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Department of Analysis, Algebra and Geometry Report AM-R8915 October

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Threshold and stability results for an age-structured epidemic model

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We study a mathematical model for an epidemic spreading in an age-structured population with age-dependent transmission coefficient. We formulate the model as an abstract Cauchy problem on a Banach space and show the existence and uniqueness of solutions. Next we derive some conditions which guarantee the existence and uniqueness for non-trivial steady states of the model. Finally the local and global stability for the steady states are examined.

1980 Mathematics Subject Classification: 92A15.
Key Words & Phrases: Age-structure, Epidemic model, Threshold theorem, Semigroup, Cauchy problem.

1. Introduction

In this paper, we consider a mathematical model for an epidemic spreading in an age-structured population where the transmission coefficient depends on age. The model is derived for an SIR disease in a constant-sized population, that is, a susceptible individual who contracts the disease will become infective but will eventually recover with permanent immunity and the total population is assumed to be in a demographically stationary state. The SIR-type age-independent epidemic model has already been investigated satisfactory, and its threshold theorem is well known (Hethcote, 1974). On the other hand, since the work of McKendrick (1926), it has been recognized that the age-structure of a population is an important factor which affects the dynamics of disease transmission. Therefore several authors introduced age-structure into their epidemic models (Hoppensteadt, 1974, 1975; Dietz, 1975; Gripenberg, 1983; Schenzle, 1984; Anderson and May, 1985; Dietz and Schenzle, 1985; Tudor, 1985; Andreasen, 1988; Busenberg, et al. 1988).

Recently, Greenhalgh (1988b) investigated the age-structured SIR-type epidemic model in case that the transmission coefficient depends on the age of both susceptibles and infectious, and he conjectured that;

(1) The threshold phenomenon can be formulated in terms of the spectral radius $r(T)$ of a certain integral operator T;
(2) An endemic steady state is possible if and only if $r(T)>1$ and if this state exists, it is unique;
(3) The equilibrium with no disease present always exists, and it is locally (in fact globally) stable if $r(T)<1$ and locally unstable if $r(T)>1$;
(4) For realistic values of parameters, the endemic equilibrium state is asymptotically stable.

Our main purpose in this paper is to prove Greenhalgh's conjecture. First, we shall formulate the model by using the McKendrick-type partial differential equation system. Then we rewrite it into an abstract Cauchy problem on a Banach space, and show the existence and uniqueness of its solutions. Next, under appropriate conditions, we shall prove the existence and uniqueness results for non-trivial steady states of this model. Finally, we investigate the local and global stability for the steady states.

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2. The basic model

We subdivide a closed population into three compartments containing susceptibles, infectives and immunes. There is no incubating class, so a person who catches the disease becomes infectious instantaneously. Moreover, we assume that the population is in a stationary demographic state. Let $N(a), 0 \leqslant a \leqslant \omega$ (the number ω denotes the life span of the population) be the density with respect to age of the total number of individuals. Under our assumption, $N(a)$ satisfies

$$N(a) = \mu^* N \exp(-\int_0^a \mu(\sigma) d\sigma), \tag{2.1}$$

where $\mu(a)$ denotes the instantaneous death rate at age a of the population, the constant N is the total size of the population and μ^* is the crude death rate. We assume that $\mu(a)$ is nonnegative, locally integrable on $[0, \omega)$ and satisfies

$$\int_0^\omega \mu(\sigma) d\sigma = +\infty.$$

The crude death rate is determined such that

$$\mu^* \int_0^\omega \ell(a) da = 1,$$

where

$$\ell(a) := \exp(-\int_0^a \mu(\sigma) d\sigma),$$

is the *survival function* which is the proportion of individuals who survive to age a. Then we have the relation

$$N(a) = \mu^* N \ell(a). \tag{2.2}$$

Next let $X(a,t), Y(a,t)$ and $Z(a,t)$ be the age-densities of respectively the susceptible, infected and immune population at time t, so that

$$N(a) = X(a,t) + Y(a,t) + Z(a,t). \tag{2.3}$$

Let γ^{-1} be the average infectious period, i.e. the probability of still being infected at the duration s elapsed since initial infection is $\exp(-s\gamma)$. Let $\beta(a,b)$ be the age-dependent *transmission coefficient*, that is, the probability that a susceptible person of age a meets an infectious person of age b and becomes infected, per unit of time. Define the *force of infection* $\lambda(a,t)$ by

$$\lambda(a,t) = \int_0^\omega \beta(a,\sigma) Y(\sigma,t) d\sigma. \tag{2.4}$$

Then the transmission from the susceptible to the infectious state is a Poisson process, i.e. the probability that a susceptible individual becomes infected during the small interval $(a, a+da)$ at time t is $\lambda(a,t)da$. Moreover we assume that the death rate of the population is not affected by the presence of the disease. Under the above assumptions, the spread of the disease can be described by the system of partial differential equations

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) X(a,t) = -\lambda(a,t) X(a,t) - \mu(a) X(a,t), \tag{2.5a}$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) Y(a,t) = \lambda(a,t) X(a,t) - (\mu(a) + \gamma) Y(a,t), \tag{2.5b}$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) Z(a,t) = \gamma Y(a,t) - \mu(a) Z(a,t), \tag{2.5c}$$

with boundary conditions

$$X(0,t) = \mu^* N,\ Y(0,t) = 0,\ Z(0,t) = 0. \tag{2.6}$$

Consider the fractions of susceptible, infectious and immune population at age a and time t;

$$x(a,t) := \frac{X(a,t)}{N(a)}, y(a,t) := \frac{Y(a,t)}{N(a)}, z(a,t) := \frac{Z(a,t)}{N(a)}.$$

Then the system (2.5a)-(2.5c) can be rewritten to a simpler form.

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a})x(a,t) = -\lambda(a,t)x(a,t), \tag{2.7a}$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a})y(a,t) = \lambda(a,t)x(a,t) - \gamma y(a,t), \tag{2.7b}$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a})z(a,t) = \gamma y(a,t), \tag{2.7c}$$

$$x(0,t) = 1,\ y(0,t) = 0,\ z(0,t) = 0, \tag{2.8}$$

where

$$\lambda(a,t) = \int_0^\omega \beta(a,\sigma)N(\sigma)y(\sigma,t)d\sigma,\ N(a) = \mu^* N\ell(a), \tag{2.9}$$

$$x(a,t) + y(a,t) + z(a,t) = 1. \tag{2.10}$$

In the following, we mainly consider the system (2.7a)-(2.7c) with initial conditions

$$x(a,0) = x_0(a),\ y(a,0) = y_0(a),\ z(a,0) = z_0(a). \tag{2.11}$$

REMARK. For the system (2.5), threshold and stability results have been investigated for special forms of the transmission coefficient β; β =constant (Dietz, 1975; Greenhalgh, 1987); $\beta(a,b) = f(a)$ (Gripenberg, 1983; Webb, 1985); $\beta(a,b) = f(a)g(b)$ (Dietz and Schenzle, 1985; Greenhalgh, 1988b). In particular, the models formulated by Gripenberg, Dietz and Schenzle treat a more general situation than the present model in the sense that the infectivity depends on the duration of an infection. However it should be noted that the existence problem of steady states for the duration-dependent model is reduced to the same kind of a nonlinear integral equation as discussed in section 4 of this paper. Tudor (1985) reduced the system (2.5) into an ordinary differential equation system by discretizing the age variable. It seems that his theoretical and numerical results support Greenhalgh's conjectures.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section we shall show that the initial-boundary value problem (2.7a)-(2.7c) has a unique solution. First we note that it suffices to consider the system in terms of only $x(a,t)$ and $y(a,t)$ since, once these functions are known, $z(a,t)$ can be obtained by $z(a,t) = 1 - x(a,t) - y(a,t)$.

First we introduce a new variable $\hat{x}$ by $\hat{x}(a,t) := x(a,t) - 1$. Then we obtain the new system for $\hat{x}$ and y

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a})\hat{x}(a,t) = -\lambda(a,t)(1 + \hat{x}(a,t)), \tag{3.1a}$$

$$(\frac{\partial}{\partial t} + \frac{\partial}{\partial a})y(a,t) = \lambda(a,t)(1 + \hat{x}(a,t)) - \gamma y(a,t), \tag{3.1b}$$

$$\hat{x}(0,t) = 0,\ \ y(0,t) = 0.$$

Let us consider the initial-boundary value problem of the system composed of (3.1a) and (3.1b) as an abstract Cauchy problem on the Banach space $X := L^1(0,\omega;C^2)$ that is the set of equivalence classes of Lebesgue integrable functions from $[0,\omega]$ to C^2 equipped with the L^1-norm. Let A be a linear operator defined by

$$(A\phi)(a) := (-\frac{d}{da}\phi_1(a), -\frac{d}{da}\phi_2(a)-\gamma\phi_2(a))^{\tau}, \tag{3.2}$$

$$\phi = (\phi_1(a),\phi_2(a))^{\tau} \in D(A),$$

where p^{τ} is the transpose of the vector p and the domain $D(A)$ is given as

$$D(A) = \{\phi \in X : \phi_i \in AC[0,\omega],\ \phi(0) = (0,0)^{\tau}\},$$

where $AC[0,\omega]$ denotes the set of absolutely continuous functions on $[0,\omega]$. Suppose that $\beta(a,b) \in L^{\infty}((0,\omega)\times(0,\omega))$. We define a nonlinear operator $F: X \to X$ by

$$F(\phi)(a) = (-(P\phi_2)(a)(1+\phi_1(a)),(P\phi_2)(a)(1+\phi_1(a)))^{\tau}, \phi \in X, \tag{3.3}$$

where P is a bounded linear operator on $L^1(0,\omega;C)$ given by

$$(Pf)(a) = \int_0^{\omega} \beta(a,\sigma)N(\sigma)f(\sigma)d\sigma. \tag{3.4}$$

Note that $Pf \in L^{\infty}(0,\omega)$ for $f \in L^1(0,\omega)$ and hence the nonlinear operator F is defined on the whole space X. Let $u(t)=(\hat{x}(\cdot,t),y(\cdot,t))^{\tau} \in X$. Then we can rewrite the initial-boundary problem (3.1a)-(3.1b) as the abstract semilinear initial value problem in X

$$\frac{d}{dt}u(t) = Au(t)+F(u(t)),\ u(0) = u_0 \in X, \tag{3.5}$$

where $u_0(a) := (\hat{x}_0(a),y_0(a))^{\tau}$, $\hat{x}_0(a) := x_0(a)-1$. It is easily seen that the operator A is the infinitesimal generator of C_0-semigroup $T(t), t\geq 0$ and F is continuously Frechet differentiable on X. Then for each $u_0 \in X$, there exists a maximal interval of existence $[0,t_0)$, and a unique continuous (mild) solution $t \to u(t;u_0)$ from $[0,t_0)$ to X such that

$$u(t;u_0) = T(t)u_0 + \int_0^t T(t-s)F(u(s;u_0))ds, \tag{3.6}$$

for all $t \in [0,t_0)$ and either $t_0 = \infty$ or $\lim_{t\uparrow t_0}\|u(t;u_0)\| = \infty$. Moreover, if $u_0 \in D(A)$, then $u(t;u_0) \in D(A)$ for $0 \leq t < t_0$ and the function $t \to u(t;u_0)$ is continuously differentiable and satisfies (3.5) on $[0,t_0)$ (Webb, p.194, Proposition 4.16).

LEMMA 3.1. *Let* $\Omega := \{(\hat{x},y) \in X : -1 \leq \hat{x}, 0 \leq y\}$ *and let* $\Omega_0 = \{(\hat{x},y) \in X : -1 \leq \hat{x} \leq 0, 0 \leq y \leq 1\}$. *Then the mild solution* $u(t;u_0), u_0 \in \Omega$ *of (3.5) enters into* Ω_0 *after finite time and the set* Ω_0 *is positively invariant.*

PROOF. From (2.7a), we have the representation

$$x(a,t) = \begin{cases} \exp(-\int_0^a \lambda(\rho,t-a+\rho)d\rho),\ t-a>0, \\ x_0(a-t)\exp(-\int_0^t \lambda(a-t+\rho,\rho)d\rho),\ a-t>0, \end{cases} \tag{3.7}$$

which shows that $\hat{x}(a,t) \geq -1$ when $x_0(a) \geq 0$. If we write (3.1b) as an abstract Cauchy problem

$$\frac{d}{dt}y(t) = By(t)+(Py(t))(1+\hat{x}(t)),\ y(0) = y_0 \in L^1(0,\omega), \tag{3.8}$$

where the operator B is defined by

$$B = -\frac{d}{da}-\gamma,\ D(B) = \{\psi\in L^1(0,\omega):\psi\in AC[0,\omega],\psi(0) = 0\},$$

then we obtain

$$y(t) = S(t)y(0)+\int_0^t S(t-s)(Py(s))(1+\hat{x}(s))ds, \tag{3.9}$$

where $S(t) := \exp(tB)$ is the C_0-semigroup generated by the closed operator B. If we assume that $\hat{x}(t)\geqslant -1, y_0\geqslant 0$, (3.9) shows that $y(t)$ is also positive because $S(t), t\geqslant 0$ is positive and $y(t)$ can be obtained by monotone iteration

$$y(t) = S(t)y(0)+\int_0^t S(t-s)(PS(s)y(0))(1+\hat{x}(s))ds + \cdots .$$

Hence we know that $u(t;u_0)\in\Omega$ for all $t\geqslant 0$ when $u_0\in\Omega$. Next let $w(t) := \hat{x}(t)+y(t)$. Then we have

$$\frac{d}{dt}w(t) = Cw(t)-\gamma y(t), w(0) = \hat{x}_0+y_0\in L^1(0,\omega), \tag{3.10}$$

where the operator C is given by

$$C = -\frac{d}{da},\ D(C) = \{\psi\in L^1(0,\omega):\psi\in AC[0,\omega],\psi(0) = 0\},$$

From (3.10), it follows that

$$w(t) = U(t)w(0)-\int_0^t U(t-s)\gamma y(s)ds\leqslant U(t)w(0), \tag{3.11}$$

where $U(t)$, $t\geqslant 0$ is the positive C_0-semigroup generated by the operator C. Since $U(t)$ is a nilpotent translation semigroup, we have $w(t)(a)\leqslant\hat{x}_0(a-t)+y_0(a-t)$, $a>t$ and $w(t)\leqslant 0$ for $t\geqslant\omega$. Then it follows that the mild solution $u(t;u_0), u_0\in\Omega$ into Ω_0 for $t\geqslant\omega$ and if $u_0\in\Omega_0$, then $u(t;u_0)\in\Omega_0$ for all $t\geqslant 0$. This completes the proof. □

By the above lemma, we know that the norm of the local solution $u(t;u_0), u_0\in D(A)\cap\Omega$, of (3.5) is finite as long as it is defined. Thus we arrive at the following result.

PROPOSITION 3.2. *The abstract Cauchy problem (3.5) has a unique global classical solution on X with respect to initial data* $u_0\in\Omega\cap D(A)$.

Therefore it follows immediately that the initial-boundary value problem (2.7)-(2.9) has a unique positive global solution with respect to the positive initial data.

4. EXISTENCE OF STEADY STATES

Let $u^* = (x^*(a),y^*(a))^T$ be the steady state solution for the equation (2.7a)-(2.7b). Then it is easy to verify the following

$$x^*(a) = \exp(-\int_0^a \lambda^*(\sigma)d\sigma), \tag{4.1a}$$

$$y^*(a) = \int_0^a \exp(-\gamma(a-\sigma))\lambda^*(\sigma)\exp(-\int_0^\sigma \lambda^*(\eta)d\eta)d\sigma, \tag{4.1b}$$

$$\lambda^*(a) = \int_0^\omega \beta(a,\sigma)N(\sigma)y^*(\sigma)d\sigma. \tag{4.1c}$$

Substituting (4.1b) into (4.1c) and changing the order of integration, we obtain an equation for $\lambda^*(a)$:

$$\lambda^*(a) = \int_0^\omega \phi(a,\sigma)\lambda^*(\sigma)\exp(-\int_0^\sigma \lambda^*(\eta)d\eta)d\sigma, \tag{4.2}$$

$$\phi(a,\sigma) := \int_\sigma^\omega \beta(a,\zeta)N(\zeta)\exp(-\gamma(\zeta-\sigma))d\zeta.$$

From (4.1c), it follows that $|\lambda^*(a)| \leq \|\beta\|_\infty \|y^*\|_1$, where $\|\cdot\|_\infty$, $\|\cdot\|_1$ denote L^∞-norm and L^1-norm respectively. Then it follows from $y^* \in L^1_+(0,\omega)$ that $\lambda^* \in L^\infty_+(0,\omega)$. It is clear that one solution of (4.2) is $\lambda^*(a) \equiv 0$, which corresponds to the equilibrium state with no disease. In order to investigate non-trivial positive solutions for (4.2), we define a nonlinear operator $\Phi(x)$ in the Banach space $E := L^1(0,\omega)$ with the positive cone $E_+ := \{\psi \in E;\ \psi \geq 0, \text{ a.e. }\}$, by

$$\Phi(x)(a) := \int_0^\omega \phi(a,\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)d\sigma,\ x \in E. \tag{4.3}$$

Since the range of Φ is included in $L^\infty(0,\omega)$, the solutions of (4.2) correspond to fixed points of the operator Φ. Observe that the operator Φ has a positive linear majorant T defined by

$$(Tx)(a) := \int_0^\omega \phi(a,\sigma)x(\sigma)d\sigma,\ x \in E. \tag{4.4}$$

Here we summarize the Perron-Frobenius theory for positive operators on the ordered Banach space as long as it is needed for our purpose. Let E be a real or complex Banach space and let E^* be its dual, i.e. the space of all linear functionals on E. The value of $F \in E^*$ at $\psi \in E$ is denoted by $<F,\psi>$. A closed subset E_+ is called a *cone* if the following holds; (1) $E_+ + E_+ \subset E_+$, (2) $\lambda E_+ \subset E_+$ for $\lambda \geq 0$, (3) $E_+ \cap (-E_+) = \{0\}$, (4) $E_+ \neq \{0\}$. We write $x \leq y$ if and only if $y - x \in E_+$ and write $x < y$ if $y - x \in E_+ \setminus \{0\}$. The cone E_+ is called *total* if the set $\{\psi - \phi;\ \psi, \phi \in E_+\}$ is dense in E. The *dual cone* E^*_+ is the subset of E^* consisting of all positive linear functionals on E, i.e. $F \in E^*_+$ if and only if $F \in E^*$ and $<F,\psi> \geq 0$ for all $\psi \in E_+$. $\psi \in E_+$ is called *nonsupporting point* (or *quasi-interior point*) if $<F,\psi> > 0$ for all $F \in E^*_+ \setminus \{0\}$. A positive linear functional $F \in E^*_+$ is called *strictly positive* if $<F,\psi> > 0$ for all $\psi \in E_+ \setminus \{0\}$. Let $B(E)$ be the set of bounded linear operators of E into E. $T \in B(E)$ is called *positive* with respect to the cone E_+ if $T(E_+) \subset E_+$. We say $T \geq S$ if $(T-S)(E_+) \subset E_+$ for $T, S \in B(E)$. We denote the spectral radius of $T \in B(E)$ by $r(T)$.

Although several formally different concepts about positivity of operators have been introduced to extend the Perron-Frobenius theory since the work of Krein and Rutman (1948), it seems that Sawashima's concepts are most natural and convenient for our purpose (see Sawashima, 1964; Marek, 1970; Heijmans, 1986):

DEFINITION 4.1. (Sawashima, 1964) A positive operator $T \in B(E)$ is called *semi-nonsupporting* if and only if for every pair $\psi \in E_+ \setminus \{0\}, F \in E^*_+ \setminus \{0\}$, there exists a positive integer $p = p(\psi, F)$ such that $<F, T^p\psi> > 0$. A positive operator $T \in B(E)$ is called *nonsupporting* if and only if for every pair $\psi \in E_+ \setminus \{0\}$, $F \in E^*_+ \setminus \{0\}$, there exists a positive integer $p = p(\psi, F)$ such that $<F, T^n\psi> > 0$ for all $n \geq p$.

The reader may refer to Sawashima (1964), Niiro and Sawashima (1966) about the proof of the following theorem:

PROPOSITION 4.2. *Let the cone E_+ be total, $T \in B(E)$ be semi-nonsupporting with respect to E_+ and let $r(T)$ be a pole of the resolvent $R(\lambda, T)$. Then the followings hold;*
(1) *$r(T) \in P_\sigma(T) \setminus \{0\}$ and $r(T)$ is a simple pole of the resolvent.*
(2) *The eigenspace corresponding to $r(T)$ is one-dimensional and the corresponding eigenvector $\psi \in E_+$ is*

a nonsupporting point. The relation $T\phi = \mu\phi$ with $\phi \in E_+$ implies that $\phi = c\psi$ for some constant c.

(3) *The eigenspace of T^* corresponding to $r(T)$ is also a one-dimensional subspace of E^* spanned by a strictly positive functional $F \in E^*_+$.*

(4) *Assume that E is a Banach lattice. If $T \in B(E)$ is nonsupporting, then the peripheral spectrum of T consists only of $r(T)$, i.e. $|\lambda| < r(T)$ for $\lambda \in \sigma(T) \setminus \{r(T)\}$.*

The following comparison theorem is due to Marek (1970).

PROPOSITION 4.3. *Suppose that E is a Banach lattice. Let S and T be positive operators in B(E).*

(1) *If $S \leqslant T$, then $r(S) \leqslant r(T)$.*

(2) *If S and T are semi-nonsupporting operators, then $S \leqslant T, S \neq T$ implies that $r(S) < r(T)$.*

After the above preparations, we first consider the nature of the majorant operator T defined by (4.4). In the following, we shall make an assumption:

ASSUMPTION 4.4.

(1) $\beta(a,\zeta) \in L^\infty_+((0,\omega) \times (0,\omega))$.

(2)

$$\lim_{h \to 0} \int_0^\omega |\beta(a+h,\zeta) - \beta(a,\zeta)| da = 0 \text{ uniformly for } \zeta \in R, \tag{4.5}$$

where β is extended by $\beta(a,\zeta) = 0$ for $a, \zeta \in (-\infty, 0) \cup (\omega, \infty)$.

(3) There exist numbers α with $\omega > \alpha > 0$ and $\epsilon > 0$ such that

$$\beta(a,\zeta) \geqslant \epsilon \text{ for almost all } (a,\zeta) \in (0,\omega) \times (\omega - \alpha, \omega). \tag{4.6}$$

Then we can prove that:

LEMMA 4.5. *Under Assumption 4.4, the operator $T: E \to E$ is nonsupporting and compact.*

PROOF. Define the positive linear functional $F \in E^*_+$ by

$$<F,\psi> = \int_0^\omega g(\sigma)\psi(\sigma)d\sigma, \quad \psi \in E, \tag{4.7}$$

where $g(\sigma)$ is given by

$$g(\sigma) := \int_\sigma^\omega s(\zeta) N(\zeta) \exp(-\gamma(\zeta - \sigma)) d\zeta, \tag{4.8}$$

where the function $s(\zeta)$ is defined as $s(\zeta) = 0$, $\zeta \in [0, \omega - \alpha)$; $s(\zeta) = \epsilon$, $\zeta \in [\omega - \alpha, \omega]$. Hence $\beta(a,\zeta) \geqslant s(\zeta)$ for almost all $(a,\zeta) \in [0,\omega] \times [0,\omega]$. Since $g(\sigma) > 0$ for all $\sigma \in [0,\omega)$, the functional F is strictly positive and

$$<F,x>e \leqslant Tx, \quad e = 1 \in E_+. \tag{4.9}$$

Then for any integer n, we have

$$T^{n+1}x \geqslant <F,x><F,e>^n e.$$

Therefore we obtain $<G,T^n x> > 0, n \geqslant 1$ for every pair $x \in E_+ \setminus \{0\}, G \in E^*_+ \setminus \{0\}$, that is, T is nonsupporting. Next observe that

$$\int_0^\omega |\phi(a+h,\sigma) - \phi(a,\sigma)| da \leqslant \mu^* N \int_0^\omega \int_0^\omega |\beta(a+h,\zeta) - \beta(a,\zeta)| da d\zeta.$$

Then it follows from Assumption 4.4 that the kernel ϕ satisfies the condition of the Lemma in the Appendix. Hence we can conclude that the operator T is compact. □

From Proposition 4.2, it follows that the spectral radius $r(T)$ of the operator T is the only positive eigenvalue with a positive eigenvector and also an eigenvalue of the dual operator T^* with a strictly positive eigenfunctional. Now we can prove the following:

PROPOSITION 4.6 (Threshold results). *Let $r(T)$ be the spectral radius of the operator T defined by (4.4). Then the following holds;*

(1) *If $r(T)\leqslant 1$, the only non-negative solution x of the equation $x = \Phi(x)$ is the trivial solution $x\equiv 0$.*

(2) *If $r(T)>1$, the equation $x = \Phi(x)$ has at least one non-zero positive solution.*

PROOF. Suppose that $r(T)\leqslant 1$. It is easily checked that $Tx-\Phi(x)\in E_+\setminus\{0\}$ for $x\in E_+\setminus\{0\}$. If there exists a $x_0\in E_+\setminus\{0\}$ being a solution of $x = \Phi(x)$, then $x_0 = \Phi(x_0)\leqslant T(x_0)$. Let $F_0^*\in E_+^*\setminus\{0\}$ be the adjoint eigenvector of T corresponding to $r(T)$. Taking duality pairing, we find $<F_0^*,T(x_0)-x_0> = (r(T)-1)<F_0^*,x_0>>0$ because $T(x_0)-x_0\in E_+\setminus\{0\}$ and F_0^* is strictly positive. Then we have $r(T)>1$, which is a contradiction. Next we assume that $r(T)>1$. Under Assumption 4.4, in the same manner as the proof of Lemma 4.5, we can see that the operator Φ is a completely continuous operator in the Banach space E. Moreover, if we define the number M_0 by

$$M_0 := \sup_{0\leqslant\sigma\leqslant\omega_0}\int_0^{\omega}\phi(a,\sigma)da,$$

the set $\Omega := \{x\in E\colon 0\leqslant x,\ \|x\|\leqslant M_0\}$ is invariant (in fact $\Phi(E_+)\subset\Omega$) under the operator Φ. We define an operator Φ_r by

$$\Phi_r(x) = \begin{cases}\Phi(x), \text{ if } \|x\|\geqslant r,\ x\in E_+,\\ \Phi(x)+(r-\|x\|)x_0, \text{ if } \|x\|\leqslant r,\ x\in E_+,\end{cases}$$

where x_0 is the positive eigenvector of T corresponding to $r(T)>1$. Then Φ_r is also completely continuous and transform the set $\Omega_r := \{x\in E;\ 0\leqslant x, \|x\|\leqslant M_0+r\|x_0\|\}$ into itself. Since Ω_r is bounded, convex and closed in E, Φ_r has a fixed point $x_r\in\Omega_r$ (Schauder's principle). Observe that the Frechet derivative of $\Phi(x)$ at $x = 0$ is the operator T and T does not have in E_+ eigenvectors corresponding to the eigenvalue one. Then we can apply the method of M.A.Krasnoselskii (1964a, Theorem 4.11), and it can be shown that the norms of these fixed points are greater than r if r is sufficiently small. That is, Φ has a positive fixed point. □

Subsequently, in order to investigate the uniqueness problem for non-trivial positive fixed points of the operator Φ, we introduce the concept of concave operators (see Krasnoselskii, 1964a,b).

DEFINITION 4.7. Let E_+ be a cone in a real Banach space E and $\leqslant$ be the partial ordering defined by E_+. A positive operator $A:E_+\to E_+$ is called a *concave operator* if there exists a $u_0\in E_+\setminus\{0\}$ which satisfies the followings;

(1) for any $x\in E_+\setminus\{0\}$ there exist $\alpha = \alpha(x)>0$ and $\beta = \beta(x)>0$ such that $\alpha u_0\leqslant Ax\leqslant\beta u_0$, that is, Ax is comparable with u_0,

(2) $A(tx)\geqslant tAx$ for $0\leqslant t\leqslant 1$ and for every $x\in E_+$ such that $\alpha(x)u_0\leqslant x\leqslant\beta(x)u_0$ $(\alpha(x)>0,\beta(x)>0)$.

Here we introduce a new class of concave operators which has at most one positive fixed point. This type of operator is closely related to the *e-sublinear operator* introduced by Amann (1972).

LEMMA 4.8. *Suppose that the operator $A:E_+\to E_+$ is monotone and concave. If for any $x\in E_+$ satisfying $\alpha_1 u_0\leqslant x\leqslant\beta_1 u_0$ $(\alpha_1 = \alpha_1(x)>0,\ \beta_1 = \beta_1(x)>0)$ and any $0<t<1$, there exists $\eta = \eta(x,t)>0$ such that*

$$A(tx) \geqslant tAx + \eta u_0, \tag{4.10}$$

then A has at most one positive fixed point.

PROOF. Suppose that $x_1 \in E_+ \setminus \{0\}$ and $x_2 \in E_+ \setminus \{0\}$ are two positive fixed points of A. From the concavity of A, we can choose positive constants α_1 and β_2 such that

$$x_1 = Ax_1 \geqslant \alpha_1 u_0 = \alpha_1 \beta_2^{-1} \beta_2 u_0 \geqslant \alpha_1 \beta_2^{-1} Ax_2 = \alpha_1 \beta_2^{-1} x_2.$$

If we define $k = \sup\{\mu : x_1 \geqslant \mu x_2\}$, then we see that $k > 0$ from the above inequality. If we assume that $0 < k < 1$, then there exists $\eta = \eta(x_2, k) > 0$ such that

$$x_1 = Ax_1 \geqslant A(kx_2) \geqslant kAx_2 + \eta u_0 \geqslant kx_2 + \eta \beta_2^{-1} Ax_2 = (k + \eta \beta_2^{-1}) x_2,$$

which contradicts the definition of k. Hence we know that $k \geqslant 1$ and $x_1 \geqslant x_2$. In the same way, we can prove $x_2 \geqslant x_1$. Thus $x_1 = x_2$. □

Here we introduce another assumption:

ASSUMPTION 4.9. For all $(a, \sigma) \in [0, \omega] \times [0, \omega]$, the inequality

$$\beta(a, \sigma) N(\sigma) - \gamma \int_\sigma^\omega \beta(a, x) N(x) \exp(-\gamma(x - \sigma)) dx \geqslant 0, \tag{4.11}$$

holds.

Then we can prove the following:

PROPOSITION 4.10. *Suppose that Assumption 4.9 holds. If $r(T) > 1$, Φ has only one positive fixed point.*

PROOF. From Lemma 4.8 and Proposition 4.6, it is sufficient to show that under Assumption 4.9, the operator Φ is a monotonic concave operator satisfying the condition (4.10). From (4.3), it follows that

$$\begin{aligned}\Phi(x)(a) &= \int_0^\omega \phi(a, \sigma)\left(-\frac{d}{d\sigma}\right)\exp\left(-\int_0^\sigma x(\eta) d\eta\right) d\sigma \\ &= \phi(a, 0) - \int_0^\omega [\beta(a, \sigma) N(\sigma) - \gamma \phi(a, \sigma)] \exp\left(-\int_0^\sigma x(\eta) d\eta\right) d\sigma.\end{aligned}$$

Then the operator Φ is monotonic under Assumption 4.9. Next observe that

$$\alpha(x) u_0 \leqslant \Phi(x)(a) \leqslant \beta(x) u_0,$$

where $u_0 \equiv 1$ and

$$\alpha(x) := \int_0^\omega g(\sigma) x(\sigma) \exp\left(-\int_0^\sigma x(\eta) d\eta\right) d\sigma,$$

$$\beta(x) := M \int_0^\omega h(\sigma) x(\sigma) \exp\left(-\int_0^\sigma x(\eta) d\eta\right) d\sigma,$$

where $M := \operatorname{ess\,sup} \beta(a, b) < \infty$, $g(\sigma)$ is given by (4.8) and $h(\sigma)$ is defined by

$$h(\sigma) := \int_\sigma^\omega N(\zeta) \exp(-\gamma(\zeta - \sigma)) d\zeta.$$

It follows that $\alpha(x) > 0$ and $\beta(x) > 0$ for $x \in E_+ \setminus \{0\}$. Moreover we obtain

$$\Phi(tx)(a) - t\Phi(x)(a) = t \int_0^\omega \phi(a, \sigma) x(\sigma) \exp\left(-\int_0^\sigma x(\eta) d\eta\right) \left[\exp\left((1 - t)\int_0^\sigma x(\eta) d\eta\right) - 1\right] d\sigma$$

$$\geq t\int_0^\omega g(\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)[\exp((1-t)\int_0^\sigma x(\eta)d\eta)-1]d\sigma,$$

from which we conclude that Φ is a concave operator and the condition (4.10) is satisfied by letting $u_0 = 1$ and

$$\eta(x,t) := t\int_0^\omega g(\sigma)x(\sigma)\exp(-\int_0^\sigma x(\eta)d\eta)[\exp((1-t)\int_0^\sigma x(\eta)d\eta)-1]d\sigma.$$

This completes the proof. □

Note that Assumption 4.9 holds if $\beta(a,\sigma)N(\sigma)$ is non-increasing as a function of σ. In fact, we have

$$\beta(a,\sigma)N(\sigma)-\gamma\int_\sigma^\omega \beta(a,x)N(x)\exp(-\gamma(x-\sigma))dx$$

$$= \gamma\int_\sigma^\omega [\beta(a,\sigma)N(\sigma)-\beta(a,x)N(x)]\exp(-\gamma(x-\sigma))dx+\exp(-\gamma(\omega-\sigma))\beta(a,\sigma)N(\sigma),$$

which is nonnegative for all $(a,\sigma)\in[0,\omega]\times[0,\omega]$ if $\beta(a,\sigma)N(\sigma)-\beta(a,x)N(x)\geq 0$ for $x\geq\sigma$. In particular, Assumption 4.9 holds if β is independent of the age of infectives σ, because $N(a)$ is a decreasing function. Another type of condition which guarantees Assumption 4.9 is given as follows:

$$\ell(a)\geq k(1-\exp(-\gamma(\omega-a))), \tag{4.12}$$

if the constant k defined by

$$k := \frac{\sup\beta(a,b)}{\inf\beta(a,b)}, \tag{4.13}$$

is finite. Since $N(a) = \mu^* N\ell(a)\leq\mu^* N$, the sufficiency of condition (4.12) follows from the inequality

$$\beta(a,\sigma)N(\sigma)-\gamma\phi(a,\sigma)\geq\inf\beta(a,b)\mu^* N[\ell(\sigma)-k(1-\exp(-\gamma(\omega-\sigma)))].$$

REMARK 4.11. No matter whether Assumption 4.9 holds, if $\beta(a,b)$ can be factorized as $f(a)g(b)$ (which is called the *proportionate mixing assumption*, see Dietz and Schenzle, 1985), it is easily seen that there always exists a unique non-trivial steady state under the condition

$$r(T) = \int_0^\omega \phi(\sigma,\sigma)d\sigma>1. \tag{4.14}$$

In this case, $f(a)$ is the eigenvector of the operator T corresponding to the spectral radius $r(T)$ (see Greenhalgh, 1988b).

5. STABILITY ANALYSIS FOR EQUILIBRIUM SOLUTIONS

In order to investigate the local stability of the equilibrium solutions $(x^*(a),y^*(a))^\tau$ of (2.7a)-(2.7b), we first rewrite (2.7a)-(2.7b) into the equation for small perturbations: Let

$$x(a,t) = x^*(a)+\zeta(a,t),\ y(a,t) = y^*(a)+\eta(a,t).$$

From (2.7), we have

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\zeta(a,t) = -\lambda(a,t)(\zeta(a,t)+x^*(a))-\lambda^*(a)\zeta(a,t), \tag{5.1a}$$

$$(\frac{\partial}{\partial t}+\frac{\partial}{\partial a})\eta(a,t) = \lambda(a,t)(\zeta(a,t)+x^*(a))+\lambda^*(a)\zeta(a,t)-\gamma\eta(a,t), \tag{5.1b}$$

where

$$\lambda(a,t) = \int_0^\omega \beta(a,\sigma)N(\sigma)\eta(\sigma,t)d\sigma,\ \lambda^*(a) = \int_0^\omega \beta(a,\sigma)N(\sigma)y^*(\sigma)d\sigma,$$

$$\zeta(0,t) = 0,\ \eta(0,t) = 0.$$

Therefore we can formulate (5.1) as an abstract semilinear problem on the Banach space X:

$$\frac{d}{dt}u(t) = Au(t)+G(u(t)),\ u(t) := (\zeta(t),\eta(t))^\tau \in X, \tag{5.2}$$

where the generator A is defined by (3.2) with the domain

$$D(A) = \{\psi\in X:\psi_i\in AC[0,\omega],\psi(0) = 0\}.$$

The nonlinear term G is defined as

$$G(u) := (-(Pu_2)(u_1+x^*)-\lambda^*u_1,(Pu_2)(u_1+x^*)+\lambda^*u_1)^\tau,$$

for $u = (u_1,u_2)^\tau\in X$, where the operator P is defined by (3.4). The linearized equation around $u=0$ is given by

$$\frac{d}{dt}u(t) = (A+C)u(t), \tag{5.3}$$

where the bounded linear operator C is the Frechet derivative of $G(u)$ at $u=0$ and given by

$$Cu = (-(Pu_2)x^*-\lambda^*u_1,(Pu_2)x^*+\lambda^*u_1)^\tau.$$

Now let us consider the resolvent equation for $A+C$;

$$(\lambda-(A+C))\phi = \psi,\ \phi\in D(A),\ \psi\in X,\ \lambda\in C. \tag{5.4}$$

Then we have

$$\phi'_1+(\lambda+\lambda^*)\phi_1 = \psi_1-(P\phi_2)x^*, \tag{5.5a}$$

$$\phi'_2+(\lambda+\gamma)\phi_2 = \psi_2+(P\phi_2)x^*+\lambda^*\phi_1. \tag{5.5b}$$

From (5.5a), we obtain

$$\phi_1(a) = \exp(-\lambda a)\Pi(a)\int_0^a[\psi_1(\sigma)-(P\phi_2)(\sigma)\Pi(\sigma)]\exp(\lambda\sigma)\Pi^{-1}(\sigma)d\sigma, \tag{5.6}$$

where $\Pi(a)$ is defined by

$$\Pi(a) := \exp(-\int_0^a\lambda^*(\sigma)d\sigma) = x^*(a).$$

From (5.5b), we have

$$\phi_2(a) = \int_0^a\exp(-(\gamma+\lambda)(a-\sigma))[\psi_2(\sigma)+(P\phi_2)(\sigma)\Pi(\sigma)+\lambda^*(\sigma)\phi_1(\sigma)]d\sigma. \tag{5.7}$$

On the other hand, from (5.6), we can write

$$\lambda^*(\sigma)\phi_1(\sigma) = \exp(-\lambda\sigma)\Pi(\sigma)\lambda^*(\sigma)\int_0^\sigma[\psi_1(\eta)-(P\phi_2)(\eta)\Pi(\eta)]\exp(\lambda\eta)\Pi^{-1}(\eta)d\eta. \tag{5.8}$$

Using (5.7) and (5.8), we obtain

$$(P\phi_2)(a) = I(a)+J(a)+K(a)+L(a), \tag{5.9}$$

where

$$I(a) := \int_0^\omega \beta(a,\sigma)N(\sigma)\int_0^\sigma \exp(-(\gamma+\lambda)(\sigma-\eta))\psi_2(\eta)d\eta d\sigma,$$

$$J(a) := \int_0^\omega \beta(a,\sigma)N(\sigma)\int_0^\sigma \exp(-(\gamma+\lambda)(\sigma-\eta))(P\phi_2)(\eta)\Pi(\eta)d\eta d\sigma,$$

$$K(a) := \int_0^\omega \beta(a,\sigma)N(\sigma)\int_0^\sigma \exp(-(\gamma+\lambda)(\sigma-\eta))\exp(-\lambda\eta)\Pi(\eta)\lambda^*(\eta)\int_0^\eta \psi_1(\zeta)\exp(\lambda\zeta)\Pi^{-1}(\zeta)d\zeta d\eta d\sigma,$$

$$L(a) := -\int_0^\omega \beta(a,\sigma)N(\sigma)\int_0^\sigma \exp(-(\gamma+\lambda)(\sigma-\eta))\exp(-\lambda\eta)\Pi(\eta)\lambda^*(\eta)\int_0^\eta (P\phi_2)(\zeta)\exp(\lambda\zeta)d\zeta d\eta d\sigma.$$

Define

$$\phi_\lambda(a,\sigma) := \int_\sigma^\omega \beta(a,\eta)N(\eta)\exp(-(\gamma+\lambda)(\eta-\sigma))d\eta, \tag{5.10}$$

then we can rewrite the above representations for I,J,K,L as

$$I(a) = \int_0^\omega \phi_\lambda(a,\sigma)\psi_2(\sigma)d\sigma,$$

$$J(a) = \int_0^\omega \phi_\lambda(a,\sigma)\Pi(\sigma)(P\phi_2)(\sigma)d\sigma,$$

$$K(a) = \int_0^\omega \phi_\lambda(a,\sigma)\exp(-\lambda\sigma)\Pi(\sigma)\lambda^*(\sigma)\int_0^\sigma \psi_1(\eta)\Pi^{-1}(\eta)\exp(\lambda\eta)d\eta d\sigma,$$

$$L(a) = -\int_0^\omega \phi_\lambda(a,\sigma)\exp(-\lambda\sigma)\Pi(\sigma)\lambda^*(\sigma)\int_0^\sigma (P\phi_2)(\eta)\exp(\lambda\eta)d\eta d\sigma,$$

If we define linear operators on the Banach space $L^1(0,\omega)$ by

$$(T_\lambda\psi)(a) := \int_0^\omega \phi_\lambda(a,\sigma)\Pi(\sigma)\psi(\sigma)d\sigma,$$

$$(U_\lambda\psi)(a) := \int_0^\omega \phi_\lambda(a,\sigma)\exp(-\lambda\sigma)\Pi(\sigma)\lambda^*(\sigma)\int_0^\sigma \psi(\eta)\exp(\lambda\eta)d\eta d\sigma,$$

$$(V_\lambda\psi)(a) := (T_\lambda\psi)(a)-(U_\lambda\psi)(a), \tag{5.11}$$

then the following expression holds:

$$(V_\lambda\psi)(a) = \int_0^\omega \chi_\lambda(a,\sigma)\psi(\sigma)d\sigma, \tag{5.12}$$

$$\chi_\lambda(a,\sigma) := \int_\sigma^\omega \Pi(x)[\beta(a,x)N(x)-\gamma\phi_\lambda(a,x)]\exp(-\lambda(x-\sigma))dx, \tag{5.13}$$

It is easy to verify the above expression if we note that

$$(U_\lambda\psi)(a) = \int_0^\omega (-\frac{\partial\Pi(\sigma)}{\partial\sigma})\phi_\lambda(a,\sigma)\exp(-\lambda\sigma)\int_0^\sigma \psi(\eta)\exp(\lambda\eta)d\eta d\sigma$$

$$= (T_\lambda\psi)(a)-\int_0^\omega \Pi(\sigma)[\beta(a,\sigma)N(\sigma)-\gamma\phi_\lambda(a,\sigma)]\int_0^\sigma \psi(\eta)\exp(-\lambda(\sigma-\eta))d\eta d\sigma.$$

From the above definitions and (5.9), it follows that

$$(P\phi_2)(a) = (T_\lambda\psi_2\Pi^{-1})(a)+(T_\lambda P\phi_2)(a)+(U_\lambda\psi_1\Pi^{-1})(a)-(U_\lambda P\phi_2)(a).$$

Hence we have

$$(P\phi_2)(a) = (I-V_\lambda)^{-1}[(T_\lambda\psi_2\Pi^{-1})(a)+(U_\lambda\psi_1\Pi^{-1})(a)]. \tag{5.14}$$

From (5.6), (5.7) and (5.14), we can conclude that

Lemma 5.1. *The perturbed operator $A+C$ has a compact resolvent and*

$$\sigma(A+C) = P_\sigma(A+C) = \{\lambda\in C\colon 1\in P_\sigma(V_\lambda)\}, \tag{5.15}$$

where $\sigma(A)$ and $P_\sigma(A)$ denote the spectrum of A and the point spectrum of A respectively.

Proof. From (5.6) and (5.14), we obtain the expression for ϕ_1

$$\phi_1(a) = J(\psi_1)(a)-K(\psi_1,\psi_2)(a),$$

where the operators J and K are defined by

$$J(\psi_1)(a) := \int_0^a G(a,\sigma)\psi_1(\sigma)d\sigma,$$

$$K(\psi_1,\psi_2)(a) := \int_0^a G(a,\sigma)\Pi(\sigma)(I-V_\lambda)^{-1}[(T_\lambda\psi_2\Pi^{-1})(\sigma)+(U_\lambda\psi_1\Pi^{-1})(\sigma)]d\sigma,$$

where $G(a,\sigma) := \exp[-\lambda(a-\sigma)]\Pi(a)\Pi^{-1}(\sigma)$. Since J is a Volterra operator with a continuous kernel, it is a compact operator on $L^1(0,\omega)$. On the other hand, in the same manner as the proof of Lemma 4.5, we can prove that T_λ and U_λ are compact for all $\lambda\in C$. Let $\Lambda := \{\lambda\in C\colon 1\in\sigma(V_\lambda)\}$. Then it follows that when $\lambda\in C\setminus\Lambda$ the operator K is a compact operator from X to $L^1(0,\omega)$. In the same way, we can prove that $\phi_2(a)$ can be represented by a compact operator from X to $L^1(0,\omega)$. Then we know that $A+C$ has a compact resolvent, so it follows that $\sigma(A+C) = P_\sigma(A+C)$ (see Kato, p. 187). From the above argument, it follows that $C\setminus\Lambda\subset\rho(A+C)$ ($\rho(A+C)$ denotes the resolvent set of $A+C$), that is, $\Lambda\supset\sigma(A+C) = P_\sigma(A+C)$. Since V_λ is a compact operator, we know that $\sigma(V_\lambda)\setminus\{0\} = P_\sigma(V_\lambda)\setminus\{0\}$ and if $\lambda\in\Lambda$, there exists an eigenfunction ψ_λ such that $V_\lambda\psi_\lambda = \psi_\lambda$. Then it is easily seen that if we define the following functions

$$\phi_1(a) = -\exp(-\lambda a)\Pi(a)\int_0^a \exp(\lambda\sigma)\psi_\lambda(\sigma)d\sigma,$$

$$\phi_2(a) = \int_0^a \exp[-\lambda(a-\sigma)][\psi_\lambda(\sigma)\Pi(\sigma)-\lambda^*(\sigma)\phi_1(\sigma)]d\sigma,$$

$(\phi_1,\phi_2)^T$ gives an eigenvector of $A+C$ corresponding to the eigenvalue λ. Then $\Lambda\subset P_\sigma(A+C)$ and we conclude that (5.15) holds. □

Lemma 5.2. *Let $T(t)$, $t\geq 0$ be the C_0-semigroup generated by the perturbed operator $A+C$. Then $T(t)$, $t\geq 0$ is eventually norm continuous and*

$$\omega_0(A+C) = s(A+C) := \sup\{\mathrm{Re}\mu\colon \mu\in\sigma(A+C)\}, \tag{5.16}$$

where $\omega_0(A+C)$ denotes the growth bound of the semigroup $T(t), t\geq 0$ and $s(A+C)$ is the spectral bound of the generator $A+C$.

Proof. We define bounded operators C_1 and C_2 by

$$C_1\phi = (-\lambda^*\phi_1, \lambda^*\phi_1)^\tau,\ C_2\phi = (-x^*(P\phi_2), x^*(P\phi_2))^\tau,\ \phi \in X.$$

Then $C = C_1 + C_2$ and $A + C_1$ generates a C_0-semigroup $S(t)$, $t \geqslant 0$. Since $S(t)$ is a nilpotent semigroup, so it is eventually norm continuous. Using Assumption 4.4 and the Lemma in the Appendix, we can prove that C_2 is a compact operator in X. Therefore, from Theorem 1.30 in the book of NAGEL (1986, p.44), $T(t)$ is also eventually norm continuous. Since the spectral mapping theorem holds for the eventually norm continuous semigroup (NAGEL, p.87), we obtain (5.16). □

If $\omega_0(A + C) < 0$, the equilibrium $u = 0$ of system (5.2) is locally exponentially asymptotically stable in the sense that there exist $\epsilon > 0$, $M \geqslant 1$, and $\gamma < 0$ such that if $x \in X$ and $\|x\| \leqslant \epsilon$, then the solution $u(t;x)$ of (5.2) exists globally and $\|u(t;x)\| \leqslant M\exp(\gamma t)\|x\|$ for all $t \geqslant 0$. This is the main part of the principle of linearized stability (Webb, 1985; Desch and Schappacher, 1986). Therefore, in order to study the stability of equilibrium states, we have to know the structure of the set of singular points $\Lambda := \{\lambda \in C:\ 1 \in P_\sigma(V_\lambda)\}$. Since $\|V_\lambda\| \to 0$ if $\mathrm{Re}\lambda \to \infty$, $I - V_\lambda$ is invertible for large values of $\mathrm{Re}\lambda$. By the theorem of Steinberg (1968), the function $\lambda \to (I - V_\lambda)^{-1}$ is meromorphic in the complex domain, and hence the set Λ is a discrete set whose elements are poles of $(I - V_\lambda)^{-1}$ of finite order.

Now we shall make use of positive operator theory once more. Our main purpose here is to determine the dominant singular point, i.e. the element of the set Λ with the largest real part. From (5.15) and (5.16), the dominant singular point gives the growth bound of the semigroup $T(t)$ generated by $A + C$. First we show that:

LEMMA 5.3. *Suppose that the following assumption holds:*

ASSUMPTION 5.4.

$$y^*(\omega) < e^{-\gamma\omega}. \tag{5.17}$$

Then the operator $V_\lambda, \lambda \in R$ is nonsupporting with respect to E_+ and the following holds:

$$\lim_{\lambda \to -\infty} r(V_\lambda) = +\infty,\quad \lim_{\lambda \to +\infty} r(V_\lambda) = 0. \tag{5.18}$$

PROOF. By changing the order of integration in the expression (5.13), it can be shown that

$$\chi_\lambda(a,\sigma) = \int_\sigma^\omega [\Pi(\zeta) - \gamma\int_\sigma^\zeta \Pi(\eta)\exp(-\gamma(\zeta-\eta))d\eta]\beta(a,\zeta)N(\zeta)\exp(-\lambda(\zeta-\sigma))d\zeta. \tag{5.19}$$

If we define

$$G_\sigma(\zeta) := \Pi(\zeta) - \gamma\int_\sigma^\zeta \Pi(\eta)\exp(-\gamma(\zeta-\eta))d\eta, \tag{5.20}$$

then the operator $V_\lambda, \lambda \in R$ is positive if $G_\sigma(\zeta) > 0$ for almost all $\zeta \in [\sigma,\omega]$, $0 \leqslant \sigma \leqslant \omega$. Since $G_\sigma(\zeta)\exp(\gamma\zeta)$ is monotone decreasing for the variable ζ, $G_\sigma(\zeta) \geqslant \exp[\gamma(\omega-\zeta)]G_\sigma(\omega)$ for all $\zeta \in [\sigma,\omega]$. From $G_\sigma(\omega) \geqslant G_0(\omega)$, we know that $G_0(\omega) > 0$ is sufficient to quarantee $G_\sigma(\zeta) > 0$ for all $0 \leqslant \sigma \leqslant \zeta \leqslant \omega$. Integrating (5.20) by parts, we have

$$G_\sigma(\zeta) = \Pi(\sigma)\exp(-\gamma(\zeta-\sigma)) - \int_\sigma^\zeta \lambda^*(\eta)\Pi(\eta)\exp(-\gamma(\zeta-\eta))d\eta.$$

Then we know that $G_0(\omega) = \exp(-\gamma\omega) - y^*(\omega)$ and the operator $V_\lambda, \lambda \in R$ is positive under Assumption 5.4. From the expression (5.19), we have for $\lambda \in R$,

$$\chi_\lambda(a,\sigma) \geqslant G_0(\omega)\phi_\lambda(a,\sigma). \tag{5.21}$$

Therefore, in order to show the nonsupporting property of $V_\lambda, \lambda \in R$, it suffices to prove that the integral operator $\hat{T}_\lambda$ defined by

$$(\hat{T}_\lambda\psi)(a) := \int_0^\omega \phi_\lambda(a,\sigma)\psi(\sigma)d\sigma, \ \psi\in E, \tag{5.22}$$

is nonsupporting. It is easy to verify the inequality

$$\hat{T}_\lambda\psi \geq <f_\lambda,\psi>e, \ e = 1\in E_+, \psi\in E_+, \tag{5.23}$$

where the linear functional f_λ is defined by

$$<f_\lambda,\psi> = \int_0^\omega[\int_\sigma^\omega s(x)N(x)\exp(-(\lambda+\gamma)(x-\sigma))dx]\psi(\sigma)d\sigma.$$

Then it follows that for all integers n

$$\hat{T}_\lambda^{n+1}\psi \geq <f_\lambda,\psi><f_\lambda,e>^n e.$$

Since f_λ is strictly positive and the constant function $e=1$ is a quasi-interior point of $L^1(0,\omega)$, it follows that $<F,\hat{T}_\lambda^n\psi>>0$ for every pair $\psi\in E_+\setminus\{0\}$, $F\in E_+^*\setminus\{0\}$. Then $\hat{T}_\lambda, \lambda\in R$ is nonsupporting. Next we show (5.18). From (5.21) and (5.23), we obtain

$$V_\lambda\psi \geq G_0(\omega)\hat{T}_\lambda\psi \geq G_0(\omega)<f_\lambda,\psi>e, \lambda\in R, \ \psi\in E_+.$$

Taking duality pairing with the eigenfunctional F_λ of V_λ that corresponds to $r(V_\lambda)$, we have

$$r(V_\lambda)<F_\lambda,\psi> \geq G_0(\omega)<F_\lambda,e><f_\lambda,\psi>.$$

If we let $\psi = e$, we arrive at the inequality

$$r(V_\lambda) \geq G_0(\omega)<f_\lambda,e>,$$

where

$$<f_\lambda,e> = \int_0^\omega\int_\sigma^\omega s(x)N(x)\exp[-(\lambda+\gamma)(x-\sigma)]dxd\sigma$$

$$= \int_0^\omega s(x)N(x)[\frac{1-e^{-(\lambda+\gamma)x}}{\lambda+\gamma}]dx \geq \epsilon\int_{\omega-\alpha}^\omega N(x)[\frac{1-e^{-(\lambda+\gamma)x}}{\lambda+\gamma}]dx.$$

Since $N(x)>0$ for $x\in[\omega-\alpha,\omega)$, we know that $\lim_{\lambda\to-\infty} r(V_\lambda) = +\infty$. On the other hand, we obtain

$$V_\lambda\psi \leq T_\lambda\psi \leq \hat{T}_\lambda\psi \leq <g_\lambda,\psi>e, \ \lambda\in R, \psi\in E_+,$$

where the positive functional g_λ is defined by

$$<g_\lambda,\psi> := M\int_0^\omega[\int_\sigma^\omega N(x)\exp[-(\lambda+\gamma)(x-\sigma)]dx]\psi(\sigma)d\sigma,$$

where $M := \text{ess sup}\beta(a,\zeta)$. Then we obtain the estimate

$$r(V_\lambda) \leq <g_\lambda,e> = M\int_0^\omega N(x)[\frac{1-e^{-(\lambda+\gamma)x}}{\lambda+\gamma}]dx,$$

from which we know that $\lim_{\lambda\to+\infty} r(V_\lambda) = 0$. This completes the proof. □

From Assumption 5.4 and the expression (5.19), the kernel $\chi_\lambda(a,\sigma)$ is strictly decreasing as a function of $\lambda\in R$. Using Proposition 4.3, we know that the function $\lambda\to r(V_\lambda)$ is strictly decreasing for $\lambda\in R$. Moreover if there exists $\lambda\in R$ such that $r(V_\lambda) = 1$, then $\lambda\in\Lambda$, because $r(V_\lambda)\in P_\sigma(V_\lambda)$. From the monotonicity of $r(V_\lambda)$ and (5.18), it is easy to see that the following holds:

LEMMA 5.5. *Under Assumption 5.4, there exists a unique* $\lambda_0 \in R \cap \Lambda$ *such that* $r(V_{\lambda_0}) = 1$, *and* $\lambda_0 > 0$ *if* $r(V_0) > 1$; $\lambda_0 = 0$ *if* $r(V_0) = 1$; $\lambda_0 < 0$ *if* $r(V_0) < 1$.

Next, by using the similar argument as Theorem 6.13 of Heijmans (1986), we can prove that λ_0 is the dominant singular point:

LEMMA 5.6. *Suppose that Assumption 5.4 holds. If there exists a* $\lambda \in \Lambda, \lambda \neq \lambda_0$, *then* $\mathrm{Re}\lambda < \lambda_0$.

PROOF. Suppose that $\lambda \in \Lambda$ and $V_\lambda \psi = \psi$. Then $|V_\lambda \psi| = |\psi|$, where $|\psi|(a) := |\psi(a)|$. From the expression (5.19), it follows that $V_{\mathrm{Re}\lambda}|\psi| \geq |\psi|$. Taking duality pairing with $F_{\mathrm{Re}\lambda} \in E_+^*$ on both sides, we have $r(V_{\mathrm{Re}\lambda}) < F_{\mathrm{Re}\lambda}, |\psi| > \geq < F_{\mathrm{Re}\lambda}, |\psi| >$, from which we conclude that $r(V_{\mathrm{Re}\lambda}) \geq 1$, because $F_{\mathrm{Re}\lambda}$ is strictly positive. Since $r(V_\lambda)$, $\lambda \in R$ is a decreasing function, we obtain that $\mathrm{Re}\lambda \leq \lambda_0$. If $\mathrm{Re}\lambda = \lambda_0$, then $V_{\lambda_0}|\psi| = |\psi|$. In fact, if we suppose that $V_{\lambda_0}|\psi| > |\psi|$, taking duality pairing with the eigenfunctional F_0 corresponding to $r(V_{\lambda_0}) = 1$ on both sides yields $< F_0, |\psi| > > < F_0, |\psi| >$ which is a contradiction. Then we can write that $|\psi| = c\psi_0$ for some constant c which we may assume to be one, where ψ_0 is the eigenfunction corresponding to $r(V_{\lambda_0}) = 1$. Hence $\psi(a) = \psi_0(a)\exp(i\alpha(a))$ for some real-valued function α. If we substitute this relation into $V_{\lambda_0}\psi_0 = |V_\lambda \psi|$, we obtain

$$\int_0^\omega \int_\sigma^\omega G_\sigma(x)\beta(a,x)N(x)\exp(-\lambda_0(x-\sigma))\psi_0(\sigma)dxd\sigma$$

$$= \left| \int_0^\omega \int_\sigma^\omega G_\sigma(x)\beta(a,x)N(x)\exp(-(\lambda_0 + i\mathrm{Im}\lambda)(x-\sigma))\psi_0(\sigma)\exp(i\alpha(\sigma))dxd\sigma \right|.$$

From Lemma 6.12 of Heijmans (1986), it follows that $-\mathrm{Im}\lambda(x-\sigma) + \alpha(\sigma) = \beta$ for some constant β. Using the relation $V_\lambda \psi = \psi$, we obtain that $\exp(i\beta)V_{\lambda_0}\psi_0 = \psi_0\exp(i\alpha(a))$, so $\beta = \alpha(a)$, which implies that $\mathrm{Im}\lambda = 0$. This completes the proof. □

PROPOSITION 5.7. *Under Assumption 5.4, the equilibrium state* (x^*, y^*) *for (2.7a) - (2.7b) is locally asymptotically stable if* $r(V_0) < 1$ *and locally unstable if* $r(V_0) > 1$.

PROOF. From Lemma 5.5 and 5.6, we conclude that $\sup\{\mathrm{Re}\lambda;\ 1 \in P_\sigma(V_\lambda)\} = \lambda_0$. Hence it follows that $s(A+C) = \sup\{\mathrm{Re}\lambda;\ 1 \in P_\sigma(V_\lambda)\} < 0$ if $r(V_0) < 1$, and $s(A+C) > 0$ if $r(V_0) > 1$. This completes the proof. □

Now we can state the local stability results for our epidemic model:

PROPOSITION 5.8. (Local stability results) *Let* $r(T)$ *be the spectral radius of the operator* T *defined by (4.4). Then the followings hold:*

(1) *If* $r(T) < 1$, *the trivial equilibrium point of (2.7a)-(2.7b) is locally asymptotically stable.*
(2) *If* $r(T) > 1$, *the trivial equilibrium point of (2.7a)-(2.7b) is locally unstable.*
(3) *If* $r(T) > 1$ *and Assumption 5.4 holds for an endemic steady state, it is locally asymptotically stable.*

PROOF. By our definition (4.4) and (5.22), note that $T = \hat{T}_0$. Since Assumption 5.4 is satisfied for the trivial steady state, it is sufficient to consider only the case that Assumption 5.4 holds. From (5.11) and (5.19), we know that U_λ, V_λ are positive operators for $\lambda \in R$ under Assumption 5.4, and it follows that

$$V_\lambda \leq T_\lambda \leq \hat{T}_\lambda \text{ for } \lambda \in R. \tag{5.24}$$

which implies that $r(V_0) \leq r(\hat{T}_0) = r(T)$, where the equality holds if and only if $\lambda^*(a) \equiv 0$, which corresponds to the trivial equilibrium state. Hence, for the trivial equilibrium state, Proposition 5.7 says that if $r(T) = r(V_0)<1$, it is locally asymptotically stable and it is locally unstable if $r(T) = r(V_0)>1$. Next we show the result (3). By Proposition 5.7, it suffices to show that $r(V_0)<1$ for the endemic equilibrium state. From (5.11), we obtain the inequality $r(V_\lambda)<r(T_\lambda), \lambda \in R$, since T_λ is nonsupporting for $\lambda \in R$ and $V_\lambda \neq T_\lambda$ when $\lambda^*(a) \neq 0$. In particular, the nonsupporting operator T_0 has an expression

$$(T_0\psi)(a) = \int_0^\omega \phi(a,\sigma)\exp(-\int_0^\sigma \lambda^*(\eta)d\eta)\psi(\sigma)d\sigma. \tag{5.25}$$

Since $\lambda^*(a)$ is a non-trivial positive solution of $x = \Phi(x)$, it follows that T_0 has a positive eigenfunction $\lambda^*(a)$ corresponding to the eigenvalue one. Since a nonsupporting operator has only one positive eigenfunction corresponding to its spectral radius, we conclude that $r(T_0) = 1$, and hence $r(V_0)<1$. This shows that the endemic equilibrium state satisfying Assumption 5.4 is locally asymptotically stable. □

We have not determined what kind of conditions could guarantee Assumption 5.4. Since it would be difficult to answer the question if we consider it under most general conditions, let us consider a simple case in the following example:

EXAMPLE 5.9. Suppose that the transmission coefficient β is constant. In this case the steady state is given by

$$x^*(a) = \exp(-\lambda^* a),$$
$$y^*(a) = \frac{\lambda^*}{\gamma-\lambda^*}(\exp(-\lambda^* a)-\exp(-\gamma a)),$$

where the constant force of infection λ^* at the steady state is given by

$$\lambda^* = \beta\int_0^\omega N(a)y^*(a)da. \tag{5.26}$$

Defining a function $f(\lambda)$ as follows;

$$f(\lambda) := \frac{\beta}{\gamma-\lambda}\int_0^\omega(\exp(-\lambda a)-\exp(-\gamma a))N(a)da. \tag{5.27}$$

Then (5.26) yields the characteristic equation $\lambda^*(1-f(\lambda^*)) = 0$. In particular, it follows that $f(0) = r(T)$. Since $f(\lambda)$ has an expression

$$f(\lambda) = \beta\int_0^\omega N(a)\exp(-\gamma a)\int_0^a \exp((\gamma-\lambda)x)dxda,$$

then $f(\lambda)$ is strictly decreasing for $\lambda \in R$. If $f(0) \leqslant 1$, $\lambda^* = 0$ is only nonnegative solution of the characteristic equation and if $f(0)>1$, there exists another possible solution which is given as a unique positive solution of the equation $f(\lambda) = 1$. If we define the critical value of the transmission rate by

$$\beta^* = \gamma(\int_0^\omega(1-\exp(-\gamma a))N(a)da)^{-1}, \tag{5.28}$$

then $f(0) = \beta/\beta^*$, the equation $f(\lambda) = 1$ has only one positive root if and only if $\beta>\beta^*$ and it has a zero solution if $\beta = \beta^*$. On the other hand we obtain

$$G_0(a) = \frac{1}{\gamma-\lambda^*}(\gamma\exp(-\gamma a)-\lambda^*\exp(-\lambda^* a)). \tag{5.29}$$

From (5.29), we know that a necessary condition to show that $G_0(\omega)>0$ is $\lambda^* < \omega^{-1} < \gamma$ (from the physical meaning, it is always that $\omega^{-1} < \gamma$). Moreover if λ^* (the total infectivity) is sufficiently small, then $G_0(\omega)>0$, that is, the inequality (5.17) holds. This situation is possible if β is sufficiently near to the critical value β^*, because in that case the positive root λ^* of $f(\lambda) = 1$ is small enough.

REMARK 5.10. Instead of Assumption 5.4, if we adopt Assumption 4.9, we can say that there is no nonnegative element in the set Λ of singular points for the endemic steady states. In fact, from Assumption 4.9 and the expression (5.13), we know that V_λ is a positive operator for $\lambda \in R_+ = [0,\infty)$. Suppose that there exists $\mu \in \Lambda \cap R_+$. Then there is a $\psi \in E_+ \setminus \{0\}$ such that $V_\mu \psi = \psi = T_\mu \psi - U_\mu \psi$. Let F_μ be the eigenfunctional corresponding to $r(T_\mu)$. Then F_μ is strictly positive, since T_μ is nonsupporting. Since $<F_\mu, U_\mu \psi> > 0$ for the endemic steady state, we obtain

$$<F_\mu,\psi> < <F_\mu, T_\mu \psi> = r(T_\mu) <F_\mu,\psi>,$$

which shows that $r(T_\mu)>1$, because $<F_\mu,\psi> > 0$. On the other hand, $r(T_\lambda)$, $\lambda \in R_+$ is a strictly decreasing function, it follows that $r(T_0) = 1 \geq r(T_\lambda)$ for $\lambda \in R_+$. This is a contradiction. Therefore we conclude that the set $\Lambda \cap R_+$ is empty. However there remains a possibility that the set Λ contains complex roots with positive real part, and hence we cannot exclude the possibility of unstable endemic steady states. Nevertheless, if $r(T)>1$ but $r(T)-1$ is small, the endemic steady states are locally stable (the principle of the exchange of stability).

Finally, in the case that $r(T)<1$, we shall prove the global stability for the trivial equilibrium state:

PROPOSITION 5.11. (Global stability result) *If $r(T)<1$, the trivial equilibrium point of (2.7) is globally stable with respect to positive initial conditions.*

PROOF. By Lemma 3.1, it suffices to show the global stability for the equation (3.5) with respect to the initial data $u_0 = (\hat{x}_0, y_0) \in \Omega_0$. As was seen in (3.8), the second element $y(t)$ of $u(t;u_0)$ of (3.5) is governed by the abstract equation

$$\frac{d}{dt} y(t) = By(t) + (Py(t))(1+\hat{x}(t)),\ y(0) = y_0 \in L^1(0,\omega),$$

which can be seen as a linear equation for $y(t)$ if we consider $\hat{x}(t)$ as a known function. If we define a bounded operator $C(t): E \rightarrow E$, $t \geq 0$ by $C(t)\phi := (P\phi)(1+\hat{x}(t))$, then we have

$$y(t) = S(t)y(0) + \int_0^t S(t-s)C(s)y(s)ds \leq S(t)y(0) + \int_0^t S(t-s)Py(s)ds, \tag{5.30}$$

because $-1 \leq \hat{x}(t) \leq 0$, $0 \leq y(t) \leq 1$ for all $t \geq 0$. Therefore we conclude that $0 \leq y(t) \leq W(t)y(0)$, where $W(t)$, $t \geq 0$ is a C_0-semigroup generated by the perturbed operator $B+P$. By the same reason as the proof of Lemma 5.2, $W(t)$ is eventually norm continuous. Moreover the resolvent $R(\lambda, B+P)$ is given by

$$R(\lambda, B+P)\psi = \int_0^a \exp[-(\lambda+\gamma)\sigma][(I-\hat{T}_\lambda)^{-1}\psi](\sigma)d\sigma,$$

where the operator $\hat{T}$ is defined by (5.22). Then we know that $B+P$ has a compact resolvent and

$$\sigma(B+P) = P_\sigma(B+P) = \{\lambda \in C:\ 1 \in \sigma(\hat{T}_\lambda)\}.$$

Let $\Sigma := \{\lambda \in C:\ 1 \in \sigma(\hat{T}_\lambda)\}$. Since $\hat{T}_\lambda, \lambda \in R$ is compact, we obtain that $\Sigma = \{\lambda \in C:\ 1 \in P_\sigma(\hat{T}_\lambda)\}$. Using similar arguments as in the proofs of Lemma 5.5 and Lemma 5.6, we knows that there exists a unique $\lambda_0 \in R \cap \Sigma$ such that $r(\hat{T}_{\lambda_0}) = 1$ and $s(B+P) = \lambda_0$. Hence if $r(T) = r(\hat{T}_0) < 1$, then $\omega_0(B+P) = s(B+P) = \lambda_0 < 0$ since $r(\hat{T}_\lambda)$ is strictly decreasing for $\lambda \in R$. That is, the semigroup

$W(t)$ is exponentially stable and $\lim_{t\to\infty} y(t) = 0$. From (2.9) and (3.7), it is easily seen that $\lim_{t\to\infty} x(t) = 1$. This completes the proof. □

Summary and Discussions

In this paper we have examined Greenhalgh's conjectures for an age-structured SIR-type epidemic model. Under the appropriate conditions, we could prove his conjectures, i.e. (1) there exists a threshold value $r(T)$ given as the spectral radius of the positive linear operator T; (2) The equilibrium with no disease is always possible and it is locally, in fact globally, stable if $r(T)<1$ and unstable if $r(T)>1$; (3) The endemic equilibrium state is possible if and only if $r(T)>1$; (4) The endemic steady state is unique if Assumption 4.9 holds and locally stable if Assumption 5.4 is satisfied. However it should be noted that the results (1)-(3) are robust to the variation of parameters, the conditions for uniqueness and stability for the endemic steady state are rather sensitive to the values of parameters. To seek more advantageous conditions to guarantee the uniqueness and stability for the endemic equilibrium state remains as an open problem. Another important question is whether destabilization of the endemic steady state could lead to the bifurcation of time-periodic solutions. This phenomenon would give an explanation for the fact that some SIR-type diseases tend to occur in regular periodic cycles (see Greenhalgh, 1988b).

On the other hand it should be also noted that the model investigated here is based on some restrictive assumptions as an epidemic model. We have assumed that;

(1) the population is in a demographically steady state;
(2) the latent period is negligibly short;
(3) the recovery rate is constant;
(4) the infectivity is independent of the duration of an infection;

The assumption of demographic steady state is appropriate for short-time argument in developed countries with low population growth rate, but in general the fact that the population growth affects the spread of disease is important in case that we consider epidemiology in populations with high growth rate or we examine the diseases with a long latent period. If we intend to take into account the latent period, it suffices to introduce the incubation class into the model. The reader may refer to McLean (1986) for more realistic model building which takes into account the incubation class and the effect of demographic growth. Most essential improvement in the model would be attained by introducing duration-dependence in the transmission process. Several authors have already introduced SIR-type age-dependent epidemic models with duration-dependent infectivity (Hoppensteadt, 1974; Gripenberg, 1983; Dietz and Schenzle, 1985). However they assume that the transmission coefficient has a special form, and hence the general case that the transmission coefficient depends on age of both susceptibles and infectious should be investigated in future.

Appendix

We shall prove the compactness criterion for a linear integral operator in the Banach space $L^1(0,\omega)$.

Lemma. *Let T be a linear integral operator from $E := L^1(0,\omega)$ to E defined by*

$$(Tf)(a) = \int_0^{\omega} G(a,\sigma) f(\sigma) d\sigma, \ f \in E,$$

where the kernel $G(a,\sigma)$ satisfies

$$G(a,\sigma) \in L^{\infty}((0,\omega)\times(0,\omega)), \tag{1}$$

$$\lim_{h\to 0} \int_0^{\omega} |G(a+h,\sigma) - G(a,\sigma)| da = 0 \tag{2}$$

uniformly for $\sigma \in R$, where $G(a,\sigma)$ is defined as $G(a,\sigma) = 0$ for $a, \sigma \in (-\infty,0)\cup(\omega,\infty)$. Then T is a *compact operator.*

PROOF. We identify the Banach space E with the subspace of $L^1(R)$ such that $E = \{\psi \in L^1(R): \psi(a) = 0 \text{ for } a \in (-\infty,0)\cup(\omega,\infty)\}$ and extend the domain of $G(a,\sigma)$ as $G(a,\sigma) = 0$ for $a,\sigma \in (-\infty,0)\cup(\omega,\infty)$. Then we can interpret T as an operator on $L^1(R)$ such that E is its invariant subspace, so it is sufficient to show that the extended operator T is compact in $L^1(R)$. Let K be a bounded subset of $L^1(R)$. Then it follows immediately that $T(K)$ is also a bounded subset. Observe that

$$\int_R |(Tf)(a+h)-(Tf)(a)|\,da \leq \int_R\int_R |G(a+h,\sigma)-G(a,\sigma)|\,|f(\sigma)|\,d\sigma da$$
$$\leq \|f\| \sup_{0\leq\sigma\leq\omega} \int_R |G(a+h,\sigma)-G(a,\sigma)|\,da.$$

Therefore it follows from the condition (2) for G that $T(K)$ is an equicontinuous family in L^1-norm. Moreover it follows from $T(K)\subset E$ that

$$\int_{|\sigma|\geq\omega} |(Tf)(\sigma)|\,d\sigma = 0,\ f\in K.$$

Thus we can apply the compactness criterion by Frechet-Kolmogorov (YOSIDA, p.275; DUNFORD & SCHWARTZ, p.298), that is, $T(K)$ is relatively compact in $L^1(R)$. Then T is a compact operator. □

ACKNOWLEDGEMENTS

I am grateful to Odo Diekmann for drawing my attention to the work of Greenhalgh and giving me valuable comments and to Hans Heesterbeek for introducing various papers related to epidemic models to me. This work was partially supported by a visitors grant from the Dutch National Science Foundation NWO.

REFERENCES

H. AMANN, On the number of solutions of nonlinear equations in ordered Banach space. *J. Func. Anal.* 11, 346-384 (1972).

R.M. ANDERSON, R.M. MAY, Age related changes in the rate of disease transmission: implications for the design of vaccination programmes. *J. Hyg. Camb.* 94, 365-436 (1985).

V.A. ANDREASEN, *Dynamical models of epidemics in age-structured populations - analysis and simplification*. Ph.D. Thesis, Cornel University 1988.

S. BUSENBERG, K. COOKE, M. IANNELLI, Endemic threshold and stability in a class of an age-structured epidemics. *SIAM J. Appl. Math.* Vol. 48, No. 6, 1379-1395 (1988).

W. DESCH, W. SCHAPPACHER, Linearized stability for nonlinear semigroups. In: Favini, A., Obrecht, E. (eds.) *Differential Equations in Banach Spaces*. (Lect. Notes Math. vol. 1223, pp. 61-73) Berlin: Springer 1986.

K. DIETZ, Transmission and control of arbovirus disease. In: Ludwig, D. Cooke, K.L. (eds.), *Proc. SIMS Conf. on Epidemiology*, pp. 104-121. Philadelphia: SIAM 1975.

K. DIETZ, D. SCHENZLE, Proportionate mixing models for age-dependent infection transmission. *J. Math. Biol.* 22, 117-120 (1985).

N. DUNFORD, J.T. SCHWARTZ, *Linear Operators Part I: General Theory*. New York: John Wiley & Sons 1957.

D. GREENHALGH, Analytical results on the stability of age-structured recurrent epidemic models. *IMA J. Math. Appl. Med. Biol.* 4, 109-144 (1987).

D. GREENHALGH, Analytical threshold and stability results on age dependent epidemic models with vaccination. *Theor. Pop. Biol.* 33, 266-290 (1988a).

D. GREENHALGH, Threshold and stability results for an epidemic model with an age-structured meeting rate. *IMA J. Math. Appl. Med. Biol.* 5, 81-100 (1988b).

G. GRIPENBERG, On a nonlinear integral equation modelling an epidemic in an age-structured population. *J. reine angew. Math.* 341, 54-67 (1983).

D. GUO, V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*. London: Academic Press 1988.

H.J.A.M. HEIJMANS, The dynamical behaviour of the age-size-distribution of a cell population. In: J.A.J. Metz, O. Diekmann, (eds.) *The Dynamics of Physiologically Structured Populations*. (Lect. Notes Biomath., vol. 68, pp. 185-202) Berlin: Springer 1986.

H.W. HETHCOTE, Asymptotic behaviour and stability in epidemic models. In: P. van den Driessche, (ed.) *Mathematical Problems in Biology*. Victoria Conference (Lect. Notes Biomath. vol. 2, pp. 83-92) Berlin: Springer 1974.

F. HOPPENSTEADT, An age dependent epidemic model. *J. Franklin Inst.* 297(5), 325-333 (1974).

F. HOPPENSTEADT, *Mathematical Theories of Populations: Demographics, Genetics and Epidemics*. Philadelphia: Society for Industrial and Applied Mathematics 1975.

T. KATO, *Perturbation Theory for Linear Operators*, 2nd Edition. Berlin: Springer 1984.

H. KNOLLE, The general age-dependent endemic with age-specific contact rate. *Biom. J.* 25, 469-475 (1983).

M.A. KRASNOSELSKII, *Positive Solutions of Operator Equations*. Groningen: Noordhoff 1964a.

M.A. KRASNOSELSKII, *Topological Methods in the Theory of Nonlinear Integral Equations*. Oxford: Pergamon Press 1964b.

M.A. KREIN, M.A. RUTMAN, Linear operators leaving invariant a cone in a Banach space. *Uspehi Math. Nauk.* 3, No. 1(23), 3-95 (1948) (Russian); English translation: *Am. Mat. Soc. Transl.* (1), 10, 199-325 (1950).

A. KUFNER, O. JOHN, S. FUCIK, *Function Spaces*, Leyden: Noordhoff 1977.

P. MARCATI, On a nonconservative hyperbolic system describing the nonlinear age dependent population growth. *Comp. & Maths. with Appls.* Vol. 11, Nos. 1-3, 207-222 (1985).

I. MAREK, Frobenius theory of positive operators: comparison theorems and applications. *SIAM J. Appl. Math.* Vol. 19, No. 3, 607-628 (1970).

A. MCKENDRICK, Applications of mathematics to medical problems. *Proc. Edin. Math.* Soc. 44, 98-130 (1926).

A. MCLEAN, Dynamics of childhood infections in high birthrate countries. In: G.W. Hoffmann, T. Hraba, (eds.) *Immunology and Epidemiology*. (Lect. Notes Biomath. vol. 65, pp. 171-197) Berlin: Springer 1986.

J.A.J. METZ, O. DIEKMANN, (eds.), *The Dynamics of Physiologically Structured Populations*. (Lect. Notes Biomath. vol. 68) Berlin: Springer 1986.

R. NAGEL, (ed.) *One-Parameter Semigroups of Positive Operators*. Lect. Notes Math. vol. 1184, Berlin: Springer 1986.

F. NIIRO, I. SAWASHIMA, On the spectral properties of positive operators in an arbitrary Banach lattice and problems of H.H. Schaefer. *Sci. Papers College Gen. Ed. Univ. Tokyo* 16, 145-183 (1966).

I. SAWASHIMA, On spectral properties of some positive operators. *Nat. Sci. Dep. Ochanomizu Univ.* 15, 53-64 (1964).

D. SCHENZLE, An age-structured model of pre- and post-vaccination measles transition. *IMA J. Math. Appl. Med. Biol.* 1, 169-191 (1984).

D. SCHENZLE, Control of virus transmission in age-structured populations. In: V. Capasso, E. Grosso, S.L. Paveri-Fontana, (eds.) *Mathematics in Biology and Medicine*. (Lect. Notes Biomath. vol. 57, pp. 171-178) Berlin: Springer 1985.

S. STEINBERG, Meromorphic families of compact operators. *Arch. Rat. Mech. An.* 31, 372-380 (1968).

D.W. TUDOR, An age-dependent epidemic model with applications to measles. *Math. Biosci.* 73, 131-147 (1985).

G.F. WEBB, *Theory of Nonlinear Age-Dependent Population Dynamics*. New York and Basel: Marcel Dekker 1985.

K. YOSIDA, *Functional Analysis*, 6th Edition. Berlin: Springer 1980.