_{s}.$$

In words this says that the action of the adjoint semigroup corresponds to translation along trajectories of the adjoint equation.

In the special case of a symmetric kernel B (so in particular for $n = 1$, the case of one equation) we can say even more. Then the equations (7.7) and (4.1) are identical and the semigroups $\{T(s)\}$ and $\{T(s)^{*}\}$ constitute just two different ways of looking at the same problem. In view of Section 4 we can interpret the condition $K(\psi) = 0$, which characterizes X_0^{*}, as a compatibility condition on an initial function.

Since the spaces we are working in are not reflexive, the process of taking adjoints does not end. In fact, readers familiar with the theory of retarded functional differential equations might wonder why we did not start directly with the problem (7.7) - (7.8), defining the semigroup by translation along the solution, and then take adjoints. This can be done, but it is technically more complicated. The semigroup thus defined is strongly continuous only on X_0^{*}, a space which depends on the specific kernel B. So even the definition of the dual space needs more care in this situation. In our opinion, starting with $\{T(s)\}$ defined on X is advantageous.

If one chooses $L_{\infty}(0,b)$ as the realization of X^{*}, instead of $L_{\infty}(-b,0)$, the correct expressions are obtained from the corresponding ones above by performing a reflection of the time axis. Among other things, this changes the appearance of the adjoint equation which, in that case, defines a solution backwards in time. Of course, everything remains essentially the same.

In conclusion of this section we refer to BURNS & HERDMAN [1] for a treatment of related problems in the context of Volterra integrodifferential systems.

8. DECOMPOSITION OF X

Since A has compact resolvent we can apply the spectral theory for such operators (HILLE & PHILLIPS [7, Section 5.14]; also see TAYLOR [17, Th. 5.8A p. 306]). It follows that for each $\lambda \in \sigma(A) = P\sigma(A)$ there exists a smallest integer $k = k(\lambda)$ such that $N(A-\lambda I)^k = N(A-\lambda I)^{k+i}$ for $i = 1,2,\ldots$. Moreover $N(A-\lambda I)^k$ is finite dimensional and

$$X = N(A-\lambda I)^k \oplus R(A-\lambda I)^k. \tag{8.1}$$

Let $p = p(\lambda) = \dim N(A-\lambda I)^k$ and let $F = F(\lambda)$ be a basis for $N(A-\lambda I)^k$. Since $N(A-\lambda I)^k$ is invariant under A (A commutes with $A-\lambda I$!) there exists a $p\times p$ constant matrix $D = D(\lambda)$ such that

$$AF = FD. \tag{8.2}$$

From the fact that $0 = (A-\lambda I)^k F = F(D-\lambda I)^k$ we infer that λ is the only eigenvalue of D.

On the one hand we have the identity $T(t)AF = T(t)FD$ and on the other $T(t)AF = \frac{d}{dt}T(t)F$. Hence $T(t)F$ satisfies the first order ordinary differential equation $\frac{d}{dt}T(t)F = T(t)FD$, which, together with the initial condition $T(0)F = IF = F$, implies that

$$T(t)F = Fe^{Dt}. \tag{8.3}$$

We observe that this formula defines the action of $T(t)$ on $N(A-\lambda I)^k$ for all $t \in (-\infty,\infty)$!

On the account of the general theory again, we know that

(i) $N(A^*-\lambda I)^k = N(A^*-\lambda I)^{k+i}$, $\quad i = 1,2,\ldots;$

(ii) $\dim N(A^*-\lambda I)^k = p;$

(iii) $R(A-\lambda I)^k = {}^{\perp}N(A^*-\lambda I)^k.$

Let $\Psi = \Psi(\lambda)$ be a basis for $N(A^*-\lambda I)^k$. From (8.1) and (iii) above we infer that $\langle\Psi,F\rangle$ is nonsingular and consequently we may assume that Ψ is constructed such that $\langle\Psi,F\rangle = I$ (here $\langle\Psi,F\rangle$ denotes the $p\times p$-matrix with entries $\langle\psi_i,f_j\rangle$ where ψ_i and f_j are the basis elements which constitute Ψ and F, respectively). Hence

$$\langle A^*\Psi,F\rangle = \langle\Psi,AF\rangle = \langle\Psi,F\rangle D = D = D\langle\Psi,F\rangle,$$

from which we conclude that

$$(8.3) \qquad A^*\Psi = D\Psi \Rightarrow \Psi(\theta) = e^{D\theta}\Psi(0),$$

and

$$(8.4) \qquad T(t)^*\Psi = \Psi_t = e^{D(t+\cdot)}\Psi(0).$$

The formula (8.4) defines the action of $T(t)^*$ on $\mathcal{N}(A^*-\lambda I)^k$ for all $t \in (-\infty,\infty)$.

At this point the subspaces occurring in the splitting (8.1) (as well as the corresponding projection operators) are described in terms of the bases F and Ψ, which are abstractly defined. In the appendix we shall show how the elements of F and Ψ can be computed in terms of quantities related to $\Delta(\lambda)$.

Next, let Λ be a finite set $\{\lambda_1,\ldots,\lambda_q\}$ of eigenvalues of A and let P denote the subspace spanned by $F(\lambda_1),\ldots,F(\lambda_q)$. Since $\mathcal{R}(A-\lambda I)^k$ is invariant under T(t) (note that A and T(t) commute on $\mathcal{D}(A)$) we can repeat the splitting described above. It follows that X is decomposed by Λ as

$$(8.5) \qquad X = P \oplus Q,$$

where both P and Q are invariant under T(t). On the finite dimensional subspace P, the action of T(t) is described by an ordinary differential equation. It remains to derive an exponential estimate for the action of T(t) on the complementary (infinite-dimensional, invariant) subspace Q (cf. HALE [5, Section 7.4]).

In general the relation between the spectrum of A and that of T(t) can be quite complicated (see, for instance, the discussion by SLEMROD [16]). However, it is known that $P\sigma(T(t)) \subset e^{tP\sigma(A)} \cup \{0\}$ (HILLE & PHILLIPS [7, Th. 16.7.2, p. 467]). We know that $T(t) = V(t)$ for $t \geq b$. Since $V(t): X \to X$ is compact, we conclude that T(t) has only point spectrum for those values of t. So we have detailed information for $t \geq b$. Next, exploiting the semigroup property one can convert this information into an estimate for T(t)f for all values of t. This is expressed in the following result, which is due to HALE [5, Lemma 4.2, p. 180].

LEMMA 8.1. *If for some* $r > 0$ *the spectral radius* $\rho = \rho(T(r)) \neq 0$ *and* β *is defined by* $\beta r = \log \rho$, *then, for any* $\varepsilon > 0$, *there is a constant* $K = K(\varepsilon) \geq 1$ *such that*

$$\|T(t)f\| \leq Ke^{(\beta+\varepsilon)t}\|f\| \qquad \textit{for all } t \geq 0.$$

This lemma is the main technical tool in the proof of the next theorem (cf. HALE [5, Th. 4.1]).

THEOREM 8.2. *For any real number* β, *let* $\Lambda = \Lambda(\beta) = \{\lambda \in P\sigma(A) \mid \mathrm{Re}\,\lambda \geq \beta\}$ *and suppose* X *is decomposed by* Λ *as in* (8.5). *Then there exist positive constants* K *and* γ *such that*

$$\|T(t)f^P\| \leq Ke^{(\beta-\gamma)t}\|f^P\|, \qquad t \leq 0.$$

$$\|T(t)f^Q\| \leq Ke^{(\beta-\gamma)t}\|f^Q\|, \qquad t \geq 0.$$

Here f^P *and* f^Q *denote the projection of an arbitrary element* f *of* X *onto* P *and* Q, *respectively.*

PROOF. First of all we observe that Λ consists of finitely many eigenvalues, $\lambda_1,\ldots,\lambda_q$ say, and that $\delta := \sup\{\mathrm{Re}\,\lambda \mid \lambda \in P\sigma(A)\setminus\Lambda\} < \beta$. Let $\tilde{T}(t)$ denote the restriction of $T(t)$ to Q. The compactness of $\tilde{T}(b)$ implies that $\rho(\tilde{T}(b)) = e^{\delta b}$ and consequently, by Lemma 8.1, for any $\varepsilon > 0$

$$\|T(t)f^Q\| \leq K(\varepsilon)e^{(\delta+\varepsilon)t}\|f^Q\|, \qquad t \geq 0.$$

If we choose $\varepsilon < \beta-\delta$ the desired result follows.

Finally, the estimate for $T(t)f^P$ follows from the explicit expression (8.3) by noting that the set of eigenvalues of the matrix $D = \mathrm{diag}(D(\lambda_1),\ldots \ldots,D(\lambda_q))$ coincides with Λ. □

We observe that it follows from the theorem above that $f = 0$ is exponentially asymptotically stable if all the roots of the characteristic equation $\det \Delta(\lambda) = 0$ have negative real parts.

In Theorem 8.2 we have obtained very useful information about autonomous problems. As a next step we investigate how one can exploit the splitting of X in the study of forced systems. More precisely, we want to obtain a decomposition of the variation-of-constants formula (cf. HALE [5, Section 7.6]).

LEMMA 8.3. *(A useful identity). Let* $\alpha,\beta \in \mathbb{R}$, $f^{\alpha} \in X$, $\psi_{\beta} \in X^{*}$ *and* $h \in L_1^{loc}(\mathbb{R})$ *be given. Let* x *denote the solution of*

$$x_{\alpha} = B{*}x_{\alpha} + f^{\alpha} + h_{\alpha}$$

and let y *denote the solution of the adjoint equation*

$$\begin{cases} y_{\beta}(t) = \int_0^b y_{\beta}(t-\tau)B(\tau)d\tau, & t \geq 0, \\ y_{\beta}(t) = \psi_{\beta}(t), & -b \leq t \leq 0. \end{cases}$$

Then

$$\langle y_{-s},f^{s}\rangle = \langle y_{-\sigma},f^{\sigma}\rangle + \int_{\sigma}^{s} y(-\tau)h(\tau)d\tau,$$

for $\alpha \leq \sigma \leq s \leq -\beta$.

PROOF. From the variation-of-constants formula (cf. Theorem 6.1)

$$\begin{aligned} f^{s} &= T(s-\sigma)f^{\sigma} + \int_{\sigma}^{s} T(s-\tau)Bh(\tau)d\tau \\ &= T(s-\sigma)f^{\sigma} + \int_{\sigma}^{s} \underset{\sim}{Q}(\cdot,s-\tau)h(\tau)d\tau, \end{aligned}$$

we deduce that we can write $\langle y_{-s},f^{s}\rangle = C_1 + C_2$ with

$$C_1 = \langle y_{-s},T(s-\sigma)f^{\sigma}\rangle = \langle T^{*}(s-\sigma)y_{-s},f^{\sigma}\rangle = \langle y_{-\sigma},f^{\sigma}\rangle$$

and

$$C_2 = \int_0^b y(-s-t) \int_\sigma^s Q(t,s-\tau)h(\tau)d\tau dt$$

$$= \int_\sigma^s \int_0^b y(-s-t)Q(t,s-\tau)dt\ h(\tau)d\tau$$

$$= \int_\sigma^s (T(s)^* y_{-s})(-\tau)h(\tau)d\tau$$

$$= \int_\sigma^s y(-\tau)h(\tau)d\tau. \qquad \square$$

Let, as before, X be decomposed by $\Lambda = \{\lambda_1,\ldots,\lambda_q\}$ as $P \oplus Q$ and let $F = (F(\lambda_1),\ldots,F(\lambda_q))$, $\Psi = (\Psi(\lambda_1),\ldots,\Psi(\lambda_q))$, $D = \mathrm{diag}(D(\lambda_1),\ldots,D(\lambda_q))$. Since $e^{Dt}\Psi(0)$ is a solution of the homogeneous adjoint equation defined for $-\infty < t < \infty$ we can write

$$(f^s)^P = F\langle\Psi,f^s\rangle$$

$$= F\{\langle\Psi_{s-\sigma},f^\sigma\rangle + \int_\sigma^s \Psi(s-\tau)h(\tau)d\tau\}$$

$$= Fe^{D(s-\sigma)}\langle\Psi,f^\sigma\rangle + \int_\sigma^s Fe^{D(s-\tau)}\Psi(0)h(\tau)d\tau$$

$$= T(s-\sigma)F\langle\Psi,f^\sigma\rangle + \int_\sigma^s T(s-\tau)F\Psi(0)h(\tau)d\tau$$

$$= T(s-\sigma)(f^\sigma)^P + \int_\sigma^s T(s-\tau)B^P h(\tau)d\tau,$$

where by definition

$$B^P = F\Psi(0).$$

Clearly this is a variation-of-constants formula involving only the P-component of f^s. Finally, using this formula, the variation-of-constants formula (6.8) and $(f^s)^Q = f^s - (f^s)^P$ we obtain a similar formula for the Q-component:

$$(f^s)^Q = T(s-\sigma)(f^\sigma)^Q + \int_\sigma^s T(s-\tau)B^Q h(\tau)d\tau,$$

where

$$B^Q = B - B^P = B - F\Psi(0).$$

In conclusion of this section we observe that the calculations above also show that the coefficients of $(f^s)^P$ with respect to the basis F satisfy an (inhomogeneous) ordinary differential equation. Indeed, if we define

$$z(s) = \langle\Psi,f^s\rangle,$$

then

$$z(s) = e^{D(s-\sigma)}z(\sigma) + \int_\sigma^s e^{D(s-\tau)}\Psi(0)h(\tau)d\tau,$$

and consequently

$$z'(s) = Dz(s) + \Psi(0)h(s).$$

9. A FREDHOLM ALTERNATIVE FOR PERIODIC SOLUTIONS

Let $\mathcal{P}(\omega)$ denote the set of ω-periodic L_1^{loc}-functions. We shall exploit the machinery developed in the preceding sections in the proof of the following theorem.

THEOREM 9.1. *For given* $h \in \mathcal{P}(\omega)$ *there exists* $x \in \mathcal{P}(\omega)$ *such that*

$$x(t) = \int_0^b B(\tau)x(t-\tau)d\tau + h(t), \tag{9.1}$$

if and only if

$$(9.2)\qquad \int_0^\omega y(-\tau)h(\tau)d\tau = 0,$$

for all ω-periodic $y \in L_\infty$ *which satisfy the homogeneous adjoint equation*

$$(9.3)\qquad y(t) = \int_0^b y(t-\tau)B(\tau)d\tau.$$

PROOF. Let $h \in P(\omega)$ be given. Let f^0 be an arbitrary element of X and let $x = x(t;f^0)$ denote the solution of

$$(9.4)\qquad x = B{*}x + f^0 + h.$$

We note that x satisfies (9.1) for $t \geq b$. So if $x \in P(\omega)$ then, on account of the periodicity, (9.1) is satisfied for all values of t. So our problem is to determine necessary and sufficient conditions on h for the existence of $f^0 \in X$ such that $x(\cdot;f^0) \in P(\omega)$. From the formula

$$(9.5)\qquad x_\omega = B{*}x_\omega + f^\omega + h_\omega,$$

we deduce that $x \in P(\omega)$ if and only if $f^\omega = f^0$. Since $f^\omega = T(\omega)f^0 + V(\omega)h$ it follows that we are interested in obtaining solutions in X of the operator equation

$$(9.6)\qquad (I-T(\omega))f^0 = V(\omega)h.$$

We recall that $T(\omega) = U(\omega) + V(\omega)$ and that $U(\omega)$ is nilpotent and $V(\omega)$ compact (this follows from the definitions in a straightforward manner). Consequently $I - T(\omega)$ has closed range and

$$\mathcal{R}(I-T(\omega)) = {}^\perp\mathcal{N}(I-T(\omega)^*).$$

So (9.6) has a solution if and only if $\langle y,V(\omega)h\rangle = 0$ for all $y \in \mathcal{N}(I-T(\omega)^*)$. Furthermore,

$$\langle y,V(\omega)h\rangle = \int_0^b y(-\tau) \int_0^\omega Q(\tau,\omega-\sigma)h(\sigma)\,d\sigma d\tau$$

$$= \int_0^\omega \int_0^b y(-\tau)Q(\tau,\omega-\sigma)\,d\tau\ h(\sigma)\,d\sigma$$

$$= \int_0^\omega (T(\omega)^*y)(-\sigma)h(\sigma)\,d\sigma$$

$$= \int_0^\omega y(-\sigma)h(\sigma)\,d\sigma.$$

Finally, we observe that there is a one-to-one correspondence (given by restriction or extension, respectively) between elements of $N(I-T(\omega)^*)$ and ω-periodic solutions of the adjoint equation (9.3). □

APPENDIX

THE RANGE AND THE NULL SPACE OF $(A-\lambda I)^k$

Our presentation follows closely HALE [5, Section 7.3]. $(A-\lambda I)f_1 = g$ implies

$$f_1(t) = e^{\lambda t}\{(I - \int_0^t e^{-\lambda s}B(s)ds)f_1(0) + \int_0^t e^{-\lambda s}g(s)ds\}.$$

By induction it follows that if $(A-\lambda I)^k f_k = g$ and $(A-\lambda I)f_\ell = f_{\ell-1}$, $\ell = k,k-1,\ldots,2$, then

$$f_k(t) = e^{\lambda t}\left\{\sum_{j=0}^{k-1} (t^j I - \int_0^t (t-s)^j e^{-\lambda s}B(s)ds)\,\frac{f_{k-j}(0)}{j!} + \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}\,e^{-\lambda s}g(s)ds\right\}. \tag{A1}$$

Taking $t = b$, using the binomial expansion for $(b-s)^j$ and interchanging the order of the summation we find

$$f_k(b) = e^{\lambda b}\sum_{m=1}^{k}\frac{b^{m-1}}{(m-1)!}\left\{\sum_{i=0}^{k-m} P_i f_{k-i-m+1}(0) + \frac{1}{(k-m)!}\int_0^b (-s)^{k-m}e^{-\lambda s}g(s)ds\right\},$$

where by definition

$$P_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\Delta(\lambda). \tag{A2}$$

Clearly $f_k \in \mathcal{D}(A-\lambda I)^k$ iff $f_m(b) = 0$ for $m = 1,\ldots,k$, i.e., iff

$$\sum_{i=0}^{m-1} P_i f_{m-i}(0) + \frac{1}{(m-1)!}\int_0^b (-s)^{m-1}e^{-\lambda s}g(s)ds = 0 \quad \text{for } m = 1,\ldots,k. \tag{A3}$$

In order to arrive at a compact formulation of this condition we introduce matrices A_k of dimension $(kn) \times (kn)$ and column-vectors F_k and G_k as follows:

$$A_k = \begin{pmatrix} P_0 & 0 & 0 & \cdots & 0 \\ P_1 & P_0 & 0 & \cdots & 0 \\ P_2 & P_1 & P_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ P_{k-1} & P_{k-2} & P_{k-3} & \cdots & P_0 \end{pmatrix}$$

$$F_k = \operatorname{col}(f_1(0),\ldots,f_k(0)),$$

$$G_k = \operatorname{col}\Big(\int_0^b e^{-\lambda s} g(s)\,ds, \int_0^b (-s)e^{-\lambda s} g(s)\,ds,\ldots,$$

$$\ldots,\frac{1}{(k-1)!}\int_0^b (-s)^{k-1} e^{-\lambda s} g(s)\,ds\Big).$$

With this notation (A3) is equivalent to the condition

(A4) $$A_k F_k + G_k = 0.$$

THEOREM A.1.

(i) $N(A-\lambda I)^k$ *consists of functions* f *of the form*

$$f(t) = e^{\lambda t}\Big\{\sum_{j=0}^{k-1} (t^j I - \int_0^t (t-s)^j e^{-\lambda s} B(s)\,ds)\frac{e_{k-j}}{j!}\Big\},$$

where $E = \operatorname{col}(e_1,\ldots,e_k)$ *satisfies* $A_k E = 0$.

(ii) $g \in R(A-\lambda I)^k$ *iff* $C_k G_k = 0$ *for all row vectors* C_k *such that* $C_k A_k = 0$.

PROOF.

(i) is a consequence of (A_1) and (A_4) in the special case $g = 0$.

(ii) is the standard solvability condition for the matrix equation (A4) as obtained from Fredholm's alternative. □

Next we calculate the null space of $(A^*-\lambda I)^k$. Clearly $(A^*-\lambda I)^k \phi_k = 0$ implies

$$\phi_k(\theta) = \sum_{m=1}^{k} c_m \frac{\theta^{m-1}}{(m-1)!} e^{\lambda\theta}.$$

Hence

$$(A^*-\lambda I)^{\ell}\phi_k(\theta) = \sum_{m=1}^{k-\ell} c_{m+\ell} \frac{\theta^{m-1}}{(m-1)!} e^{\lambda\theta},$$

and the condition $(A^*-\lambda I)^{\ell}\phi_k \in \mathcal{D}(A^*)$ for $\ell = 0,\dots,k-1$, can be written as

$$\sum_{m=1}^{k-\ell} c_{m+\ell}P_{m-1} = 0 \qquad \text{for } \ell = 0,\dots,k-1,$$

or, in other words, as

$$C_kA_k = 0,$$

where by definition $C_k = \text{row}(c_1,\dots,c_k)$. Thus we obtain

<u>THEOREM A.2.</u>

(i) $N(A^*-\lambda I)^k$ *consists of functions* ϕ *of the form*

$$\phi(\theta) = \sum_{j=1}^{k} c_j \frac{\theta^{j-1}}{(j-1)!} e^{\lambda\theta},$$

where $C = \text{row}(c_1,\dots,c_k)$ *satisfies* $CA_k = 0$.

(ii) $g \in R(A-\lambda I)^k$ *iff* $\langle\phi,g\rangle = 0$ *for all* $\phi \in N(A^*-\lambda I)^k$.

In the special case of one equation ($n = 1$) things simplify a little. Suppose $P_i = 0$ for $i = 0,1,\dots,M$ and $P_{M+1} \neq 0$, i.e.,

$$\int_0^b e^{-\lambda s}B(s)\,ds = 1, \qquad \int_0^b s^i e^{-\lambda s}B(s)\,ds = 0 \qquad \text{for } i = 1,\dots,M$$

and

$$\int_0^b s^{M+1}e^{-\lambda s}B(s)\,ds \neq 0.$$

From (A3) we deduce that $g \in R(A-\lambda I)^k$ iff

$$\int_0^b s^m e^{-\lambda s} g(s)\,ds = 0 \qquad \text{for } m = 0,1,\ldots,\min(M,k-1).$$

Moreover, it follows at once from Theorem A.1(i) that the dimension of the generalized eigenspace equals the multiplicity of λ as a zero of det $\Delta(\lambda)$. This is true in the general case as well, but the proof requires some more linear algebraic manipulations, We refer to LEVINGER [11] for a proof in the context of retarded functional differential equations which applies, mutatis mutandis, to the present situation equally well.

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