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CONVOLUTION OPERATORS FOR FOURIER-JACOBI EXPANSIONS

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Convolution operators for Fourier-Jacobi expansions

by

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1. Introduction

1.1. In some recent work [3], [4], the author has used the convolution structure for Jacobi series, introduced by Askey and Wainger [2], in order to study the summation of Jacobi series by classical summability methods. Many of these summability methods, in fact, can be interpreted as convolution operators and it is possible to investigate the order of approximation of these operators by the same techniques as are used for trigonometric convolution operators. In this paper some new summability kernels are introduced, which can be written in a simple closed form by means of Jacobi polynomials. Even in the case of Fourier series ($\alpha = \beta = -\frac{1}{2}$) these kernels induce new approximation processes. The saturation order and the saturation class of these processes are obtained.

1.2. By X we denote one of the following function spaces on $[-1,1]$: the space C of continuous functions with the norm ($x = \cos\theta$)

$$||f||_C = \sup_{0\leq\theta\leq\pi} |f(\cos\theta)|$$

or the L^p spaces ($1 \leq p < \infty$) with respect to the weight function

$$\rho^{(\alpha,\beta)}(\theta) = (\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1} \qquad (\alpha \geq \beta \geq -\tfrac{1}{2}) \tag{1.1}$$

endowed with the norm

$$||f||_p = [\int_0^\pi |f(\cos\theta)|^p \rho^{(\alpha,\beta)}(\theta)\, d\theta]^{1/p}.$$

Functions belonging to X can be expanded in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$, the polynomials which are orthogonal with respect to (1.1). If we take

$$R_n^{(\alpha,\beta)}(\cos\theta) = \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)} ,$$

then with $f \in X$ we associate

$$(1.2) \qquad f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta),$$

where

$$(1.3) \qquad f^{\wedge}(n) = \int_0^{\pi} f(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta, \qquad (n=0,1,\ldots),$$

and

$$(1.4) \qquad \omega_n^{(\alpha,\beta)} = \Big[\int_0^{\pi} \{R_n^{(\alpha,\beta)}(\cos\theta)\}^2\, \rho^{(\alpha,\beta)}(\theta)\, d\theta\Big]^{-1}$$

$$= \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\alpha+1)} .$$

Askey and Wainger [2] have introduced a generalized translation and a convolution structure for Jacobi series. The translation $T_\phi f$ maps a function $f \in X$ with the expansion (1.2) into

$$(1.5) \qquad T_\phi\, f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)} \cos\phi).$$

Gasper [7] has shown that T_ϕ is a positive operator and consequently has operator norm 1. For $f_1, f_2 \in L^1$ the convolution $f_1 * f_2$ is defined by

$$(1.6) \qquad (f_1 * f_2)(\cos\theta) = \int_0^{\pi} T_\phi\, f_1(\cos\theta)\, f_2(\cos\phi)\, \rho^{(\alpha,\beta)}(\phi)\, d\phi$$

and has the following properties.

1.3. Theorem. For f_1, f_2, $f_3 \in L^1$ and $g \in X$

$$(1.7)\quad \begin{cases} \text{i)} & f_1 * f_2 = f_2 * f_1, \\ \text{ii)} & f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3, \\ \text{iii)} & \|f_1 * g\|_X \le \|f_1\|_1 \; \|g\|_X, \\ \text{iv)} & (f_1 * f_2)^\wedge(n) = f_1^\wedge(n) \; f_2^\wedge(n). \end{cases}$$

In [3] we have defined a summability kernel.

1.4. Definition. Let $K_\lambda \in L^1$, $\lambda > 0$, satisfying the conditions

(a) $$\int_0^\pi K_\lambda(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta = 1,$$

(b) $$K_\lambda(\cos\theta) \ge 0, \qquad \lambda > 0,\ 0 \le \theta \le \pi,$$

(c) $$\lim_{\lambda\to\infty} K_\lambda^\wedge(n) = 1, \qquad n = 0,1,\dots\ .$$

Then we call K_λ a positive summability kernel. If instead of b we merely have

(b') $$\int_0^\pi |K_\lambda(\cos\theta)|\, \rho^{(\alpha,\beta)}(\theta)\, d\theta \le N, \text{ uniformly in } \lambda,$$

with $N \ge 1$, we call K_λ a quasi-positive kernel.
Condition c is often replaced by

(c') $$\lim_{\lambda\to\infty} \int_h^\pi |K_\lambda(\cos\theta)|\, \rho^{(\alpha,\beta)}(\theta)\, d\theta = 0, \quad \text{for each } h,\ 0 < h \le \pi.$$

We have ([3], theorems 3.3 and 3.4)

1.5. Theorem. If $f \in X$ and $K_\lambda \in L^1$, $\lambda > 0$, satisfies the conditions a, b (or b') and c (or c') of definition 1.4, then

$$||K_\lambda * f||_X \leq N||f||_X, \qquad \text{uniformly in } \lambda,$$

and

$$\lim_{\lambda\to\infty} ||K_\lambda * f - f||_X = 0.$$

The convolution operators $\{K_\lambda, \lambda>0\}$ are said to be saturated if there exists a positive non-increasing function $\phi(\lambda)$ on $0 < \lambda < \infty$ with $\lim_{\lambda\to\infty} \phi(\lambda) = 0$ such that

i) $$||K_\lambda * f - f||_X = o(\phi(\lambda)) \qquad (\lambda\to\infty)$$

if and only if f belongs to some 'trivial' subspace of X;

ii) there exists a 'non-trivial' element $f_0 \in X$ satisfying

$$||K_\lambda * f_0 - f_0||_X = O(\phi(\lambda)) \qquad (\lambda\to\infty).$$

The function $\phi(\lambda)$ is then called the saturation order and the set $F(X, K_\lambda)$, which consists of all the elements of X which satisfy ii, is called the saturation class or Farvard class of $\{K_\lambda\}$.

The Lipschitz classes with respect to the generalized translation are defined by

$$(1.8)\ \mathrm{Lip}(\gamma, X) = \{f \in X : \exists\, c > 0,\ \sup_{0<\psi<\phi} ||T_\psi f - f||_X \leq c\phi^\gamma\}, \qquad (0 < \gamma \leq 2).$$

If for $f \in X$ with the expansion (1.2) there exists an element $Af \in X$ such that

$$(1.9) \qquad Af \sim \sum_{n=0}^{\infty} n(n+\alpha+\beta+1)\ f^\wedge(n)\ \omega_n^{(\alpha,\beta)}\ R_n^{(\alpha,\beta)}(\cos\theta),$$

then we say that $f \in D(A)$ and we call A the operator which maps D(A) into X by $f \to Af$. The operator A is the realization in A of the differential operator

$$-\frac{1}{\rho^{(\alpha,\beta)}(\theta)}\ \frac{d}{d\theta}\ \{\rho^{(\alpha,\beta)}(\theta)\ \frac{d}{d\theta}\}$$

with boundary conditions $\frac{d}{d\theta} = 0$ at $\theta = 0$ and $\theta = \pi$ in view of the differential equation for Jacobi polynomials

$$-\frac{1}{\rho^{(\alpha,\beta)}(\theta)} \frac{d}{d\theta} \{\rho^{(\alpha,\beta)}(\theta) \frac{d}{d\theta} R_n^{(\alpha,\beta)}(\cos\theta)\} = n(n+\alpha+\beta+1)\, R_n^{(\alpha,\beta)}(\cos\theta).$$

There is a close connection between the generalized translation operator and the operator A, as Löfström and Peetre showed in [8]. For $f \in D(A)$, in fact, we have

$$\lim_{\phi\to 0^+} \left|\left|\frac{f - T_\phi f}{C_1(\phi)} - Af\right|\right|_X = 0,$$

where

$$C_1(\phi) = \int_0^\phi \frac{1}{\rho^{(\alpha,\beta)}(\theta)} \left(\int_0^\theta \rho^{(\alpha,\beta)}(\tau)\, d\tau\right) d\theta = O(\phi^2), \qquad \phi \to 0^+.$$

Moreover, if the K function norm, introduced by Peetre [9] is given by

$$K(\phi,f;X,D(A)) = \inf_{\substack{f=f_0+f_1\\ f_0\in X\\ f_1\in D(A)}} (||f_0||_X+\phi||f_1||_{D(A)}),$$

where

$$||f||_{D(A)} = ||f||_X + ||Af||_X,$$

then it can be shown (Löfström-Peetre [8], Bavinck [3]) that the spaces

$$(X,D(A))_{\theta,\infty;K} = \{f \in X\colon \sup_{\phi>0} \phi^{-\theta}\, K(\phi,f;X,D(A)) < \infty\}, \qquad 0 < \theta \leq 1,$$

coincide with the spaces Lip$(2\theta,x)$, defined by (1.8).

Notation: We shall use the notation $a_n \approx b_n$ $(n\to\infty)$ if there are positive numbers c_1 and c_2 such that $c_1 a_n \leq b_n \leq c_2 a_n$.

2. An oscillating kernel

The following formula is due to Szegö ([11], section 9.4)

$$(2.1)\quad R_n^{(\alpha+k+1,\beta)}(\cos\theta) =$$

$$= \frac{\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+k+2)\Gamma(\alpha+1)}{\Gamma(n+\alpha+\beta+k+2)\Gamma(n+\alpha+k+2)\Gamma(k+1)} \sum_{\nu=0}^{n} \frac{\Gamma(n+\nu+\alpha+\beta+k+2)\Gamma(n-\nu+k+1)}{\Gamma(n+\nu+\alpha+\beta+2)\Gamma(n-\nu+1)} \omega_\nu^{(\alpha,\beta)} R_\nu^{(\alpha,\beta)}(\cos\theta)$$

$$(k > 0).$$

We shall study here the polynomial kernel $J_{n,1}^{(\alpha+k+1,\beta)}(\cos\theta)$, $n \geq 0$, defined by

$$(2.2)\quad J_{n,1}^{(\alpha+k+1,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+k+2)\Gamma(k+1)}{\Gamma(n+\beta+1)\Gamma(n+k+1)\Gamma(\alpha+k+2)\Gamma(\alpha+1)} R_n^{(\alpha+k+1,\beta)}(\cos\theta)$$

$$= O(n^{2\alpha+2})\, R_n^{(\alpha+k+1,\beta)}(\cos\theta).$$

We first show that $J_{n,1}^{(\alpha+k+1,\beta)}(\cos\theta)$ satisfies the conditions a, b' and c' in definition 1.4. Taking the term $\nu = 0$ in (2.1) we see that condition a is satisfied. For the proof of condition b' we use the well-known estimates (Szegö [11] (7.32.5))

$$(2.3)\quad R_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{\alpha-\frac{1}{2}}\, O(n^{-\frac{1}{2}-\alpha}) & \text{if } cn^{-1} \leq \theta \leq \frac{\pi}{2}, \\ O(1) & \text{if } 0 \leq \theta \leq cn^{-1}. \end{cases}$$

Then

$$\int_0^\pi |R_n^{(\alpha+k+1,\beta)}(\cos\theta)|\, \rho^{(\alpha,\beta)}(\theta) = \int_0^{1/n} + \int_{1/n}^{\pi/2} + \int_{\pi/2}^{\pi} = I_1 + I_2 + I_3.$$

$$I_1 = O(1)\int_0^{1/n} \theta^{2\alpha+1}\, d\theta = O(n^{-2\alpha-2}),$$

$$I_2 = O(n^{-\frac{1}{2}-\alpha-k-1})\int_{1/n}^{\pi/2} \theta^{\alpha-k-\frac{1}{2}}\, d\theta = O(n^{-2\alpha-2}), \qquad \text{if } k > \alpha + \tfrac{1}{2},$$

$$I_3 = O(n^{\beta-\alpha-k-1}) \int_0^{\pi/2} |R_n^{(\beta,\alpha+k+1)}(\cos\theta)|\ \rho^{(\alpha,\beta)}(\theta)\ d\theta = O(n^{-\alpha-k-\frac{1}{2}}).$$

Hence $\displaystyle\int_0^{\pi} |J_{n,1}^{(\alpha+k+1,\beta)}(\cos\theta)|\ \rho^{(\alpha,\beta)}(\theta)\ d\theta \le M,$ if $k > \alpha + \frac{1}{2}$.

For the proof of c' we take $n^{1/2} > \frac{1}{h}$. Then for $k > \alpha + \frac{1}{2}$

$$O(n^{2\alpha+2}) \int_h^{\pi} |R_n^{(\alpha+k+1,\beta)}(\cos\theta)|\ \rho^{(\alpha,\beta)}(\theta)\ d\theta \le$$

$$\le O(n^{2\alpha+2})\Big(\int_{n^{-\frac{1}{2}}}^{\pi/2} + \int_{\pi/2}^{\pi}\Big) =$$

$$= O(n^{\frac{1}{2}(\alpha+\frac{1}{2}-k)}) + O(n^{(\alpha+\frac{1}{2}-k)}) = o(1).$$

When we use the notation $\lambda_n = n(n+\alpha+\beta+1)$, the kernel $J_{n,1}^{(\alpha+k+1,\beta)}(\cos\theta)$ has the following representation in the case k is an integer

$$(2.4)\qquad J_{n,1}^{(\alpha+k+1,\beta)}(\cos\theta) = \sum_{\nu=0}^{n} \prod_{j=0}^{k-1} \Big(1 - \frac{\lambda_\nu}{\lambda_{n+j+1}}\Big)\ \omega_\nu^{(\alpha,\beta)}\ R_\nu^{(\alpha,\beta)}(\cos\theta).$$

This representation shows that this kernel is essentially a generalization of the typical means (see Butzer-Nessel [6], p.262). Also, as is shown by Szegö [11], section 9.41, this kernel is closely related to the Cesàro means. In the case of Fourier series ($\alpha = \beta = -\frac{1}{2}$) we can choose $k = 1$ and we get

$$(2.5)\qquad J_{n,1}^{(\frac{3}{2},-\frac{1}{2})}(\cos\theta) = \frac{1}{\pi}\Big\{1 + 2\sum_{\nu=1}^{n} \Big(1 - \Big(\frac{\nu}{n+1}\Big)^2\Big)\cos\nu\theta\Big\},$$

which are typical means. From the relation

$$R_n^{(\alpha+1,\beta)}(\cos\theta) = \frac{(\alpha+1)}{2n+\alpha+\beta+2}\ \frac{R_n^{(\alpha,\beta)}(\cos\theta) - R_{n+1}^{(\alpha,\beta)}(\cos\theta)}{\sin^2\frac{\theta}{2}}$$

we derive

$$J_{n,1}^{(\frac{3}{2},-\frac{1}{2})}(\cos\theta) = \frac{1}{4\pi(n+1)^2}\left[\frac{(2n+3)\sin(2n+1)\frac{\theta}{2} - (2n+1)\sin(2n+3)\frac{\theta}{2}}{(\sin\frac{\theta}{2})^3}\right].$$

For the other values of $k > 0$, the kernel $J_{n,1}^{(\frac{1}{2}+k,-\frac{1}{2})}(\cos\theta)$ differs slightly from the Riesz means.

For the convolution of a function $f \in X$ with the kernel (2.2) the following relations are valid.

$$(2.6)\quad J_{n,1}^{(\alpha+k+2,\beta)}*f - J_{n-1,1}^{(\alpha+k+2,\beta)}*f =$$

$$= \frac{(k+1)(2n+k+\alpha+\beta+2)}{n(n+\alpha+\beta+1)(n+k+1)(n+\alpha+\beta+k+2)} J_{n,1}^{(\alpha+k+1,\beta)}*Af,$$

where A is the operator defined by (1.9). In the case k = integer, formula (2.6) is easy to check by using the representation (2.4). Furthermore, we have

$$(2.7)\quad J_{n,1}^{(\alpha+k+2,\beta)}*f - J_{n,1}^{(\alpha+k+1,\beta)}*f = -\frac{1}{(n+k+1)(n+\alpha+\beta+k+2)} J_{n,1}^{(\alpha+k+1,\beta)}*Af.$$

By theorem 1.5 and by repeated application of (2.6) we obtain

$$f - J_{n,1}^{(\alpha+k+2,\beta)}*f =$$

$$= (k+1)\sum_{l=n+1}^{\infty} \frac{(2l+k+\alpha+\beta+2)}{l(l+\alpha+\beta+1)(l+k+1)(l+\alpha+\beta+k+2)} J_{l,1}^{(\alpha+k+1,\beta)}*Af.$$

Then, by (2.7)

$$f - J_{n,1}^{(\alpha+k+1,\beta)}*f = -\frac{1}{(n+k+1)(n+\alpha+\beta+k+2)} J_{n,1}^{(\alpha+k+1,\beta)}*Af$$

$$+ (k+1)\sum_{l=n+1}^{\infty} \frac{(2l+k+\alpha+\beta+2)}{l(l+\alpha+\beta+1)(l+k+1)(l+\alpha+\beta+k+2)} J_{l,1}^{(\alpha+k+1,\beta)}*Af.$$

If we put

$$c_n = k \sum_{l=n+1}^{\infty} \frac{(2l+k+\alpha+\beta+2)}{l(l+\alpha+\beta+1)(l+k+1)(l+\alpha+\beta+k+2)} = O(n^{-2}), \qquad (n\to\infty),$$

then

$$||c_n^{-1}(f-J_{n,1}^{(\alpha+k+1,\beta)}*f) - Af||_X$$

$$\leq \frac{(k+1)}{k} \sup_{l\geq n+1} ||J_{l,1}^{(\alpha+k+1,\beta)}*Af - Af||_X +$$

$$+ \frac{c_n^{-1}}{(n+k+1)(n+\alpha+\beta+k+2)} ||J_{n,1}^{(\alpha+k+1,\beta)}*Af - Af||_X +$$

$$+ ||Af||_X \left|\frac{1}{k} - \frac{c_n^{-1}}{(n+k+1)(n+\alpha+\beta+k+2)}\right| =$$

$$= o(1), \qquad (n\to\infty).$$

Hence, $\lim_{n\to\infty} [c_n^{-1} \{f-J_{n,1}^{(\alpha+k+1,\beta)}*f\} - Af] = 0$ in X.

Since for the operator A an inequality of the Bernstein type is valid (see Stein [10]), there is a constant M such that

$$||A(J_{n,1}^{(\alpha+k+1,\beta)}*f)||_X \leq Mn^2||f||_X.$$

Therefore, the approximation process $J_{n,1}^{(\alpha+k+1,\beta)}*f$ satisfies all the conditions of the general theorems on approximation processes in Banach spaces treated in Berens [5] (see also Butzer-Nessel [6], 13.4.1). It follows that the process $J_{n,1}^{(\alpha+k+1,\beta)}*f$ is saturated with order n^{-2} and the saturation class is $\mathrm{Lip}(2,X) = (X,D(A))_{1,\infty;K}$. The spaces of non-optimal approximation can also be characterized in terms of the intermediate spaces $(X,D(A))_{\theta,q;K}$.

3. A positive kernel

We now consider the positive polynomial kernel $J_{n,2}^{(\alpha+k+1,\beta)}(\cos\theta)$, which is given by

$$(3.1)\qquad J_{n,2}^{(\alpha+k+1,\beta)}(\cos\theta) = c_n[R_n^{(\alpha+k+1,\beta)}(\cos\theta)]^2,$$

where

$$(3.2)\qquad c_n^{-1} = \int_0^\pi [R_n^{(\alpha+k+1,\beta)}(\cos\theta)]^2\,\rho^{(\alpha,\beta)}(\theta)\,d\theta.$$

The following useful estimate can be derived by means of (2.3) and some asymptotic formulas for Jacobi polynomials (see Szegö [11], (7.34.1) for similar estimates). For $\alpha+k+1$, $\alpha+l$, β each greater than $-\frac{1}{2}$, we have for $n \to \infty$

$$(3.3)\qquad \int_0^\pi [R_n^{(\alpha+k+1,\beta)}(\cos\theta)]^2\rho^{(\alpha+l,\beta)}(\theta)d\theta \approx \begin{cases} n^{-2\alpha-2l-2} & , 2l < 2k+1, \\ n^{-2\alpha-2k-3}\log n, & 2l = 2k+1, \\ n^{-2\alpha-2k-3} & , 2l > 2k+1. \end{cases}$$

From (3.3) it follows that $c_n \approx n^{2\alpha+2}$, if $k > -\frac{1}{2}$.

For the kernel (3.1) the conditions a and b of definition 1.4 are trivially satisfied. We verify condition c', choosing $n > h^{-2}$. By (2.3)

$$O(n^{2\alpha+2}) \int_h^\pi [R_n^{(\alpha+k+1,\beta)}(\cos\theta)]^2\,\rho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$= O(n^{2\alpha+2})\,[\int_{n^{-\frac{1}{2}}}^{\pi/2} + \int_{\pi/2}^{\pi}\,] =$$

$$= O(n^{-2k-1})\,[\int_{n^{-\frac{1}{2}}}^{\pi/2} \theta^{-2k-2}\,d\theta + \int_0^{\pi/2} d\theta] = o(1) \qquad (n \to \infty,\ k > -\tfrac{1}{2}).$$

Hence for $k > -\frac{1}{2}$ the convolution of a function $f \in X$ with $J_{n,2}^{(\alpha+k+1,\beta)}$ ($k > -\frac{1}{2}$) approximates the function f in the X norm as $n \to \infty$. The trigonometric moments of order σ for a kernel K_λ ($\cos\theta$) are defined by

$$T(K_\lambda;\sigma) = \int_0^\pi (\sin\frac{\theta}{2})^\sigma K_\lambda(\cos\theta)\,\rho^{(\alpha,\beta)}(\theta)\,d\theta =$$

$$= \int_0^\pi K_\lambda(\cos\theta)\,\rho^{(\alpha+\frac{\sigma}{2},\beta)}(\theta)\,d\theta.$$

Thus, formula (3.3) enables us to survey the asymptotic behavior of the trigonometric moments of the kernel $J_{n,2}^{(\alpha+k+1,\beta)}$. If $k > \frac{1}{2}$, then it follows that

$$T(J_{n,2}^{(\alpha+k+1,\beta)};4) = o(T(J_{n,2}^{(\alpha+k+1,\beta)};2)).$$

We may conclude by Bavinck [4], theorems 1.5 and 2.3, that the process $J_{n,2}^{(\alpha+k+1,\beta)}$, $k > \frac{1}{2}$, is saturated with order n^{-2} and that the saturation class is Lip(2,X). The kernel $J_{n,2}^{(\alpha+k+1,\beta)}$, with k sufficiently large, shows the same behavior as the higher order Jackson kernel. We have, in fact,

$$T(J_{n,2}^{(\alpha+k+1,\beta)};\sigma) \approx n^{-\sigma} \qquad (0 \leq \sigma < 2k+1).$$

For the Jackson kernel of order r

$$L_{n,r}(\theta) = \lambda_{n,r}^{-1} \left(\frac{\sin n\,\theta/2}{\sin\theta/2}\right)^{2r}$$

the same relation is valid (see Bavinck [3], (4.19))

$$T(L_{n,r};\sigma) \approx n^{-\sigma} \qquad (2r > 2\alpha+\sigma+2),$$

but for larger values of α a high order of the Jackson kernel is necessary to compete with the kernel $J_{n,2}^{(\alpha+k+1,\beta)}$.
In the case of Fourier series ($\alpha = \beta = -\frac{1}{2}$) the kernel $J_{n,2}^{(\frac{1}{2},-\frac{1}{2})}$ (the case $k = 0$) coincides with the Fejér kernel, but the relatively simple kernel

$J_{n,2}^{(\frac{3}{2},-\frac{1}{2})}$, which has the same optimal properties as the Jackson kernel has never been considered, as far as the author knows.

The case k = 1 will be studied in some more detail. In this case the constant c_n (see (3.2)) and the trigonometric moments can be computed explicitly by means of Parseval's formula. Substituting k = 1 in formula (2.4), we have

$$\int_0^\pi [J_{n,1}^{(\alpha+2,\beta)}(\cos\theta)]^2\, \rho^{(\alpha,\beta)}(\theta)\, d\theta = \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right)^2 \omega_\nu^{(\alpha,\beta)}.$$

The sum at the right-hand side can be evaluated, if one uses (2.4) for different values of k at the point θ = 0. After some calculation one obtains

$$c_n^{-1} = \frac{\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+3)\Gamma(\alpha+1)}{(\alpha+3)\Gamma(n+\alpha+\beta+3)\Gamma(n+\alpha+3)}\,(2n^2 + 2n(\alpha+\beta+3) + (\alpha+3)(\alpha+\beta+2)).$$

The second and the fourth trigonometric moments are easily computed.

$$T(J_{n,2}^{(\alpha+2,\beta)};2) = c_n \frac{(2n+\alpha+\beta+3)}{(\alpha+2)\ \omega_n^{(\alpha+2,\beta)}} = \frac{(\alpha+3)(\alpha+1)}{p_{2n}}$$

and

$$T(J_{n,2}^{(\alpha+2,\beta)};4) = c_n \frac{1}{\omega_n^{(\alpha+2,\beta)}} = \frac{(\alpha+3)(\alpha+2)(\alpha+1)}{(2n+\alpha+\beta+3)\ p_{2n}},$$

where

$$p_{2n} = 2n^2 + 2n(\alpha+\beta+3) + (\alpha+3)(\alpha+\beta+2).$$

The author has not succeeded in calculating the Fourier-Jacobi coefficients of the kernel $J_{n,2}^{(\alpha+2,\beta)}$, except the first few, which follow from the trigonometric moments. In the case $\alpha = \beta = -\frac{1}{2}$, k = 1, we have

$$J_{n,2}^{(\frac{3}{2},-\frac{1}{2})} = \frac{15}{\pi(n+1)(2n+1)(2n+3)(4n^2+8n+5)} \left[\frac{(2n+3)\sin(2n+1)\frac{\theta}{2} - (2n+1)\sin(2n+3)\frac{\theta}{2}}{4(\sin\frac{\theta}{2})^3}\right]^2$$

and the Fourier coefficients can be calculated by squaring the series (2.5), since we have

$$\cos n\theta \cos m\theta = \tfrac{1}{2}(\cos(n+m)\theta + \cos(n-m)\theta).$$

A simple representation of the product $R_n^{(\alpha,\beta)}(\cos\theta)\, R_m^{(\alpha,\beta)}(\cos\theta)$ in the general case is not known. However, in some important special cases (α,α), $(\alpha,-\frac{1}{2})$ and $(\alpha+1,\alpha)$ the coefficients $a(k,m,n)$ in the representation

$$R_n^{(\alpha,\beta)}(x)\, R_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)\, R_k^{(\alpha,\beta)}(x)$$

are available (see Askey [1], p.11).

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